0. Introduction

Let us consider in a domain $\Omega$ of $\mathbb{R}^n$ solutions of the differential inequality

$$|\Delta u(x)| \leq V(x)|u(x)|, \quad x \in \Omega,$$

where $V$ is a non smooth, positive potential.

We are interested in global unique continuation properties. That means that $u$ must be identically zero on $\Omega$ if it vanishes on an open subset of $\Omega$.

There is an extensive literature on the matter, mainly to relax the local integrability condition required to the potential $V$. When $L^p_{loc}$ classes are considered, $p \geq n/2$ is a necessary and sufficient condition for the strong unique continuation property [JK] (see [K] for references). In this paper we shall consider some spaces introduced by Morrey [M], which have been recently used by C. Fefferman and D.H. Phong [FP] in studying the eigenvalues of Schrödinger operators; these spaces contain $L^{p/2}_{loc}$.

We say that $V \in F_p^{\lambda}$, with $\lambda = 2p - n$ in classical notation [P], if

$$\|V\|_{F_p^{\lambda}} = \sup_{Q} |Q|^{2/n-p} \left( \int_{Q} |V|^p \right)^{1/p} < \infty$$

where the $\sup$ is taken over all cubes in $\mathbb{R}^n$ and $|Q| = \text{Volume of } Q$. Notice $F^p \subseteq F^q$ if $p \geq q$.

In this paper we prove that any solution of (1) has the global unique continuation property if $V \in F^p_{loc}$ and $p > (n-2)/2$.

Very recently T. Wolf has obtained the same result with a different approach.

We would like to thank C. Kenig for telling us about T. Wolf's result.

This improves the previously known results where $p > \frac{(n-1)}{2}$ (see [CS] and [ChR]).

The point to obtain this improvement is that in the above works the Carleman estimate is seen as a consequence of a uniform Sobolev inequality (see [KRS]).

$$\|u\|_{L^2(V)} \leq C\|V\|_{F^p} \|(\Delta + a_j \partial / \partial x_j + b)u\|_{L^2(V^{-1})},$$

(2)
where $C$ is independent of the linear perturbation of the Laplacian. Nevertheless, we prove directly the Carleman estimate

$$\|e^{r^2} u\|_{L^2(V)} \leq C \|V\|_{F^p} \|e^{r^2} \Delta u\|_{L^2(V^{-1})},$$

where $C$ is independent of $r$ for $r$ in $(\tau_0, \infty)$.

As we shall see while (2) is based on the restriction theorem for the Fourier Transform on the $(n-1)$-dimensional sphere, together with classical theory of weights, our proof follows from a detailed analysis of the multiplier associated to (3) which just involves the restriction theorem in dimension $n-2$. Therefore the assumption in $p$ comes from the restriction operator in the sphere. We think that this is just a technical obstruction and the restriction theorem should be true for $p \geq 1$. Notice that we are close in the case $n = 4$. We also remark that $F^1_{loc}$ contains the so called Kato-Stummel class which B. Simon has conjectured is enough to assure unique continuation (see [S]).

In the sequel we denote by $H^2_{loc}(\Omega)$ the classical Sobolev space, and

$$A u_Q f = (1/|Q|) \int_Q f.$$

We define the local Morrey class as the functions $W$ such that

$$\|W\| = \sup_{y \in \Omega} \limsup_{r \to 0} \|\lambda B(y, r) W(y)\|_{F^p} < \infty.$$

The main theorem is:

**Theorem 1.** Let $u \in H^2_{loc}(\Omega), n \geq 3$, be a solution of (1), then there exists an $\varepsilon > 0$, only depending on $p$ and $n$, such that if $V \in F^2_{loc}, \|V\|_{F^p} < \varepsilon, p > (n - 2)/2$, and $u$ vanishes in an open subdomain of $\Omega$, then $u$ must be zero everywhere in $\Omega$.

