SOME CHARACTERIZATIONS OF REGULAR MODULES

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Abstract

Let $M$ be a left module over a ring $R$. $M$ is called a Zelmanowitz-regular module if for each $x \in M$ there exists a homomorphism $f: M \to R$ such that $f(x)x = x$. Let $Q$ be a left $R$-module and $h: Q \to M$ a homomorphism. We call $h$ locally split if for each $x \in M$ there exists a homomorphism $g: M \to Q$ such that $h(g(x)) = x$. $M$ is called locally projective if every epimorphism onto $M$ is locally split. We prove that the following conditions are equivalent:

1. $M$ is Zelmanowitz-regular.
2. Every homomorphism into $M$ is locally split.
3. $M$ is locally projective and every cyclic submodule of $M$ is a direct summand of $M$.

As generalizations of the concept of Von Neumann's regular rings to the module case, there have been considered three types of modules by Fieldhouse [1], Ware [4] and Zelmanowitz [5], each called regular. The Fieldhouse-regular module was defined to be a module whose submodules are pure submodules and the Ware-regular module was defined as a projective module in which every cyclic submodule is a direct summand, while a left module $M$ over a ring $R$ is called a Zelmanowitz-regular module if for each $x \in M$ there exists a homomorphism $f: M \to R$ such that $f(x)x = x$. Now we introduce a notion of locally split homomorphisms to show that a module is Zelmanowitz-regular if and only if every homomorphism into the module is locally split, and by making use of this we prove that Zelmanowitz-regular modules are characterized as locally projective modules whose cyclic submodules are direct summands. For convenience (but at the risk of confusion), we call a module regular if every cyclic submodule of it is a direct summand. Thus, in this terminology, a module is Ware-regular or Zelmanowitz-regular if and only if it is projective regular or locally projective regular respectively. Moreover we shall see that every regular module is Fieldhouse-regular and that Ware-regular and Zelmanowitz-regular modules are also characterized as projective Fieldhouse-regular and locally projective Fieldhouse-regular modules respectively.

Let $R$ be a ring with identity element. By a module we shall throughout mean a unital left $R$-module, unless otherwise specified. Let $Q$ and $M$ be modules, and let $h: Q \to M$ be a $(R-) \text{ homomorphism}$. $h$ is called locally split if for any $x_0 \in h(Q)$ there exists a homomorphism $q: M \to Q$ such that $h(q(x_0)) = x_0$.  


Proposition 1. Let \( h : Q \to M \) be a locally split homomorphism. Then, for any finite number of \( x_1, x_2, \ldots, x_n \in h(Q) \), there exists a homomorphism \( q : M \to Q \) such that \( h(q(x_i)) = x_i \) for \( i = 1, 2, \ldots, n \).

Proof: In order to prove by induction, suppose that \( n > 1 \) and our assertion is true for \( n - 1 \) (instead of \( n \)). Then there exists a \( q_1 : M \to Q \) such that \( h(q_1(x_n)) = x_n \). Since \( x_n = h(q_1(x_n)) \) is in \( h(Q) \), there is a \( q_2 : M \to Q \) such that \( h(q_2(x_n - h(q_1(x_n)))) = x_n - h(q_1(x_n)) \). Let \( q = q_1 + q_2 \circ h \circ q_1 : M \to Q \). Then \( h(q(x_n)) = h(q_1(x_n)) + h(q_2(x_n)) - h(q_2(h(q_1(x_n)))) = h(q_1(x_n)) + h(q_2(x_n)) = x_n \), and \( h(q(x_i)) = h(q_1(x_i)) + h(q_2(x_i)) - h(q_2(h(q_1(x_i)))) = x_i \) for \( i = 1, 2, \ldots, n - 1 \). Thus \( q \) is a desired homomorphism.

Let \( N \) be a submodule of a module \( M \). \( N \) is called locally split in \( M \) if the inclusion map \( N \to M \) is locally split, i.e., for any \( x_0 \in N \) there exists a homomorphism \( s : M \to N \) such that \( s(x_0) = x_0 \).

