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# DISCONTINUITY OF THE PRODUCT IN MULTIPLIER ALGEBRAS

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*Abstract*

Entire functions operate in complete locally $A$-convex algebras but not continuously. Actually squaring is not always continuous. The counter-example, we give, is a multiplier algebra.

## 1. Introduction

W. Żelazko constructs ([13]) an example of a non $m$-convex algebra on which all entire functions operate. Doing so he solves a problem stated in [12]. The example turns out to be a uniformly $A$-convex algebra. The problem in question had also been solved in [10].

We obtain that entire functions do not operate continuously. Actually we show by a counter-example, which is a multiplier algebra, that the product is not (globally) continuous in general. On the other hand we obtain here that it is always sequentially continuous in any unital and complete locally $A$-convex algebra. In the Uniformly $A$-convex case it is hypocontinuous.

## 2. Sequential continuity and hypocontinuity of the product

Let $E$ be a locally convex algebra with a topology defined by the family of semi-norms $(p_\lambda)_{\lambda\in\Lambda}$. Then $E$ is said to be locally $A$-convex ([4]) if for every $x \in E$ and every $\lambda \in \Lambda$ there exist $M(\lambda, x) > 0$, $N(\lambda, x) > 0$ such that

$$p_\lambda(xy) \leq M(\lambda, x)p_\lambda(y) \qquad (y \in E).$$
$$p_\lambda(yx) \leq N(\lambda, x)p_\lambda(y) \qquad (y \in E).$$

A locally $A$-convex algebra is said to be Uniformly $A$-convex algebra if $M(\lambda, x)$ and $N(\lambda, x)$ can be chosen independently of $\lambda$ ([5]).

Observe that the so-called locally $m$-convex algebras ([9]) are particular cases of the above definition. Some examples of all these classes of algebras and relationships among them can be seen in ([4]) and ([5]).

Let $E$ be a commutative Banach algebra without order i.e if $xy = 0\ (y \in E)$ then $x = 0$. The multiplier algebra $M(E)$ of $E$ is the space of linear operators $T$ verifying $T(x.y) = xT(y)(x, y \in E)$. Endowed with the strong topology given by the family of semi-norms $(P_x)_{x\in E}$, where $p_x(T) = \|Tx\|(x \in E)$, $M(E)$ is a unital, complete, uniformly locally $A$-convex algebra which is not always $m$-convex since it is a generalization (see [1, p. 139]) of the algebra $C_b(R)$, of [4], that is not $m$-convex.

We can endow any unital locally $A$-convex algebra with a locally $m$-convex topology $M(\tau)$ finer than $\tau$ by putting $q_\lambda(x) = \sup\{p_\lambda(x.y) : p_\lambda(y) \leq 1\}$. This topology is complete when $\tau$ is. Moreover $\tau$ and $M(\tau)$ have the same bounded sets ([2]); indeed it suffices to show that bounded sets for $\tau$ are bounded for $M(\tau)$, but this follows from the fact that any barrel in a complete locally convex space is bornivorous.

**Proposition 2.1.** *Entire functions operate on any unital and compete locally $A$-convex algebra.*

*Proof:* $(E, M(\tau))$ is a unital complete locally $m$-convex algebra and entire functions operate on such algebras ([9]); the result follows since $\tau$ is coarser than $M(\tau)$. ■

**Proposition 2.2.** *In any unital and complete locally $A$-convex algebra $(E, \tau)$, the product is always sequentially continuous.*

*Proof:* Let $(x_n)_n$ and $(y_n)_n$ be two sequences converging to zero in $(E, \tau)$. Since $\tau$ and $M(\tau)$ have the same bounded sets, $(q_\lambda(x_n))_n$ is bounded for every $\lambda$. Then the conclusion follows from the relation $p_\lambda(x.y) \leq q_\lambda(x).p_\lambda(y)$, for every $x$ and $y$. ■

We can endow any unital and complete uniformly $A$-convex algebra $(E, \tau)$ with a Banach algebra norm $\|\cdot\|$ finer than $\tau$, by putting $\|x\| = \sup\{q_\lambda(x) : \lambda \in \Lambda\}$; it is Cochran's norm ([5]). And, as before, we can show that $\tau$ and $\|\cdot\|$ have the same bounded sets. This has, as a consequence, the proposition 2 of [13].

Now recall that, in a locally convex algebra $(E, \tau)$. The multiplication is said to be left (right) hypocontinuous if for each neighborhood $U$ of $O$ and any bounded set $B$ there exists a neighborhood $V$ of $o$ such that $B.V \subset U(V.B \subset U)$. The multiplication, in $E$, is called hypocontinuous if it is left as well as right hypocontinuous.