The proof is related to a restriction theorem for the Fourier Transform, obtained in [CS] and [ChR], for which we are going to give an easy proof. Let us define, for this purpose, the Morrey classes; we say that $V$ is in $F^{\alpha,p}$ if

$$\|V\|_{\alpha,p} = \sup_{r \neq 0} r^{\alpha} (Av_{B(x, r)} V)^{1/p} < \infty,$$

where the sup is taken on all the balls contained in $\Omega$. This notation corresponds to $E^{-\alpha,p}$ in [P], $1 \leq \alpha \leq n/p$. Also $F^{2,p} = F^p$.

**Theorem 2.** Let $d\sigma$ be the uniform measure on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, and $(d\sigma)^\wedge$ its Fourier transform, let $V \in F^{\alpha,p}, p > (n - 1)/(2(\alpha - 1))$, and consider the operator

$$T f(x) = (d\sigma)^\wedge * f(x).$$
Then there exists a constant $C$ such that
\[ \| Tf \|_{L^2(V)} \leq C \| V \|_{F^p} \| f \|_{L^2(V^{-1})} \]
for any $f$ in $C_0^\infty$.

It would be interesting to understand how this theorem is related to the one in [V] for mixed norm introduced by Rubio de Francia in the study of Bochner-Riesz operators [R].

1. The Carleman estimate

It is standard to obtain Theorem 1 as a consequence of the following Carleman estimate. This reduction can be seen in the case of $L^2$ weighted estimates in [CS] or [ChR].

**Theorem (1.1).** There exists a constant $C > 0$ such that for $V$ in $F^p$, $p > (n - 2)/2$, the inequality
\[ \| e^{\tau x^\mu} u \|_{L^2(V)} \leq C \| V \|_{F^p} \| e^{\tau x^\mu} \Delta u \|_{L^2(V^{-1})}, \]
holds for every $u$ in $C_0^\infty$ and $\tau$ in $\mathbb{R}$.

**Proof.** We can reduce to the case $\tau = 1$ in the following way:

Take $f(x) = e^{\tau x^\mu} u(x)$, then (1.1) reduces to
\[ \| f \|_{L^2(V)} \leq C \| V \|_{F^p} \| P_1(D) f \|_{L^2(V^{-1})}, \]

where $P_1(D)$ has symbol $P_1(\xi) = |\xi|^2 - \tau^2 + i\tau \xi_n$.

The change of variable $f(x) = g(x)$ reduces (1.2) to
\[ \| g \|_{L^2(V)} \leq C \| V \|_{F^p} \| P_1(D) g \|_{L^2(V^{-1})}, \]

where $V(x) = V(\frac{x}{\tau})$, since $\| V \|_{F^p} \leq \tau^2 \| V \|_{F^p}$.

Consider the inverse operator given by the Fourier multiplier
\[ (T^* g)(\xi) = \frac{1}{P_1(\xi)} g(\xi). \]

Our theorem reduces to prove that $T : L^2(V^{-1}) \to L^2(V)$ for $V$ in $F^p, p > (n - 2)/2$.

We are going to use a decomposition of $T$ in the phase space. Consider first
\[ P_1(\xi)^{-1} = (\varphi_1(\xi) + \varphi_2(\xi) + \varphi_3(\xi)) P_1(\xi)^{-1} = \sum_{i=1}^3 m_i(\xi), \]
where \( \varphi_i \) is in \( C_0^\infty \), \( i = 1, 2 \); \( \text{supp} \, \varphi_1 \subset \{ |\xi| < 1/2 \}, \varphi_1 \equiv 1 \) in \( \{ |\xi| < 1/4 \} \); \( \text{supp} \, \varphi_2 \subset \{ |\xi| > 2 \}, \varphi_2 \equiv 1 \) in \( \{ |\xi| > 3 \} \).

The Fourier multiplier corresponding to \( m_1 \) has a kernel rapidly decreasing and hence satisfies the inequality. For \( m_3 \) just observe that it behaves like \( |\xi|^{-2} \) and by known results, see [FeP], satisfies the inequality for \( V \) in \( F^p \) with \( p > 1 \).