Proposition 2. Let \( h : Q \to M \) be a homomorphism. Denote by \( h' \) the epimorphism \( Q \to h(Q) \) regarded \( h \) as a map onto \( h(Q) \). Then \( h \) is locally split if and only if \( h' \) is locally split and \( h(Q) \) is locally split in \( M \).

Proof: Let \( x_0 \) be any element of \( h(Q) \). Suppose that \( h \) is locally split. Then there exists a homomorphism \( q : M \to Q \) such that \( h(q(x_0)) = x_0 \). This implies that the homomorphism \( s = h \circ q : M \to h(Q) \) satisfies \( s(x_0) = x_0 \), and thus \( h(Q) \) is locally split in \( M \). On the other hand, if we denote by \( q' : h(Q) \to Q \) the restriction of \( q \) to \( h(Q) \) then we have \( h'(q'(x_0)) = h(q(x_0)) = x_0 \), which shows that \( h' \) is locally split. Suppose conversely that \( h(Q) \) is locally split in \( M \) and \( h' \) is also locally split. This means that there exist homomorphisms \( s : M \to h(Q) \) and \( q' : h(Q) \to Q \) such that \( s(x_0) = x_0 \) and \( h'(q'(x_0)) = x_0 \). Let \( q = q' \circ s : M \to Q \). Then we have \( h(q(x_0)) = h'(q'(s(x_0))) = h'(q'(x_0)) = x_0 \). Thus \( h \) is locally split.

Proposition 3. Let \( M \) be a module. Then every locally split submodule of \( M \) is pure in \( M \), while every locally split epimorphism from \( M \) is pure, i.e., the kernel of the epimorphism is pure in \( M \).

Proof: Let \( N \) be a locally split submodule of \( M \). Let \( x_1, x_2, \ldots, x_n \in M \) satisfy the system of linear equations \( r_1 x_1 + r_2 x_2 + \cdots + r_m x_m = v_i \) (\( i = 1, 2, \ldots, m \)), where each \( r_{ij} \in R \) and \( v_i \in N \). Then, by applying Proposition 1 to \( v_1, v_2, \ldots, v_m \) and the inclusion map \( N \to M \) (instead of \( x_1, x_2, \ldots, x_n \) and \( h : Q \to M \)), we can find a homomorphism \( s : M \to N \) such that \( s(v_i) = v_i \) for \( i = 1, 2, \ldots, m \). We have then \( r_{11} s(x_1) + r_{12} s(x_2) + \cdots + r_{1n} s(x_n) = s(v_1) = v_1 \) (\( i = 1, 2, \ldots, m \)). Since each \( s(x_i) \) is in \( N \), this shows that \( N \) is pure in \( M \) by Cohn's theorem.

Let next \( h : M \to M' \) be an epimorphism and \( N \) the kernel of \( h \). Let \( x_1, x_2, \ldots, x_n \in M \) satisfy the system of linear equations \( r_1 x_1 + r_2 x_2 + \cdots + r_m x_m = v_i \) (\( i = 1, 2, \ldots, m \))
\[ r_{in}x_n = v_i \quad (i = 1, 2, \ldots, m), \]
where \( r_{ij} \in R \) and \( v_i \in N \). Then we have
\[ r_{11}h(x_1) + r_{12}h(x_2) + \cdots + r_{1n}h(x_n) = h(v_1) = 0 \quad (i = 1, 2, \ldots, m). \]
Suppose that \( h \) is locally split. Then since each \( h(x_j) \) is in \( h(M) = M' \), by applying Proposition 1 to
\( h(x_1), h(x_2), \ldots, h(x_n) \) and \( h : M \to M' \) (instead of \( x_1, x_2, \ldots, x_n \) and \( h : Q \to M' \)), we find a homomorphism \( q : M' \to M' \) such that \( h(q(h(x_j))) = h(x_j) \), i.e., \( x_j - q(h(x_j)) \in N \) for \( j = 1, 2, \ldots, n \). From the above equalities it follows now
\[ r_{11}(x_1 - q(h(x_1))) + r_{12}(x_2 - q(h(x_2))) + \cdots + r_{1n}(x_n - q(h(x_n))) = v_i \quad (i = 1, 2, \ldots, m). \]
This implies that \( N \) is pure in \( M \) again by Cohn's theorem. \( \blacksquare \)

**Remark.** The notion of locally split submodules was introduced by Ramamurthi and Rangaswamy [2] by the name of strongly pure submodules, and they actually obtained the first half of the preceding proposition.

