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AN EXPLICIT EXPRESSION FOR THE K_r FUNCTIONALS OF INTERPOLATION BETWEEN L^p SPACES

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Introduction and Notation

When dealing with interpolation spaces by real methods one is lead to compute (or at least to estimate) the K-functional associated to the couple of interpolation spaces. This concept was first introduced by J.Peetre (see [8], [9]) and some efforts have been done to find explicit expressions of it for the case of Lebesgue spaces. It is well known that for the couple consisting of L^1 and L^∞ on $[0,\infty)$ K is given by $K(t;f,L^1,L^\infty)=\int_0^t f^*$ where f^* denotes the non increasing rearrangement of the function f.

In [7] Nilsson and Peetre computed the K-functional also between spaces L^p and L^q when $1\leq p<q<\infty$. More recently the two first named authors obtained an explicit expression for a suitable modification of the K-functional for the case (L^p,L^M) where L^M stands for an Orlicz space (see [1]).

The aim of this paper is to answer a question raised by J. Peetre to the authors and to extend the results in [1] and [7] for the more general case of the K_r-functionals between L^p spaces. The notion of K_r-functional was introduced in [4] by Holmsted and Peetre obtaining also some estimates for those functionals between general compatible couples of interpolation spaces.

We shall write L^p for the Lebesgue space $L^p([0,\infty))$. For $f\in L^p+L^q, 1\leq p<q<\infty$, $t>0$ and $1\leq r\leq\infty$ we define the K_r-functional by

$$K_r(t;f)=\inf\left(\|g\|_p^r+t^r\|h\|_q^r\right)^{1/r}$$

where the infimum runs over all possible decompositions $f=g+h$ with $g\in L^p$ and $h\in L^q$. (Obviously K_∞ will mean

$$K_\infty(t;f)=\inf\max\{\|g\|_p,t\|h\|_q\})$$

Note that $r=1$ corresponds to the classical definition of K- functional. The reader is referred to [2], [3] and [9] for background on interpolation spaces.

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Clearly the K_r-functional defines a rearrangement invariant norm on $L^p + L^q$ equivalent to the natural one. Hence $K_r(t; f) = K_r(t; f^*)$ and this allows us to restrict ourselves to non negative and non increasing functions $f \in L^p + L^q$.

The paper is divided into three sections and one appendix. Section I is devoted to show the existence of extremal functions which minimize the K_r-functionals. In section II we give a procedure to get such extremal functions for the cases $1 \leq r < \infty$ and finally the case $r = \infty$ is considered in section III. In the appendix we introduce and compute the functionals $K_{r,s}$ and $\mathcal{K}_{r,s}$. We apply these results to the isometric problem of interpolation between the couple of spaces (L^p, L^q).

Let us mention that the method we use in sections II and III is actually a simplified version of the calculus of variations, but we shall include some proofs for the sake of completeness. As it happens in [1] and [7] the solutions for $p = 1$ or $p > 1$ are essentially different. In the case $p = 1$ the extremal decomposition of f is achieved by a horizontal slicing of the function f.

I. Existence of extremal solutions

In the sequel $1 \leq p < q < \infty$ and f will denote a non negative and non increasing fixed function on $[0, \infty)$. It is very easy to see that

$$K_r(t; f) = \inf \left(||g||_p^r + t^r ||f - g||_q^r \right)^{1/r}$$

where the infimum is taken over all functions $g \in L^p$ with $0 \leq g \leq f$.