We may decompose \( m_2 \) as a finite sum of operators the worst of which is given by the multiplier

\[
\tilde{m}(\xi) = p_1(\xi)^{-1} \psi_1(|\xi'|^2 - 1) \psi_2(\xi_n),
\]

with \( \xi' = (\xi_1, ..., \xi_{n-1}) \), \( \text{supp} \, \psi_2 \subset [-1, 1] \), \( \text{supp} \, \psi_1 \subset [-1/4, 1/4], \psi_1 \in C_0^\infty \).

Now we may write

\[
\tilde{m}(\xi) = \sum_{j=1}^{\infty} \tilde{m}_j(\xi),
\]

for \( \tilde{m}_j(\xi) = a_j(\xi) \psi_1 \left( \frac{|\xi'|^2 - 1}{\delta^2} \right) \psi_2 \left( \frac{\xi_n}{\delta} \right) \), \( \delta = 2^{-j} \), with appropriate \( a_j \) with \( \delta^{-1} < |a_j| < 2\delta^{-1} \).

Hence we may reduce our inequality to the study of the operator \( K_{\delta} \) given by a Fourier multiplier which has \( L^\infty \) norm as \( \delta^{-1} \) and is supported in the “torus” \( |\xi'| - 1 < 2\delta, |\xi_n| < \delta \). It is enough to prove:

**Lemma.** For \( 0 < \delta < 1/2 \) and \( T_{\delta} \) defined by

\[
(T_{\delta})^\wedge(\xi) = m(\xi)f^\wedge(\xi),
\]

where

\[
m(\xi) = \varphi \left( \frac{1 - |\xi'|}{\delta} \right) \varphi \left( \frac{\xi_n}{\delta} \right), \quad \varphi \subset [-1, 1], \varphi \in C_0^\infty,
\]

the following inequalities hold:

(i) \[
\left( \int |T_{\delta}f|^2 \{1/2 \right) \leq C\delta |\log \delta||V||_{F^{p_0}} \left( \int |f|^2 \{1/2 \right) \right)^{1/2}, \quad p_0 = (n - 2)/2.
\]

(ii) \[
\left( \int |T_{\delta}f|^2 \{1/2 \right) \leq C\delta^{1+\epsilon}||V||_{F^p} \left( \int |f|^2 \{1/2 \right) \right)^{1/2}, \quad \text{with } 0 < \epsilon < 1 - (n-2)/2p.
\]

**Proof:** Let us call \( K(x) = m^\wedge(x) \) and consider \( \{ \psi_j \} \) a smooth partition of unity

\[
1 = \sum_{j=0}^{\infty} \psi_j, \quad \text{supp} \, \psi_j \subset (2^{j-1}, 2^{j+1}) \quad j = 1, 2, \ldots
\]
Define $T_j f = K_j * f$, where $K_j(x) = \psi_j(|x'|)K(x)$ and $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We shall obtain a good estimate for $K_j$ which will allow us to sum in $j$.

On one hand observe that a straightforward calculation gives $|m_j(\xi)| = |(K_j)^{\wedge}(\xi)| \leq C \min\{2^j \delta, 1\}$ and, as a consequence,

$$
(1.3) \quad \left( \int |T_j f|^2 \right)^{1/2} \leq C \min\{2^j \delta, 1\} \left( \int |f|^2 \right)^{1/2}.
$$

On the other hand for any natural number $m$ there exists a constant $C_m$ such that

$$
(1.4) \quad |K_j(x)| \leq C_m \delta^2 2^{-j(n-2)/2} (1 + \delta |x_n|)^{-m} (1 + \delta^2)^{-m}.
$$

Consider first the case $0 < j \leq 1 + \lfloor \log_2 \delta \rfloor$. For $k \in \mathbb{Z}$ we define

$$
K_{jk}(x) = K_j(x) \cdot \chi_{[k \delta^{-1}, (k+1) \delta]}(x_n).
$$

Then

$$
|K_{jk}(x)| \leq C_m \delta^2 2^{-j(n-2)/2} (1 + |k|)^{-m}.
$$

Finally we can make in $\mathbb{R}^n$ a grid with parallelepipeds $\{Q_v\}$ such that the dimension of $Q_v$ are $2^j \times \ldots \times 2^j \times \delta^{-1}$.