**Theorem 4.** Let \( M \) be a left \( R \)-module. Then the following conditions are equivalent:

1. \( M \) is a Zelmanowitz-regular module.
2. Every homomorphism into \( M \) (from any module) is locally split.
3. Every homomorphism \( R \to M \) is locally split.

**Proof:** (1) \( \Rightarrow \) (2): Let \( Q \) be a module and \( h : Q \to M \) a homomorphism. Let \( x_0 \) be any element of \( h(Q) \). Choose a \( z_0 \in Q \) such that \( h(z_0) = x_0 \). Since \( M \) is Zelmanowitz-regular, there exists a homomorphism \( f : M \to R \) such that \( f(x_0)z_0 = x_0 \). Define a homomorphism \( q : M \to Q \) by \( q(x) = f(x)z_0 \) for \( x \in M \). Then we have \( h(q(x_0)) = f(x_0)h(z_0) = f(x_0)x_0 = x_0 \), which shows that \( h \) is locally split.

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1): Let \( x_0 \) be any element of \( M \). Let \( g : R \to M \) be the homomorphism defined by \( g(r) = rx_0 \) for \( r \in R \). Then \( g \) is locally split, so that there exists a homomorphism \( f : M \to R \) such that \( x_0 = g(f(x_0)) = f(x_0)x_0 \). This shows that \( M \) is Zelmanowitz-regular. \( \blacksquare \)

Now we call a module \( M \) a regular module if every submodule of \( M \) is locally split in \( M \).

**Proposition 5.** Let \( M \) be a module. Then the following conditions are equivalent:

1. \( M \) is a regular module.
2. Every finitely generated submodule of \( M \) is a direct summand of \( M \).
3. Every cyclic submodule of \( M \) is a direct summand of \( M \).

**Proof:** (1) \( \Rightarrow \) (2): Let \( N = Rx_1 + Rx_2 + \cdots + Rx_n \) be a finitely generated submodule of \( M \). Since \( M \) is regular, \( N \) is locally split and therefore, by applying Proposition 1 to the inclusion map \( N \to M \) (instead of \( h : Q \to M \)), we can find a homomorphism \( s : M \to N \) such that \( s(x_i) = x_i \) for \( i = 1, 2, \ldots, n \).
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1, 2, ..., n, or equivalently, \( s(x) = x \) for all \( x \in N \). This implies that \( N \) is a direct summand of \( M \).

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1): Let \( N \) be a submodule of \( M \). Let \( x_0 \) be any element of \( N \). Then the cyclic submodule \( Rx_0 \) is a direct summand of \( M \), which means that there is a homomorphism \( s : M \to Rx_0(\subseteq N) \) such that \( s(x_0) = x_0 \). Thus \( N \) is locally split in \( M \).

It is to be pointed out that every submodule of a regular module is regular too, and every regular module is Fieldhouse-regular, i.e., every submodule is a pure submodule.

A module \( M \) is called locally projective if every epimorphism onto \( M \) (from any module) is locally split. It follows from Proposition 3 that every locally projective module is flat, since a flat module is characterized as a module onto which every epimorphism is pure. The notion of locally projective modules was introduced by Zimmermann-Huisgen [6] and also by Raynaud and Gruson [3] under the name of flat strict Mittag-Leffler modules. Their definitions are apparently different from the above one. But the following proposition implies that all the definitions coincide (if compared with [6], Theorem 2.1 and [3], Proposition 2.3.4), and we will give a proof to the proposition for completeness:

**Proposition 6.** Let \( M \) be a left \( R \)-module. Then the following conditions are equivalent:

(1) \( M \) is locally projective.

(2) For any finitely generated submodule \( M_0 \) of \( M \), there exist a finitely generated free left \( R \)-module \( F \) and homomorphisms \( f : M \to F \) and \( g : F \to M \) such that \( g(f(x)) = x \) for all \( x \in M_0 \).

