

VECTOR VALUED MEASURES OF BOUNDED MEAN OSCILLATION

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0. Introduction

The duality between H^1 and BMO , the space of functions of bounded mean oscillation (see [JN]), was first proved by C. Fefferman (see [F], [FS]) and then other proofs of it were obtained. Using the atomic decomposition approach ([C], [L]) the author studied the problem of characterizing the dual space of H^1 of *vector-valued* functions. In [B2] the author showed, for the case $\Omega = \{|z| = 1\}$, that the expected duality result H^1 - BMO holds in the vector valued setting if and only if X^* has the Radon-Nikodym property. If we want to get a duality result valid for all Banach spaces we may consider vector valued measures (see [BT], where the vector valued L_p case is treated, for an explanation) and therefore to deal with the general case it was necessary to consider a new space of vector valued measures closely related to BMO (see [B1]).

In this paper we shall study such space in little more detail and we shall consider the H^1 - BMO duality for vector-valued functions in the more general setting of spaces of homogeneous type (see [CW]).

Throughout the paper X will stand for a Banach space, Ω will be a space of homogeneous type (see definition in the preliminary section) and we write $L_p(\Omega, X)$ for the space of measurable functions on Ω with values in X such that $\|f(x)\|$ belongs to $L_p(\Omega)$. As usual C will denote a constant not necessarily the same at each occurrence.

1. Preliminaries

A space of homogeneous type Ω is a topological space endowed with a Borel measure m and a quasi-distance d , that is $d : X \times X \rightarrow \mathbf{R}^+$ with

a)
$$d(x, y) = d(y, x),$$

b)
$$d(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

c)
$$d(x, y) \leq K(d(x, z) + d(z, y)).$$

*Partially supported by the grant C.A.I.C.Y.T. PB85-0338

and we assume that the balls $B_r(x) = \{y \in \Omega : d(x,y) < r\}$ form a basis of open neighborhoods of the point x and there exists a constant A satisfying

$$(1.0) \quad m(B_r(x)) \leq A m(B_{r/2}(x))$$

From (1.0) we can assume that $0 < m(B) < \infty$ for every ball B (otherwise m would be identically 0 or ∞) and therefore m is a σ -finite measure on Ω . Denote by Σ_0 the ring of bounded measurable sets. The σ -finiteness condition implies that the σ -algebra generated by Σ_0 coincides with the Borel σ -algebra that we shall denote by Σ .

Let us now recall the notion of atom with values in X . Given $1 < p \leq \infty$, a function a in $L_p(\Omega, X)$ is called (X, p) -atom if

a) the support is contained in a ball $B = B_r(x_0)$

$$b) \quad \left(\frac{1}{m(B)} \int_B \|a(x)\|^p dm(x) \right)^{1/p} \leq \frac{1}{m(B)} \quad (p < \infty)$$

$$b') \quad \|a(x)\| \leq \frac{1}{m(B)} \quad m - a.e. \quad (p = \infty)$$

$$c) \quad \int_B a(x) dm(x) = 0$$

In the case $m(\Omega) < \infty$ the constant function $\frac{1}{m(\Omega)} b$, where $b \in X$ with $\|b\| = 1$, is also considered as a (X, p) -atom.

Note that the atoms are in the unit ball of $L_1(\Omega, X)$.

Following [CW] we define $H_p^1(\Omega, X)$ as the space of functions f in $L_1(\Omega, X)$ admitting an atomic decomposition

$$(1.1) \quad f = \sum_{j=0}^{\infty} \lambda_j a_j$$

where the a_j 's are (X, p) -atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. (The convergence of (1.1) is taken in $L_1(\Omega, X)$).

We get a Banach space if we consider the norm

$$\|f\|_{H_p^1} = \inf \sum_{j=0}^{\infty} |\lambda_j|$$

where the infimum is taken over all representations $f = \sum_{j=0}^{\infty} \lambda_j a_j$.

The same arguments as in [CW] show that, in fact, for $1 < p, r \leq \infty$

$$(1.2) \quad H_p^1(\Omega, X) = H_r^1(\Omega, X) \text{ (with equivalent norms) .}$$

Let us also recall the definition of vector-valued BMO. Let $1 \leq q < \infty$, an X -valued function which is locally in $L_q(\Omega, X)$ is said to belong to $BMO_q(\Omega, X)$ provided that

$$(1.3) \quad \sup_{\text{ball } B} \left(\frac{1}{m(B)} \int_B \|g(x) - g_B\|^q dm(x) \right)^{1/q} \leq C$$

where $g_B = \frac{1}{m(B)} \int_B g(x) dm(x)$.