Call $f_v = f \cdot \chi_{Q_v}$. Then

$$
\int |K_{jk} * f|^2 w = \int |K_{jk} * \sum_v f_v|^2 w
\leq C \sum_v \int |K_{jk} * f_v|^2 w
\leq C \left( \sup_v \int_{Q_{v^*}} w \right) \sum_v \|K_{jk} * f_v\|_2 \|f_v\|_2 \|f_v\|_\infty (Q_{v^*}),
$$

where $Q_{v^*}$ is a parallelepiped with the same center as $Q_v$ and side ten times bigger than the sides of $Q_v$. By (1.4) and Young's inequality

$$
\leq C_m \delta^2 2^{-j(n-2)} (1 + |k|)^{-2m} \left( \sup_v \int_{Q_{v^*}} w \right) \sum_v \left( \int |f_v| \right)^2
\leq C_m \delta^2 2^{-j(n-2)} (1 + |k|)^{-2m} \left( \sup_v \int_{Q_{v^*}} w \right)^2 \int |f|^2 w^{-1}.
$$

Now observe that if $w = V^{p_0}$ and $V \in F_{p_0}^m$, then

$$
\sup_v \int_{Q_{v^*}} w \leq C (2^j \delta)^{-1} 2^{2j} \|V\|_{F_{p_0}^m}.
$$
Thus,
\[
\left( \int |K_j \ast f|^2 V^{p_0} \right)^{1/2} \leq C \delta 2^{-j(n-4)/2} \|V\|_{F_{p_0}}^{p_0} \left( \int |f|^2 V^{-p_0} \right)^{1/2}.
\]

Interpolation with (1.3) gives
\[
\left( \int |K_j \ast f|^2 V \right)^{1/2} \leq C \|V\|_{F_{p_0}} \left( \int |f|^2 V^{-1} \right)^{1/2}, \text{ if } 0 \leq j \leq 1 + [\log 1/\delta].
\]

In the case \( j \geq 1 + [\log 1/\delta] \), let us define \( K_{jk} \) as \( K_j(x) \chi_{[k2^j,(k+1)2^j]}(x_n) \), with \( k \in \mathbb{Z} \). Now for \( j \) fixed we consider in \( \mathbb{R}^n \) a grid of cubes of side \( 2^j \). Repeating the above process we obtain
\[
\left( \int |K_j \ast f|^2 V^{p_0} \right)^{1/2} \leq C \delta^{2(1-m)} 2^{-j((n-2)/2+2m-2)} \|V\|_{F_{p_0}}^{p_0} \left( \int |f|^2 V^{-p_0} \right)^{1/2}.
\]

Again interpolation with (1.3) gives for \( j \geq 1 + [\log 1/\delta] \)
\[
\left( \int |K_j \ast f|^2 V \right)^{1/2} \leq C 2^{-j} \|V\|_{F_{p_0}} \left( \int |f|^2 V^{-1} \right)^{1/2}.
\]

Adding up in \( j \) we prove (i).

In order to prove (ii) we proceed as follows:

Define \( K_j(x) = \psi_j(\delta|x|)K(x) \), with \( \psi_j \) as above \( j = 0, 1, \ldots \) and the support of \( K_j \subset B(0, 2^{1+1}/\delta^{-1}) \). Then fix \( j \) and construct a grid of cubes \( \{Q_v\} \) of side \( 2^j 1 \). Then it is enough to prove the estimate for \( f_v = f \chi_{Q_v} \).