(3) For any \( x_0 \in M \), there exist a finite number of homomorphisms \( f_i : M \to R \) \( (i = 1,2,\ldots,n) \) and elements \( y_i \in M \) \( (i = 1,2,\ldots,n) \) such that

\[
\sum_{i=1}^{n} f_i(x_0) y_i = x_0.
\]

**Proof:** (1) \( \Rightarrow \) (2): Let \( Q \) be a free \( R \)-module having an epimorphism \( h : Q \to M \). Then \( h \) is locally split, so that, by applying Proposition 1 to the finite number of generators of \( M_0 \), we can find a homomorphism \( q : M \to Q \) such that \( h(q(x)) = x \) for all \( x \in M_0 \). Since the image \( q(M_0) \) of \( M_0 \) is a finitely generated submodule of \( Q \), there exists a finite subset \( \{u_1,u_2,\ldots,u_n\} \) of the free basis of \( Q \) such that \( q(M_0) \) is contained in the finitely generated free submodule \( F = Ru_1 + Ru_2 + \cdots + Ru_n \) of \( Q \). Since \( F \) is a direct summand of \( Q \), there exists a homomorphism \( p : Q \to F \) such that \( p(z) = z \) for all \( z \in F \). Let \( f = p \circ q : M \to F \), and let \( g : F \to M \) be the restriction of \( h \) to \( F \). Then they clearly satisfy \( g(f(x)) = x \) for all \( x \in M_0 \).

(2) \( \Rightarrow \) (3): Let \( x_0 \in M \). Since \( Rx_0 \) is finitely generated, there exist a finitely generated free \( R \)-module \( F \) and homomorphisms \( f : M \to F \), \( g : F \to M \) such that \( g(f(x_0)) = x_0 \). Let \( u_1,u_2,\ldots,u_n \) be a free basis of \( F \). Then we can, for
each i, define a homomorphism $f_i : M \rightarrow R$ by $f_i(x) = f_i(x)u_1 + f_2(x)u_2 + \cdots + f_n(x)u_n$ for $x \in M$. Let $y_i = g(u_i) \in M$ for $i = 1, 2, \ldots, n$. Then we have $x_0 = g(f(x_0)) = f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n$.

(3) $\Rightarrow$ (1): Let $Q$ be any $R$-module having an epimorphism $h : Q \rightarrow M$. Let $x_0 \in M$, and let $f_i : M \rightarrow R$ and $y_i \in M$ ($i = 1, 2, \ldots, n$) be as in (3). Let $z_i \in Q$ be such that $h(z_i) = y_i$ for each i, and define a homomorphism $q : M \rightarrow Q$ by $q(x) = f_1(x)z_1^{-1} - f_2(x)z_2 + \cdots + f_n(x)z_n$ for $x \in M$. Then we have that $h(q(x_0)) = f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$. Thus $h$ is locally split, so that $M$ is locally projective.

**Proposition 7.** Let $M$ be a locally projective module, and let $N$ be a pure submodule of $M$. Then $N$ is locally projective and is locally split in $M$.

**Proof:** Let $x_0$ be any element of $N$. By the preceding proposition, there exist homomorphisms $f_i : M \rightarrow R$ and elements $y_i \in M$ ($i = 1, 2, \ldots, n$) such that $f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$. Since $N$ is pure in $M$, we can find elements $v_1, v_2, \ldots, v_n$ in $N$ such that $f_1(x_0)v_1 + f_2(x_0)v_2 + \cdots + f_n(x_0)v_n = x_0$ according to Cohn's criterion. Now we define a homomorphism $s : M \rightarrow N$ by $s(x) = f_1(x)v_1 + f_2(x)v_2 + \cdots + f_n(x)v_n$ for $x \in M$. Then we have that $s(x_0) = x_0$. Thus $N$ is locally split in $M$. On the other hand, if we denote by $g_i$ the restriction of $f_i$ to $N$ then clearly we have that $g_1(x_0)v_1 + g_2(x_0)v_2 + \cdots + g_n(x_0)v_n = x_0$, which shows that $N$ is locally projective.