Let us denote by

$$\|g\|_{*,q} = \sup \left\{ \left(\frac{1}{m(B)} \int_B \|g(x) - g_B\|^q dm(x) \right)^{1/q} : B \text{ ball} \right\}$$

When $m(\Omega) = \infty$ then $\|g\|_{BMO_q} = \|g\|_{*,q}$ gives a norm on the set of equivalence classes of functions which differ by a constant in X .

For $m(\Omega) < \infty$ we consider the norm $\|g\|_{BMO_q} = \|g\|_{*,q} + \left\| \int_{\Omega} g(x) dm(x) \right\|$.

Let us recall now a few definitions about vector-valued measures we shall use later on. Let (Ω, Σ, m) be any σ -finite measure space, A a measurable set and $1 < p < \infty$. Given a vector valued measure G , we denote by $|G|$ the variation of G , that is

$$(1.4) \quad |G|(A) = \sup \left\{ \sum_{i=1}^n \|G(E_i)\| : (E_i) \text{ partition of } A \right\}$$

and by $|G|_p(A)$ the p -variation on A , that is

$$(1.5) \quad |G|_p(A) = \sup \left\{ \left(\sum_{i=1}^n \frac{\|G(E_i)\|^p}{m(E_i)^{p-1}} \right)^{1/p} \right\}$$

where the supremum is taken over all finite partitions (E_i) of disjoint measurable sets contained in A with $m(E_i) > 0$.

For the case $p = \infty$ we shall denote by $V^\infty(\Omega, X)$ the space of X -valued measures G satisfying

$$(1.6) \quad \|G(E)\| \leq C m(E) \text{ for all measurable set } E$$

Defining the norm by the infimum of the constants satisfying (1.6) we get a Banach space.

Remark 1.1. It is not hard to see that in fact $\|G(E_i)\|$ can be replaced by $|G|(E_i)$ in the definition of p -variation. (See Lemma 1 in [B3])

Remark 1.2. If G is a vector valued measure defined on Σ_0 which is absolutely continuous with respect to m , that is $\lim_{m(E) \rightarrow 0} G(E) = 0$, then it can be extended to a measure on Σ , being still absolutely continuous with respect to m . (See [D],[DU])

We refer the reader to ([DU], [D]) and to ([J], [GC-RF]) for general theory and the properties we shall use about vector valued measures and Hardy spaces respectively.

2. Vector valued measures of bounded mean oscillation

Definition 2.1. Let $1 \leq q < \infty$. Given a countably additive measure G defined on Σ and with values in X , it is said that G belongs to $MBMO_q(\Omega, X)$ if

$$(2.1) \quad |G|_{*,q} = \sup \left\{ \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|^q \frac{m(E_i)}{m(B)} \right)^{1/q} \right\} < \infty$$

where the supremum is taken over all balls B and over all finite partitions of B in pairwise disjoint measurable sets E_i with $m(E_i) > 0$.

When $m(\Omega) = \infty$ then $\|G\|_{MBMO_q} = |G|_{*,q}$ gives a norm on the set of equivalence classes of measures: $G_1 \sim G_2$ if there is b in X such that $G_1(E) - G_2(E) = b m(E)$ for all measurable set E .

For $m(\Omega) < \infty$ we consider the norm $\|G\|_{MBMO_q} = |G|_{*,q} + \|G(\Omega)\|$.

It is obvious that if $1 < q_1 < q_2 < \infty$ then

$$(2.2) \quad V^\infty(\Omega, X) \subset MBMO_{q_2}(\Omega, X) \subset MBMO_{q_1}(\Omega, X)$$

Remark 2.1. Let us assume G belong to $MBMO_q(\Omega, X)$. Given a ball B and a measurable set $E \subset B$, it is quite immediate to find a constant C_B depending on B satisfying

$$(2.3) \quad \|G(E)\| \leq C_B \max(m(E), m(E)^{1-1/q})$$

Suppose we consider $B_n = \{y \in \Omega : d(x_0, y) < n\}$ and denote by G_{B_n} the measure G concentrated on B_n , that is $G_{B_n}(E) = G(E \cap B_n)$. A glance at (2.3) allows us to say that for any $1 < q < \infty$ if G belongs to $MBMO_q(\Omega, X)$ then G_{B_n} are necessarily absolutely continuous with respect to m and this clearly implies that also G is absolutely continuous with respect to m . (Recall that for vector-measures on σ -algebras it suffices to check that they vanish on m -null sets).

