ON THE MAXIMALITY OF THE SUM OF TWO MAXIMAL MONOTONE OPERATORS

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Abstract

In this paper we deal with the maximal monotonicity of \( A + B \) when the two maximal monotone operators \( A \) and \( B \) defined in a Hilbert space \( X \) are satisfying the condition: \( \bigcup_{\lambda \geq 0} \lambda (\text{dom} B - \text{dom} A) \) is a closed linear subspace of \( X \).

In this note we study the maximal monotonicity of the sum of two maximal monotone operators by introducing a new weakened condition. The classical theorem of Rockafellar [5] and Brezis [3] tell us that \( A + B \) is a maximal monotone operator whenever \( A \) and \( B \) are so and \( \text{dom} A \cap \text{int}(\text{dom} B) \neq \emptyset \). Attouch [1] dealt with the same problem with the condition: \( 0 \in \text{int}(\text{dom} A - \text{dom} B) \).

Our idea is to use Attouch and Brezis assumption kind, see [2]:

\[ \bigcup_{\lambda \geq 0} \lambda (\text{dom} B - \text{dom} A) \text{ is a closed linear subspace.} \]

Let \( X \) be a real Hilbert space with the norm \( \| \cdot \| \), and scalar product \( \langle \cdot, \cdot \rangle \).

Definition 4.1. A multivalued operator \( A \) in \( X \) is said to be monotone if for every \( x_1, x_2 \in X \) and every \( y_1 \in A x_1 \) and \( y_2 \in A x_2 \) one has

\[ \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0. \]

\( A \) is maximal monotone if it is maximal, relatively to the inclusion, in the set of all monotone operators.

Given \( A \) a maximal monotone operator in \( X \), we shall denote by \( \text{dom} A \) its domain (i.e. \( x \in \text{dom} A \) if \( Ax \neq \emptyset \)), respectively by

\[ A_\lambda = (1/\lambda)(I - J_\lambda^A) \text{ and } J_\lambda^A = (I + \lambda A)^{-1} \text{ for } \lambda > 0, \]

its Yosida approximation and resolvante and by \( A^0 \) its minimal section (i.e. \( A^0 x \) is the projection of zero on \( Ax \)). See Brezis [3] for more details.
Theorem 4.2. Let $X$ be a Hilbert space, $A$ and $B$ be two maximal monotone operators such that $\text{dom}A \cap \text{dom}B \neq \emptyset$ and $\mathbb{R}_+(\text{dom} B - \text{dom} A)$ is a closed linear subspace of $X$. Then $A + B$ is maximal monotone.

Proof: Without loss of generality we can assume that $0 \in \text{dom}A \cap \text{dom}B$. Indeed, since there exists $x_0 \in \text{dom}A \cap \text{dom}B$, then the operators $A_{x_0}$ and $B_{x_0}$ defined by $A_{x_0}(x) = A(x_0 - x)$ and $B_{x_0}(x) = B(x_0 - x)$ are maximal monotone operators and satisfying: $0 \in \text{dom}A_{x_0} \cap \text{dom}B_{x_0}$ and 

$$\mathbb{R}_+(\text{dom}B_{x_0} - \text{dom}A_{x_0}) = \mathbb{R}_+(\text{dom} B - \text{dom} A).$$

Let $x \in X$ and $\lambda > 0$, then, cf. [4], proposition 2.6 and lemma 2.6, $A + B_\lambda$ is a maximal monotone operator. Let $u_\lambda$ be a solution of the inclusion $x \in u_\lambda + Au_\lambda + B_\lambda u_\lambda$.

From [4], lemma 2.5 and the fact that $\text{dom}A \cap \text{dom}B \neq \emptyset$, we deduce that the set $\{u_\lambda; \lambda > 0\}$ is bounded and included in $\mathbb{R}_+(\text{dom} B - \text{dom} A)$. Now from [4], theorem 2.4, we shall conclude that $A + B$ is maximal monotone provided we show that $\sup_{\lambda > 0} \|B_\lambda u_\lambda\| < +\infty$.

Indeed, let us fix some $y \in X$, and prove that $\sup_{\lambda > 0} \langle B_\lambda u_\lambda, y \rangle < +\infty$.