**Remark.** That $N$ is locally projective in Proposition 7 was mentioned in [6, p. 236].

**Theorem 8.** Let $M$ be a module. Then the following conditions are equivalent:

(1) $M$ is a Zelmanowitz-regular module.

(2) $M$ is a locally projective regular module.

(3) $M$ is locally projective and Fieldhouse-regular (i.e., every submodule of $M$ is pure in $M$).

**Proof:** (1) $\Rightarrow$ (2): If $M$ is Zelmanowitz-regular, it follows from Theorem 4 that every epimorphism onto $M$ is locally split and every monomorphism into $M$ is locally split, which mean that $M$ is locally projective and regular respectively. (Another proof for the local projectivity of $M$ can be obtained directly from Proposition 6, for that for any $x_0 \in M$ there exists an homomorphism $f : M \rightarrow R$ such that $f(x_0)x_0 = x_0$ implies that $M$ satisfies the condition (3) in Proposition 6 with $n = 1, f_1 = f$ and $y_1 = x_0$. That a Zelmanowitz-regular module is regular, i.e., every cyclic submodule of the module is a direct summand, is also proved in [5, Theorem 1.6].

(2) $\Rightarrow$ (3) is a consequence of the fact, due to Proposition 3, that every locally split submodule is a pure submodule.
$(3) \Rightarrow (1)$: Let $Q$ be a module and $h : Q \rightarrow M$ a homomorphism. Since $h(Q)$ is a pure submodule of $M$ by assumption, it follows from Proposition 7 that $h(Q)$ is locally projective and is locally split in $M$. Regarding $h$ as a map onto $h(Q)$ we have an epimorphism $h' : M \rightarrow h(Q)$, but the local projectivity of $h(Q)$ implies that $h'$ is locally split. Therefore, by Proposition 2, $h$ is locally split. Thus, $M$ is Zelmanowitz-regular according to Theorem 4.

**Remark 1.** Although we throughout assume that $R$ has an identity element, the paper [5] deals with modules over rings without identity element.

**Remark 2.** It is pointed out in [6] that over a regular ring a module is locally projective if and only if it is Zelmanowitz-regular. But this can be regarded as a particular case of Theorem 8, because over a regular ring every module is flat and hence is Fieldhouse-regular.

In this connection, we would like to mention of some properties of regular modules and locally projective modules:

1. **If $M$ is a regular $R$-module then its Jacobson radical $J(M)$ is zero, and if $M$ is a faithful regular $R$-module then the Jacobson radical $J(R)$ of $R$ is zero.**

The proof is actually given in [4], though regular modules in [4] mean projective regular modules in the present paper. Namely, if $x_0$ is in $J(M)$ then $Rx_0$ is a direct summand small submodule of $M$ and therefore $x_0 = 0$, which implies $J(M) = 0$. Since $J(R)M \subseteq J(M)$, it follows $J(R) = 0$ if $M$ is faithful and regular.

2. **If $M$ is a locally projective $R$-module then $J(R)M \subseteq J(M)$.**

For, let $x_0$ be in $J(M)$; then by Proposition 6 there exist a finite number of homomorphisms $f_i : M \rightarrow R$ and elements $y_i$ in $M$ ($i = 1, 2, \ldots, n$) such that $f_1(x_0)y_1 + f_2(x_0)y_2 + \cdots + f_n(x_0)y_n = x_0$. Let $L$ be a maximal left ideal of $R$. Then its inverse image by $f_i$ is either equal to $M$ or a maximal submodule of $M$ and therefore contains $J(M)$, or equivalently, $f_i(J(M)) \subseteq L$. Since this is true for every maximal left ideal $L$, it follows $f_i(J(M)) \subseteq J(R)$ and in particular $f_i(x_0) \in J(R)$. This is true for each $i = 1, 2, \ldots, n$, so that we have $x_0 \in J(R)M$. Thus we know that $J(M) \subseteq J(R)M$.

3. **A module $M$ is Fieldhouse-regular if (and only if) every finitely generated submodule of $M$ is pure in $M$.**

This is because Cohn's criterion for purity is concerned only with finite number of elements.