Proposition 2.1. Let $1 \leq q < \infty$, g be locally in $L_q(\Omega, X)$ and G be an X -valued measure such that $G(E) = \int_E g(x) dm(x)$ for all measurable bounded set E .

Then g belongs to $BMO_q(\Omega, X)$ if and only if G belongs to $MBMO_q(\Omega, X)$.
Moreover $\|G\|_{MBMO_q} = \|g\|_{BMO_q}$.

Proof: Given any ball B , consider $G_B(E) = G(E \cap B) - \frac{G(B)}{m(B)} m(E \cap B)$. Observe that

$$\sup \left\{ \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|^q \frac{m(E_i)}{m(B)} \right)^{1/q} : (E_i) \text{ partition of } B \right\}$$

coincides with the q -variation of G_B on Ω divided by $m(B)^{1/q}$ and G_B is a measure represented by the function $(g - g_B)\chi_B$, that is

$$G_B(E) = \int_E (g(x) - g_B)\chi_B \, dm(x).$$

Therefore the proposition follows from the equality between the q -variation and the norm in L_q of the function which represents the measure (see [D]). ■

Remark 2.2. In general it is not true that any measure in $MBMO_q(\Omega, X)$ is representable by a function, this depends on the Radon-Nikodym property. We refer the reader to [B1] for the case $\Omega = \{|z| = 1\}$, but a similar result and proof can be established also in this general setting.

Proposition 2.2. *Let $1 \leq q < \infty$. G belongs to $MBMO_q(\Omega, X)$ if and only if there exists a family of vectors in X , say $\{a_B: B \text{ ball}\}$, such that*

$$(2.4) \quad \sup \left\{ \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - a_B \right\|^q \frac{m(E_i)}{m(B)} \right)^{1/q} \right\} < \infty$$

where the supremum is taken over all balls B and over all finite partitions of B in pairwise disjoint measurable sets E_i with $m(E_i) > 0$

Proof: The direct implication is obvious by taking $a_B = \frac{G(B)}{m(B)}$. To show the converse let us assume that we have $\{a_B: B \text{ ball}\}$ with the above property, and notice that

$$\|a_B - \frac{G(B)}{m(B)}\| \leq C$$

for all B (simply take the partition of B given only by B).

Therefore for any B and any partition

$$\begin{aligned} & \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|^q \frac{m(E_i)}{m(B)} \right)^{1/q} \leq \\ & \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - a_B \right\|^q \frac{m(E_i)}{m(B)} \right)^{1/q} + \left(\sum_{i=1}^n \|a_B - \frac{G(B)}{m(B)}\|^q \frac{m(E_i)}{m(B)} \right)^{1/q} \leq C \quad \blacksquare \end{aligned}$$

As in the case of functions we can define an equivalent norm in $MBMO_q(\Omega, X)$. Let us take

$$(2.5) \quad |G|'_{*,q} = \sup_{\text{ball } B} \left\{ \inf_{a \in X} \frac{1}{m(B)^{1/q}} |G - a m|_q(B) \right\}.$$

Note that essentially the same argument as in Proposition 2.2. shows the following

$$(2.6) \quad |G|'_{*,q} \leq |G|_{*,q} \leq C |G|'_{*,q}$$

Proposition 2.3. *Let $1 < q < \infty$. If G belongs to $MBMO_q(\Omega, X)$ then there exists a non negative function ϕ in $BMO_q(\Omega)$ such that*

$$|G|(E) = \int_E \phi(x) dm(x).$$

Moreover $\|\phi\|_{BMO_q} \leq C \|G\|_{MBMO_q}$.