If $y \in \mathbb{R}_+ (\text{dom} B - \text{dom} A)$, there exists $\alpha > 0$, $a \in \text{dom} A$ and $b \in \text{dom} B$ such that $y = \alpha (b - a)$. Hence

$$\langle B_\lambda u_\lambda, y \rangle = \alpha (\langle B_\lambda u_\lambda, b \rangle - \langle B_\lambda u_\lambda, a \rangle).$$

Since $B_\lambda$ is monotone then $\langle B_\lambda u_\lambda - B_\lambda b, u_\lambda - b \rangle \geq 0$ and consequently

$$\langle B_\lambda u_\lambda, y \rangle \leq \alpha (\langle B_\lambda u_\lambda, u_\lambda - a \rangle + \|B_\lambda b\| \cdot \|u_\lambda - b\|).$$

Here we use the fact $\sup_{\lambda > 0} \|B_\lambda b\| = \|B^0 b\|$, see [4, prop. 2.6]. On the other hand, since $x \in u_\lambda + Au_\lambda$, there exists $y_\lambda \in Au_\lambda$ such that $B_\lambda u_\lambda = x - u_\lambda - y_\lambda$. It follows that:

$$\langle B_\lambda u_\lambda, y \rangle \leq \alpha (\langle x - u_\lambda - y_\lambda, u_\lambda - a \rangle + \|B^0 b\| \cdot \|u_\lambda - a\|).$$

From the monotonicity of $A$ we derive $\langle y_\lambda - A^0 a, u_\lambda - a \rangle \geq 0$. Hence

$$\langle B_\lambda u_\lambda, y \rangle \leq \alpha (\|u_\lambda - a\| (\|x - u_\lambda\| + \|A^0 a\|) + \|u_\lambda - b\| \cdot \|B^0 b\|) = f(y)$$

Thus $(B_\lambda u_\lambda, y) \leq f(y) < +\infty$ for every $y \in \mathbb{R}_+(\text{dom} B - \text{dom} A)$.

*If $y \notin \mathbb{R}_+(\text{dom} B - \text{dom} A)$, we shall have $(B_\lambda u_\lambda, y) = 0$. Indeed, since $H = \mathbb{R}_+(\text{dom} B - \text{dom} A)$ is a closed linear subspace of $X$, then $X = H \oplus H^\perp$ (i.e. $H \cap H^\perp = \{0\}$ and $X = H + H^\perp$). On the other hand we have $B_\lambda u_\lambda \in B(J_\lambda^B u_\lambda)$, then $J_\lambda^B u_\lambda \in \text{dom} D \subset H$, and since $u_\lambda \in H$ we get $B_\lambda u_\lambda = \frac{1}{\lambda} (u_\lambda - J_\lambda^B u_\lambda) \in H$. Hence $(B_\lambda u_\lambda, y) = 0$, since $B_\lambda u_\lambda \in H$ and $y \in H^\perp$. We then have for every $y \in X$, $\sup_{\lambda > 0} (B_\lambda u_\lambda, y) < +\infty$, and from the Banach-Steinhaus theorem, we derive that $\{B_\lambda u_\lambda; \lambda > 0\}$ is bounded in $X$, which completes the proof of the theorem.

Remark 4.3. When $\text{dom} A$ and $\text{dom} B$ are convex, we can omit the assumption $\text{dom} A \cap \text{dom} B \neq \emptyset$, since $\mathbb{R}_+(\text{dom} B - \text{dom} A)$ is a closed linear subspace of $X$ provided $0 \in (\text{dom} B - \text{dom} A)$.
Theorem 4.4. Under the assumptions of theorem 4.3, if we assume that \( \mathbb{R}_+(\text{co}(\text{dom}B) - \text{co}(\text{dom}A)) \) is a closed linear subspace, where \( \text{co}(\text{dom}A) \) is the convex hull of \( \text{dom}A \), then \( A + B \) is still maximal monotone.

The proof of this theorem is similar to that of theorem 4.2.

Remark 4.5. It is clear that the assumption in theorem 4.4 is weaker than the condition of Rockafellar [5], Brezis [3], \( \text{int}(\text{dom}A) \cap \text{dom}B \neq \emptyset \) and the condition of Attouch [1] that is: \( 0 \in \text{int}(\text{dom}B - \text{dom}A) \). More generally we can obtain the same result when \( 0 \in \text{ri}(\text{co}(\text{dom}B - \text{co}(\text{dom}A)) \) (the relative interior) since this condition implies that of theorem 4.4.

References


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