Let g be a non negative measurable function on $[0, \infty)$ such that $0 \leq g \leq f$. We define the functional $\Phi(g; f; t; r)$ or simply $\Phi(g)$ by

$$\Phi(g) = \begin{cases} ||g||_p^r + t^r ||f - g||_q^r, & \text{if } 1 \leq r < \infty; \\ \max\{||g||_p, t||f - g||_q\}, & \text{if } r = \infty. \end{cases}$$

It is clear that $0 \leq \Phi(g) \leq \infty$ and

$$K_r(t; f) = \inf \{\Phi(g)^{1/r}; 0 \leq g \leq f, g \in L^p\}.$$

The main result of this section is the following theorem

I.1 Theorem. *Let $1 \leq r \leq \infty$ and $f \in L^p + L^q$. There exists a non increasing function $g \in L^p, 0 \leq g \leq f$, such that $K_r(t; f) = \Phi(g)^{1/r}$. If $1 < r \leq \infty$, this function g is unique and $f - g$ is also non increasing.*

In order to prove this theorem we need several lemmas. We begin by establishing the existence of extremal solutions.

I.2 Lemma. *There exists a function $g \in L^p, 0 \le g \le f$, such that $K_r(t;f) = \Phi(g)^{1/r}$.*

Proof: Let α be defined by $\alpha = \inf\{\Phi(g); 0 \le g \le f, g \in L^p\}$. We can choose a sequence $(g_n)_n$ in L^p such that $0 \le g_n \le f$ and $\alpha = \lim_{n\to\infty}\Phi(g_n)$. As $(f-g_n)_n$ is a bounded sequence in L^q we may suppose, by passing to subsequence if necessary, that $(f-g_n)_n$ converges weakly to a function $h \in L^q$. Then $(g_n)_n$ is a weakly Cauchy sequence in L^p and so $w-\lim g_n = f-h \in L^p$. Hence if $g = f-h$ we have

$$\Phi(g) \le \liminf \|g_n\|_p^r + t^r \liminf \|f-g_n\|_q^r = \lim_{n\to\infty} \Phi(g_n) = \alpha.$$

The same arguments can be modified for $r = \infty$. ■

Remark. Let us point out that the same ideas used in preceding lemma may be applied to K_r-functionals between a larger class of interpolation spaces. Actually, if (A_0, A_1) is a compatible couple of Banach spaces let us define K_r on $A_0 + A_1$ by

$$K_r(t;f) = \inf\left(\|g\|_{A_0}^r + t^r\|h\|_{A_1}^r\right)^{1/r}$$

where the infimum runs all possible decompositions $f = g+h$ with $g \in A_0$ and $h \in A_1$. If we suppose that A_0 is weakly sequentially complete, A_1 is reflexive and $A_0^* \cap A_1^*$ is dense in A_0^*, then there exist $g \in A_0$ and $h \in A_1$ such that $f = g+h$ and $K_r(t;f) = \left(\|g\|_{A_0}^r + t^r\|h\|_{A_1}^r\right)^{1/r}$.

I.3 Definition. A non increasing function g in L^p is an *extremal solution* of the functional $K_r(t;f)$ if $0 \le g \le f$ and $K_r(t;f) = \Phi(g)^{1/r}$ ($K_r(t;f) = \Phi(g)$ if $r = \infty$).

Next we will study the uniqueness of the extremal solutions.

I.4 Lemma. *Let $1 < r \le \infty$. If $K_r(t;f) = \Phi(g_1)^{1/r} = \Phi(g_2)^{1/r}$, then $g_1 = g_2$.*

Proof: It is very easy to check that the function $g = \dfrac{g_1+g_2}{2}$ also verifies $K_r(t;f) = \Phi(g)^{1/r}$ for all $1 \le r < \infty$. Indeed, by using Minkowski's inequality we have

$$\begin{aligned}\Phi(g)^{1/r} &\le \frac{1}{2}\left[(\|g_1\|_p + \|g_2\|_p)^r + (t\|f-g_1\|_q + t\|f-g_2\|_q)^r\right]^{1/r} \\ &\le \frac{1}{2}\left[\Phi(g_1)^{1/r} + \Phi(g_2)^{1/r}\right] = K_r(t;f)\end{aligned} \quad (*)$$

and then $\|g_1+g_2\|_p = \|g_1\|_p + \|g_2\|_p$, $\|(f-g_1)+(f-g_2)\|_q = \|f-g_1\|_q + \|f-g_2\|_q$.