Proof: Since G is countably additive and m -continuous then the same is true for the variation of G , $|G|$. Therefore using the Radon-Nikodym theorem there exists a non negative measurable function ϕ which represents the measure $|G|$. To show that ϕ belongs to $BMO_q(\Omega)$, we shall use Propositions 2.2 and 2.1. We simply have to find a family of real numbers $\{a_B\}$ such that

$$\sup\left\{\left(\sum_{i=1}^n \left|\frac{|G|(E_i)}{m(E_i)} - a_B\right|^q \frac{m(E_i)}{m(B)}\right)^{1/q}\right\} < \infty$$

Take $a_B = \frac{\|G(B)\|}{m(B)}$, and observe that

$$\left|\frac{|G|(E)}{m(E)} - \frac{\|G(B)\|}{m(B)}\right| \leq \left|G - \frac{G(B)}{m(B)}\right|(E)$$

Then

$$\begin{aligned} & \sup\left\{\left(\sum_{i=1}^n \left|\frac{|G|(E_i)}{m(E_i)} - \frac{\|G(B)\|}{m(B)}\right|^q \frac{m(E_i)}{m(B)}\right)^{1/q}\right\} \leq \\ & \sup\left\{\frac{1}{m(B)^{1/q}} \left(\sum_{i=1}^n \left(\left|G - \frac{G(B)}{m(B)}\right|(E_i)\right)^q m(E_i)^{1-q}\right)^{1/q}\right\} \leq |G|_{*,q} \end{aligned}$$

The last inequality follows from Remark 1.1. ■

3. The theorem and its proof

In the sequel $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. In this section we shall achieve the duality result between $H_p^1(\Omega, X)$ and $MBMO_q(\Omega, X^*)$. We shall need several lemmas before we prove the result. The next result was done in [B1] for the circle and for $q = 2$, and here we present a different approach which is valid for general spaces of homogeneous type. The author would like to point out that a similar and independent proof of the following lemma has been obtained by T. Wolniewicz (personal communication).

Lemma 3.1. *Let G be a measure in $MBMO_q(\Omega, X)$. Then for each integer $n \in \mathbb{N}$ we can find a measure G_n in $V^\infty(\Omega, X)$ and a constant C_n satisfying $|G_n|_{*,q} \leq C_n$ and such that*

$$(3.1) \quad |G|_{*,q} \leq \lim_{n \rightarrow \infty} C_n \leq K|G|_{*,q}$$

$$(3.2) \quad \lim_{n \rightarrow \infty} G_n(E) = G(E) \quad \text{for all measurable bounded set } E.$$

Proof: Using Proposition 2.3 we first get a function ϕ in $BMO_q(\Omega)$.

Denote by $\Omega_n = \{x \in X : \phi(x) > n\}$ and $\phi_n(x) = \min(1, n/\phi(x))$. Let us define now

$$(3.3) \quad G_n(E) = \int_E \phi_n(x) dG(x) \quad (E \in \Sigma_0)$$

Notice that

$$\|G_n(E)\| \leq |G_n|(E) \leq \int_E \phi_n(x) d|G|(x) \leq \int_E \phi_n(x) \phi(x) dm(x) \leq n m(E)$$

This, using Remark 1.2., allows to extend G_n to Σ and shows that G_n belongs to $V^\infty(\Omega, X)$.

On the other hand

$$(3.4) \quad G(E) - G_n(E) = \int_{E \cap \Omega_n} (1 - \phi_n(x)) dG(x)$$

Therefore if E is contained in some ball B

$$\|G(E) - G_n(E)\| \leq 2 \int_{E \cap \Omega_n} \phi(x) dm(x)$$

Since $\phi \chi_B$ is in $L_1(\Omega)$ then taking limit as $n \rightarrow \infty$ shows (3.2).

From (2.6) we have finally to estimate $m(B)^{-1/q} |G_n - a m|_q(B)$ for all balls B . Using (3.4) we have that for any $E \subset B$

$$\|G(E) - G_n(E)\| \leq \int_{E \cap \Omega_n} (1 - n/\phi(x)) d|G|(x)$$

If $\|a\| \leq n$ then

$$\|G(E) - G_n(E)\| \leq \int_{E \cap \Omega_n} (\phi(x) - n) dm(x) \leq \int_{E \cap \Omega_n} (\phi(x) - \|a\|) dm(x)$$