**Proposition 9.** Let $M$ be a Zelmanowitz-regular module, and let $S$ be the endomorphism ring of $M$. Then, as an $S$-module, $M$ is Zelmanowitz-regular too, and the Jacobson radical $J(S)$ of $S$ is zero.

**Proof:** We consider $M$ a right $S$-module and hence a two-sided $R$-$S$-module; thus $st = t \circ s$ for all $s, t \in S$. Let $x_0$ be an element of $M$. Then there exists a homomorphism $f : M \rightarrow R$ such that $f(x_0)x_0 = x_0$. Let $y \in M$. Then the mapping $x \mapsto f(x)y$ for $x \in M$ is an endomorphism of $M$, which we denote by $\bar{y} \in S$. If $s \in S$, we have $f(x)(ys) = (f(x)y)s$ for all $x \in M$, i.e., $\bar{ys} = y\bar{s}$.
This implies that the mapping \( y \mapsto y \) for \( y \in M \) is a homomorphism \( M \rightarrow S \) as \( S \)-modules. If we denote this by \( g \) then we have \( f(x)y = xg(y) \) for all \( x, y \in M \). (In the notation in [5], \( g(y) = [f, y] \) for all \( y \in M \).) It follows in particular that \( x_0 = f(x_0)x_0 = x_0g(x_0) \). This shows that the \( S \)-module \( M \) is Zel'manowitiz-regular. Since \( M \) is a faithful \( S \)-module, we have \( J(S) = 0 \) according to the above mentioned property 1.

Now, clearly a locally projective module is projective if it is finitely generated, but this is true even if it is countably generated:

**Proposition 10.** Every countably generated locally projective module is projective.

**Proof:** If we observe the fact that every locally projective module is a Mittag-Leffler module, our proposition can be regarded as a particular case of [3], Corollaire 2.2.2. But we shall for completeness give a proof which is valid for our case. Let \( M \) be a locally projective \( R \)-module with countable generators \( x_1, x_2, x_3, \ldots \). Let \( M_1 = Rx_1 \). By Proposition 6 there exist a finitely generated free \( R \)-module \( F_1 \) and homomorphisms \( f_1 : M \rightarrow F_1, g_1 : F_1 \rightarrow M \) such that \( g_1(f_1(x)) = x \) for all \( x \in M_1 \). Let next \( M_2 = g_1(F_1) + Rx_2 \). Since \( M_2 \) is finitely generated, again by Proposition 6, there exist a finitely generated free \( R \)-module \( F_2 \) and homomorphisms \( f_2 : M \rightarrow F_2, g_2 : F_2 \rightarrow M \) such that \( g_2(f_2(x)) = x \) for all \( x \in M_2 \). In this way, for each \( n > 1 \), we can find a finitely generated free \( R \)-module \( F_n \) and homomorphisms \( f_n : M \rightarrow F_n, g_n : F_n \rightarrow M \) such that \( g_n(f_n(x)) = x \) for all \( x \in M \). \( = g_{n-1}(F_{n-1}) + Rx_n \). But this is clearly equivalent to that \( g_n(f_n(g_{n-1}(y))) = g_{n-1}(y) \) for all \( y \in F_{n-1} \) and \( g_n(f_n(x_n)) = x_n \). From this follows then that \( g_n \circ f_n \circ g_{n-1} = g_{n-1} \) whence \( g_{n-1}(F_{n-1}) \subseteq g_n(F_n) \) and \( x_n \in g_n(F_n) \). Thus we have an ascending chain \( g_1(F_1) \subseteq g_2(F_2) \subseteq g_3(F_3) \subseteq \ldots \) of submodules of \( M \) whose union is equal to \( M \). For simplicity, we put \( s_n = g_n \circ f_n : M \rightarrow g_n(F_n) \) for each \( n \). Then \( s \) is an endomorphism of \( M \) satisfying \( s_n \circ g_{n-1} = g_{n-1} \) and hence \( s_n \circ s_{n-1} = s_{n-1} \) for each \( n > 1 \). Moreover we point out that \( s_n \circ g_r = g_r \) and \( s_n \circ s_r = s_r \) whenever \( n > r \), because if \( r < n \) then \( g_r(F_r) \subseteq g_{n-1}(F_{n-1}) \) and so \( s_n(g_r(y)) = g_r(y) \) for all \( y \in F_r \).