Take \( V \in F_p \) and \((n-2)/2 = p_0 < p < \infty\), let us call \( w = V^{p/p_0} \), then
\[
\left( \int |T_j f_v|^2 w \right)^{1/2} \leq \left( \int_{Q_{v*}} |T_j f_v|^2 w \right)^{1/2} = \left( \int |T_j f_v|^2 w_v \right)^{1/2},
\]

where \( w_v = w \chi_{Q_v} \); then \( w_v \in F^{p_0} \) and
\[
\|w_v\|_{F^{p_0}} \leq C \|V\|_{F^{p/p_0}} (2^j \delta^{-1})^{2(1-p/p_0)} \text{ and then by (i)}
\]
\[
\left( \int |T_j f_v|^2 w \right)^{1/2} \leq C \delta \|\log \delta\| (2^j \delta^{-1})^{2(1-p/p_0)} \|V\|_{F^{p_0}}^{p_0} \left( \int |f_v|^2 w^{-1} \right)^{1/2}.
\]

But also
\[
\left( \int |T_j f_v|^2 \right)^{1/2} \leq C \left( \int |f_v|^2 \right)^{1/2}, \text{ and by interpolation } \left( \int |T_j f_v|^2 V \right)^{1/2}
\]
\[
\leq C \delta^{2-p/p_0} \|\log \delta\|^{p_0/2} (2^{-j(1-p/p_0)} \|V\|_{F^{p_0}} \left( \int |f_v|^2 V^{-1} \right)^{1/2},
\]

and (ii) is proved. 

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\]
2. The Restriction theorem

We give the proof of theorem 2. Let us remark again that this theorem is contained in [CS] and [ChR], but the simplicity of our proof justifies to write it here.

Proof of theorem 2: It is known that

\[ K(x) = (d\sigma)^\wedge(x) = |x|^{-\left(\frac{n}{2} - 1\right)} J_{\frac{n}{2} - 1}(|x|), \]

where \( J_\lambda \) designs the Bessel function of order \( \lambda \). Then decompose

\[ K(x) = \sum_{j=0}^{\infty} K_j(x) \]

with

\[ K_j(x) = (d\sigma)^\wedge(x) \psi_j(|x|), \quad j = 1, 2, ..., \supp \psi_j \subset [2^{j-1}, 2^{j+1}]; \]

\[ K_0(x) = (d\sigma)^\wedge(x) \psi(|x|), \quad \supp \psi \subset [-1, 1]. \]

The classical P. Tomas, estimate for the Fourier Transform of \( K_j(x) \) gives us the boundedness of \( T_j = K_j \ast \) from \( L^2 \) to \( L^2 \) with norm \( 2^j \).

We can repeat the argument in the proof of theorem 1 and obtain, for \( w = V^p \),

\[ T_j : L^2(w^{-1}) \rightarrow L^2(w) \text{ with norm bounded by } 2^{-j(n-1)/2} \left( \sup_{Q_j} \int_{Q_j} w \right) \]

where \( Q_j \) is a cube in the grid in \( \mathbb{R}^n \) of side \( 2^j \). Since \( V \in F^{\alpha,p} \), we obtain

\[ \|T_j\|_{L^2(w^{-1}) \rightarrow L^2(w)} \leq C \cdot 2^j (\alpha^{p-(n-1)/2}) \|V\|_{p,p}. \]

Interpolation gives

\[ \|T_j\|_{L^2(V^{-1}) \rightarrow L^2(V)} \leq C (2^j)^{-\frac{n-1+\alpha(1-\alpha)}{2}} \|V\|_{p,p}, \]

the sum is convergent if \( p > \frac{n-1}{2(\alpha-1)}. \)

It is an open question if the above operator send \( L^2(V^{-1}) \) to \( L^2(V) \) for \( V \in F^{\alpha,p}, p < (n-1)/2 \). The answer to this question would be the corner stone to extend unique continuation properties to potential in \( F^p \) for \( p \leq (n-2)/2 \).

References


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