Therefore we have

$$(3.5) \quad |G_n - G|_q(B) \leq |G - a m|_q(B \cap \Omega_n)$$

Though $|G|_q$ is not a measure for $q > 1$ the q -variation is subadditive and therefore we get that for all $\|a\| \leq n$

$$(3.6) \quad m(B)^{-1/q} |G_n - a m|_q(B) \leq 2 m(B)^{-1/q} |G - a m|_q(B)$$

Denoting now by

$$D_n = \sup_{\text{ball } B} \inf_{\|a\| \leq n} \{m(B)^{-1/q} |G - a m|_q(B)\}$$

we get (3.1) for $C_n = 2C D_n$ where C is the constant appearing in (2.6). ■

Notice that $V^\infty(\Omega, X^*)$ can be obviously identified with the dual of $L_1(\Omega, X)$. Indeed any measure G in $V^\infty(\Omega, X^*)$ defines a functional T_G acting on X -valued simple functions (which are dense in $L_1(\Omega, X)$) by the formula

$$(3.7) \quad T_G\left(\sum_{i=1}^n a_i \chi_{E_i}\right) = \sum_{i=1}^n \langle G(E_i), a_i \rangle$$

where \langle, \rangle means duality between X and X^* .

Lemma 3.2. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and G belong to $V^\infty(\Omega, X^*)$. Then*

$$(3.8) \quad |T_G(f)| \leq C \|G\|_{MBMO_q} \|f\|_{H_p^1} \text{ for all } f \text{ in } H_p^1(\Omega, X).$$

Proof: Let us first take a "simple atom" in $H_p^1(\Omega, X)$, that is $s = \sum_{i=1}^n b_i \chi_{E_i}$, $E_i \subset B$ for some ball B , $\sum_{i=1}^n b_i m(E_i) = 0$ and $\sum_{i=1}^n \|b_i\|_X^p m(E_i) \leq m(B)^{1-p}$.

For such an atom we can write

$$T_G\left(\sum_{i=1}^n b_i \chi_{E_i}\right) = \sum_{i=1}^n \langle G(E_i), b_i \rangle = \sum_{i=1}^n \left\langle G(E_i) - \frac{G(B)}{m(B)} m(E_i), b_i \right\rangle$$

Therefore

$$\begin{aligned} |T_G(s)| &\leq \sum_{i=1}^n \left\| G(E_i) - \frac{G(B)}{m(B)} m(E_i) \right\|_{X^*} m(E_i)^{-1/p} m(E_i)^{1/p} \|b_i\|_X \leq \\ &\leq \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^q m(E_i)^{1/q} \right)^{1/q} \left(\sum_{i=1}^n \|b_i\|_X^p m(E_i) \right)^{1/p} \leq \\ &\leq \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^q \frac{m(E_i)}{m(B)} \right)^{1/q} \leq |G|_{*,q} \end{aligned}$$

For a general atom a supported in B in $H_p^1(\Omega, X)$ we can use approximation by simple functions in $L_p(\Omega, X)$, and find a sequence of simple functions d_k supported in B converging to a in $L_p(\Omega, X)$, and take the sequence $s_k = (d_k - \int_B d_k(x) dm(x))\chi_B$ which clearly also converges to a in $L_p(\Omega, X)$. Hence $\|s_k\|_p \leq 2\|a\|_p$ for k large enough, and therefore $s_k/2$ are "simple atoms".

Using now that T_G is continuous as operator on $L_1(\Omega, X)$, and that s_k converges to a in $L_1(\Omega, X)$, then

$$(3.9) \quad |T_G(a)| = \lim_{k \rightarrow \infty} |T(s_k)| = 2 \lim_{k \rightarrow \infty} |T(s_k/2)| \leq 2|G|_{*,q}$$

For a general function f , take any representation of f in $H_p^1(\Omega, X)$, say $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where the a_j are (X, p) -atom and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$ and notice that (3.8) follows from (3.9) and the fact that the series $f = \sum_{j=0}^{\infty} \lambda_j a_j$ is absolutely convergent in $L_1(\Omega, X)$ what implies that $T_G(f) = \sum_{j=0}^{\infty} \lambda_j T_G(a_j)$. ■

Theorem 3.1. *Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$(3.10) \quad (H_p^1(\Omega, X))^* = MBMO_q(\Omega, X^*) \text{ (equivalent norms)}$$

Proof: Let us take G in $MBMO_q(\Omega, X^*)$, and define as above

$$T_G\left(\sum_{i=1}^n b_i \chi_{E_i}\right) = \sum_{i=1}^n \langle G(E_i), b_i \rangle$$