Let \( F \) be the direct sum of all \( F_n \)'s. Then \( F \) is a countably generated free \( R \)-module. The homomorphisms \( g_n : F_n \rightarrow M \) for \( n = 1, 2, 3, \ldots \) together define a homomorphism \( g : F \rightarrow M \) in the natural manner. The image \( g(F) \) is the sum of all \( g_n(F_n) \)'s and hence is equal to \( M \), because even their union is \( M \). Thus \( g \) is an epimorphism. In order to prove that \( M \) is projective, it is therefore sufficient to show that \( g \) splits, i.e., there exists a homomorphism \( f : M \rightarrow F \) such that \( g \circ f = 1 \), the identity map of \( M \). Let now \( q_n : F_n \rightarrow F \) be the canonical embedding for \( n = 1, 2, 3, \ldots \). Then we have \( g \circ q_n = g_n \) for each \( n \). We shall construct a homomorphism \( h_n : F_n \rightarrow F \) for each \( n \) such that \( g \circ h_n = g_n \) and \( h_n \circ f_n \circ g_{n-1} = h_{n+1} \circ f_{n+1} \circ g_n \) if \( n > 1 \). For this purpose, let first \( h_1 = q_1 \). Then \( g \circ h_1 = g_1 \). Suppose \( n > 1 \) and there is given an \( h_n : F_n \rightarrow F \) such
that \( g \circ h_n = g_n \). We define \( h_{n+1} = (h_n \circ f_n + q_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_{n+1} \). Then we have

\[
\begin{align*}
  g \circ h_{n+1} &= (g \circ h_n \circ f_n + g \circ g_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_{n+1} \\
  &= (g_n \circ f_n + g_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_{n+1} \\
  &= (s_n + s_{n+2} \circ (1 - s_n)) \circ g_{n+1} = g_{n+1}.
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
  h_{n+1} \circ f_{n+1} \circ g_{n-1} &= (h_n \circ f_n + q_{n+2} \circ f_{n+2} \circ (1 - s_n)) \circ g_{n-1} \\
  &= h_n \circ f_n \circ g_{n-1} + q_{n-2} \circ f_{n-2} \circ g_{n-1} - g_{n-2} \circ f_{n-2} \circ s_n \circ g_{n-1} = h_n \circ f_n \circ g_{n-1}.
\end{align*}
\]

Thus by induction we get a desired sequence of homomorphisms \( h_n(n = 1, 2, 3, \ldots) \).

Let \( x \in M \). Then there exists an \( n > 1 \) such that \( x \in g_{n-1}(F_{n-1}) \) i.e., \( x = g_{n-1}(y) \) for some \( y \in F_{n-1} \). We have then that \( h_n(f_n(x)) = h_n(f_n(g_{n-1}(y))) = h_{n+1}(f_{n+1}(g_{n-1}(y))) = h_{n+1}(f_{n+1}(x)) \). Moreover, since \( x \in g_n(F_n) \) in this case, by replacing \( n \) by \( n+1 \) we should have that \( h_{n+1}(f_{n+1}(x)) = h_{n+2}(f_{n+2}(x)) \).

Continuing in this way, we confirm that \( h_n(f_n(x)) = h_m(f_m(x)) \) for every \( m > n \). This shows that \( h_n(f_n(x)) \) is independent of the choice of \( n \) so far as \( x \) is in \( g_{n-1}(F_{n-1}) \). Thus by defining \( f(x) = h_n(f_n(x)) \) for \( x \in M \) we have a homomorphism \( f : M \to F \), which satisfies \( g(f(x)) = g_n(f_n(x)) = x \) (since \( x \in g_{n-1}(F_{n-1}) \)). This completes our proof. \(
\square
\)

It is to be pointed out that the preceding proposition can be regarded as a generalization of [5, Corollary 1.7].

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References