**Proposition 2.3.** *In a unital and complete uniformly $A$-convex algebra $(E, \tau)$ the product is always hypocontinuous.*

*Proof:* By ([6, proposition 9, p. 155]) it is sufficient to prove that if we consider the map $u : x \to L_x$, where $L_x(y) = xy\ (x, y \in E)$ then, for every bounded set $B$ in $E$, the set $u(B)$ is equicontinuous. But this follows from the relation $p_\lambda(xy) \leq \|x\|. \ p_\lambda(y)(x, y \in E, \lambda \in \Lambda)$ and the fact that $\tau$ and $\|\cdot\|$ have the same bounded sets. ∎

## 3. Discontinuity of the product

Our counter-example is the multiplier algebra of an $H^*$-algebra. It goes along the lines of ([7, problem 111]).

An $H^*$-algebra is a Banach algebra $E$, with involution $*$, which is a Hilbert space under a scalar product $\langle .,. \rangle$ such that.

a) $\|x\|^2 = \langle x, x \rangle$, for every $x$ in $E$

b) $\|x^*\| = \|x\|$, for every $x$ in $E$

c) $x^*.x \neq 0$, for every $x$ in $E \backslash \{0\}$.

d) $\langle x.y, \ z \rangle = \langle y, \ x^*.z \rangle = \langle x, z.y^* \rangle$, for every $x, y, z$ in $E$.

In such commutative algebras there always exists a complete orthogonal system of idempotent and self-adjoint elements (we can even suppose more but this is sufficient for our needs; cf [3]).

**Counter-example 5.1.** Let $(E, \|\cdot\|)$ be a commutative infinite dimensional $H^*$-algebra. Without loss of generality suppose $E$ separable. Let then $(e_1, e_2, \ldots, e_k \ldots)$ be a complete orthogonal system of self-adjoint and indempotent elements. The set $\{\sqrt{k}.e_k : k = 1, 2, \ldots\}$ is unbounded since $\|e_k\| \geq$ $(k = 1, 2, \ldots)$ and it contains the null vector in its weak closure (cf., [7, problem 28]). Hence there exists a net $(k_i)_i$ of positive integers such that $\sqrt{k_i}.e_{k_i}$ converges weakly to zero (it cannot be a sequence).

For each $k$, consider the multiplier $A_k$ defined by $A_k(x) = k'.e_k.x$ where $k'$ is the fourth root of $k$. We have $\|A_{k_i}(x)\|^2 = \langle \sqrt{k_i}.e_{k_i}, \ x.x^* \rangle$, hence $\|A_{k_i}(x)\| \rightarrow 0$. But $A^4_{k_i}(x) = k_i.e_{k_i}.x$, and. if therefore we take in particular $x = (1, \ldots, n^{-1}, \ldots)$, then $A^4_{k_i}(x) = e_{k_i}$. So $\|A^4_{k_i}(x)\| \geq 1$. Whence squaring is not continuous.

**Remarks.**

1. In many interesting situations the product is continuous. This is the case for the example $C_b(R)$ of [4]. It is also so for any multiplier algebra $M(E)$ where $E$ admits factorization, i.e for every $z \in E$, there exist $x$ and $y$ in $E$ such that $z = x.y$.

2. In connection with the previous remark, we can notice that there cannot exist a Banach algebra $E$ admitting a bounded approximate identity and such that the set $N = \{x : x^2 = 0\}$ is $\beta$-dense in $E$. Indeed the product, and in particular squaring, is continuous, hence $\{T\epsilon M(E) : T^2 = 0\}$ in $\beta$-closed. It is also $\beta$-dense since it contains $N$ and $E$ is $\beta$-dense in $M(E)$ for it admits a bounded approximate identity. Therefore any element of $M(E)$ should be nilpotent. But this contradicts the fact that $M(E)$ is always unital.

3. Incidentally we get that an absolutely convergent series in a complete locally convex space does not necessarily define a continuous mapping. Indeed if $(E, \tau)$ is a locally uniformly $A$-convex algebra whose topology $\tau$ is given by a

family of seminorms $(P_\lambda)_\lambda$, there exists a Banach algebra norm $\|\cdot\|$ and $\alpha > 0$ such that $P_\lambda(x) \leq \alpha.\|x\|$, for every $\lambda$ and every $x$. Now if $f(z) = \sum a_n.z^n$ is an entire function, then, for every $\lambda$, $P_\lambda(\sum a_n.x^n) \leq \alpha.\sum |a_n|.\|x\|^n$. Hence the series is absolutely convergent; but the previous counter-example shows that the map $x \to f(x)$ is not always continuous.

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