From the definition of $H_p^1(\Omega, X)$ we can easily see that simple functions with support in balls are dense in the space, therefore it is enough to see that

$$(3.11) \quad |T_G\left(\sum_{i=1}^n b_i \chi_{E_i}\right)| \leq C |G|_{*,q} \left\| \left(\sum_{i=1}^n b_i \chi_{E_i}\right) \right\|_{H_p^1}$$

To see (3.11) we first invoke Lemma 3.1 to find a sequence of measures G_n in $V^\infty(\Omega, X^*)$, that according to (3.2) verifies $\lim_{n \rightarrow \infty} T_{G_n}(s) = T_G(s)$ for all simple function supported in a ball.

Secondly we use Lemma 3.2, together with (3.1) to get

$$|T_G(s)| \leq \lim_{n \rightarrow \infty} |T_{G_n}(s)| \leq C \lim_{n \rightarrow \infty} |G_n|_{*,q} \|s\|_{H_p^1} \leq$$

$$C \lim_{n \rightarrow \infty} C_n \|s\|_{H_p^1} \leq C |G|_{*,q} \|s\|_{H_p^1}.$$

For the converse we shall deal first with the case $m(\Omega) < \infty$. Let us take now a functional T in $(H_p^1(\Omega, X))^*$. Since constant functions are also considered

as X -atoms in the case of finite measure we have that $a_{\chi_E} \in H_p^1(\Omega, X)$, what allows us to define the following X^* valued measure.

$$(3.12) \quad \langle G(E), a \rangle = T(a_{\chi_E}) \quad (a \in X)$$

Given a ball B and a partition of B , say $\{E_i\}$, of pairwise disjoint sets, using the duality $(L^p(X))^* = L^q(X^*)$, we have

$$\begin{aligned} & \left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^q \frac{m(E_i)}{m(B)} \right)^{1/q} = \\ & \left(\sum_{i=1}^n \left\| \left(\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right) \left(\frac{m(E_i)}{m(B)} \right)^{1/q} \right\|_{X^*}^q \right)^{1/q} = \\ & \sup \left\{ \sum_{i=1}^n \left\langle \left(\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right) \left(\frac{m(E_i)}{m(B)} \right)^{1/q}, b_i \right\rangle : \sum_{i=1}^n \|b_i\|_X^p = 1 \right\}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} & \left| \sum_{i=1}^n \left\langle \left(\frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right) \left(\frac{m(E_i)}{m(B)} \right)^{1/q}, b_i \right\rangle \right| = \\ & \frac{1}{m(B)^{1/q}} \left| \sum_{i=1}^n \left\langle \frac{G(E_i)}{m(E_i)^{1/p}}, b_i \right\rangle - \left\langle \frac{G(B)}{m(B)}, \sum_{i=1}^n m(E_i)^{1/q} b_i \right\rangle \right| = \\ & = \frac{1}{m(B)^{1/q}} |T(\sum_{i=1}^n m(E_i)^{-1/p} b_i \chi_{E_i}) - T(b \chi_B)| \end{aligned}$$

where $b = \frac{1}{m(B)} (\sum_{i=1}^n m(E_i)^{1/q} b_i)$.

Denote by $a = \frac{1}{2m(B)^{1/q}} (\sum_{i=1}^n m(E_i)^{-1/p} b_i \chi_{E_i} - b \chi_B)$. It is elementary to show that if $\sum_{i=1}^n \|b_i\|_X^p = 1$ then a is a (X, p) -atom.

Therefore we obtain

$$\left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^q \frac{m(E_i)}{m(B)} \right)^{1/q} \leq 2 |T(a)| \leq 2 \|T\|$$

This shows $|G|_{*,q} \leq 2 \|T\|$. Since T and T_G coincide over simple atoms, we have $T = T_G$.

On the other hand

$$\|G(\Omega)\| \leq \sup \{ |T(b \chi_\Omega)| : \|b\| \leq 1 \} \leq m(\Omega) \|T\|$$

and this finishes the proof for the finite measure case.

Let us deal now with the case of $m(\Omega) = \infty$. Take a functional T in $(H_p^1(\Omega, X))^*$ and a ball B in Ω . Let us consider the following space

$$L_0^p(B, X) = \{f \in L_p(\Omega, X) : \text{supp } f \subset B \text{ and } \int_B f(x) dm(x) = 0\}$$

The following function is an (X, p) -atom

$$a(x) = \frac{f(x)}{m(B)^{1/q} \|f\|_p} \text{ for } f \in L_0^p(B, X).$$

hence

$$\|f\|_{H_p^1} \leq m(B)^{1/q} \|f\|_p$$

and therefore

$$\|Tf\| \leq \|T\| m(B)^{1/q} \|f\|_p$$

This shows that T defines a bounded functional on $L_0^p(B, X)$ and hence from the Hahn-Banach extension theorem, we get an element in the dual of $L_p(B, X)$. The characterization of the dual space $(L_p(B, X))^*$ in terms of X^* -valued measures of bounded q -variation allows us to find a measure G_B with values in X^* verifying

$$(3.13) \quad T(f) = \int_B f dG_B \quad f \in L_0^p(B, X)$$

(Note that this measure is uniquely determined up to a measure $F(E) = \xi m(E \cap B)$ for some $\xi \in X^*$). Now if we take an increasing sequence of balls converging to Ω , say B_n , and we determine G_{B_n} by the assumption $G_{B_n}(B_1) = 0$, then we can construct a vector-valued measure on Σ_0 , given by $G(E) = G_{B_n}(E)$ for $E \subset B_n$. It is clear that G_{B_n} are absolutely continuous and hence the same is true for G . Now from remark 1.2 we get an extension to Σ .

$$\left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^q \frac{m(E_i)}{m(B)}\right)^{1/q} = \sup_{\|f\|_p=1} \left| \frac{1}{m(B)^{1/q}} \int_B f d\left(G - \frac{G(B)}{m(B)} m\right) \right|$$

For each $f \in L_p(B, X)$, consider $a = \frac{1}{2m(B)^{1/q}}(f - f_B)\chi_B$ and therefore

$$\left(\sum_{i=1}^n \left\| \frac{G(E_i)}{m(E_i)} - \frac{G(B)}{m(B)} \right\|_{X^*}^q \frac{m(E_i)}{m(B)}\right)^{1/q} = \sup_a 2|T(a)| \leq 2\|T\|$$

This completes the proof. ■

Remark 3.1. For $1 < p, r < \infty$,

$$MBMO_q(\Omega, X) = MBMO_r(\Omega, X) \text{ with equivalent norms}$$

For dual spaces follows from the theorem and (1.1), and the general case is consequence of the embedding $X \subset X^{**}$

References

- [B1] O. BLASCO, Hardy spaces of vector-valued functions: Duality, *Trans. Amer. Math. Soc.* **308** (1988), 495–507.
- [B2] O. BLASCO, On the dual space of $H_B^{1,\infty}$, *Colloquium Math.* **55** (1988), 253–259.
- [B3] O. BLASCO, Boundary values of vector-valued functions in Orlicz-Hardy classes, *Arch. Math.* **49** (1987), 434–439.
- [BT] S. BOCHNER, A.E. TAYLOR, Linear functionals on certain spaces of abstractly-valued functions, *Annals of Math.* **39** (1938), 913–944.
- [C] R.R. COIFMAN, A real variable characterization of H_p , *Studia Math.* **51** (1974), 269–274.
- [CW] R.R. COIFMAN, G. WEISS, Extension of Hardy spaces and their use in Analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
- [DU] J. DIESTEL, J.J. UHL, Vector Measures, *Amer. Math. Soc. Mathematical Surveys* **15** (1977).
- [D] N. DINCULEANU, “*Vector measures*,” Pergamon Press, New York, 1967.
- [F] C. FEFFERMAN, Characterizations of bounded mean oscillation, *Bull. Amer. Math. Soc.* **77** (1971), 587–588.
- [FS] C. FEFFERMAN, E.M. STEIN, H_p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
- [GC-RF] J. GARCIA-CUERVA, J.L. RUBIO DE FRANCIA, “*Weighted norm inequalities and related topics*,” North-Holland, Amsterdam, 1985.
- [JN] F. JOHN, L. NIRENBERG, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415–426.
- [J] J.L. JOURNE, “*Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*,” Lectures Notes in Math. **994**, Springer-Verlag, Berlin, 1983.
- [L] R.H. LATTE, A characterization of $H_p(\mathbb{R}^n)$ in terms of atoms, *Studia Math.* **62** (1978), 93–101.

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