If $1 < r < \infty$ the vectors in $\mathbb{R}^2$ $(\|g_1\|_p, t\|f-g_1\|_q)$ and $(\|g_2\|_p, t\|f-g_2\|_q)$ are also colinear. Since L^q is strictly convex ($q > 1$) we obtain that, for instance,

$f - g_1 = \lambda(f - g_2)$ for some $\lambda \geq 0$. Thus, $||g_1||_p = \lambda||g_2||_p$ and $\Phi(g_1) = \lambda^r\Phi(g_2)$ which implies that $\lambda = 1$ and consequently $g_1 = g_2$.

Suppose now that $r = \infty$. We realize that $K_\infty(t; f) = \Phi(g)$ implies that $||g||_p = t||f - g||_q$. Indeed, if it were $||g||_p > t||f - g||_q$, we choose some positive a such that the set $A = \{x \in [0, \infty); g(x) > a\}$ has $m(A) > 0$. Take $0 < \delta$ small enough verifying

$$t||f - g||_q < t||f - g + \delta\chi_A||_q < ||g - \delta\chi_A||_p < ||g||_p$$

Hence $\Phi(g - \delta\chi_A) = ||g - \delta\chi_A||_p < \Phi(g)$ and consequently $K_\infty(t; f) < \Phi(g)$. (An analogous argument works if we suppose $||g||_p < t||f - g||_q$).

Let g_1, g_2 be two functions in L^p such that $K_\infty(t; f) = \Phi(g_1) = \Phi(g_2)$. We may repeat the arguments used for $1 < r < \infty$ and we obtain

$$\begin{aligned}\Phi\left(\frac{g_1 + g_2}{2}\right) &\leq \frac{1}{2}\max\{||g_1||_p + ||g_2||_p, t||f - g_1||_q + t||f - g_2||_q\} \\ &= \Phi(g_1)\end{aligned}$$

We therefore have $||f - g_1 + f - g_2||_q = ||f - g_1||_q + ||f - g_2||_q$ and then $f - g_1 = \lambda(f - g_2)$ which also implies $g_1 = g_2$. ■

Remark. In the case $r = 1$ different solutions could be obtained. We realize that if g_1 and g_2 are two different solutions any other function g in the segment defined by g_1 and g_2 is also a solution.

I.5 Lemma. *i) If $0 \leq g \leq f$ then $0 \leq g^* \leq f$ and $\Phi(g^*) \leq \Phi(g)$ (g^* is the non increasing rearrangement of g). ii) If the functional $K_r(t; f)$ has only one extremal solution g then $f - g$ is non increasing.*

Proof: i) We apply the proposition 1 of [6] (which is also valuable in the interval $(0, \infty)$) and then we have

$$\int_o^a (f - g^*)^* \leq \int_o^a (f - g)^*.$$

This implies that $||f - g^*||_q \leq ||f - g||_q$ and so $\Phi(g^*) \leq \Phi(g)$.

ii) If $K_r(t; f) = \Phi(g)$ then $g = g^*$. The function g_1 defined by $g_1 = f - (f - g)^*$ satisfies $\Phi(g_1) \leq \Phi(g^*)$ by i). Thus $g_1 = g^*$ and so $f - g^* = (f - g)^*$ which implies that the function $f - g$ is non increasing. ■

Remark. The lemma is also true if we consider the corresponding K_r functional between a couple of rearrangement invariant function spaces.

Proof of the theorem I.1: It is obvious from the preceding lemmas. ■

II. Determination of the extremal solutions when $1 \le r < \infty$

The main tool for this part is the following lemma.

II.1 Lemma. *Let g be an extremal solution of $K_r(t;f)$ and assume* supp $g = [0,a]$, $0 \le a \le \infty$. *The following assertions are true:*

i) *Either $g < f$ a.e. on* supp f, *or $r = 1$ and $g = f$, a.e. on* supp f.

ii) *If $p > 1$, either $0 < g(x)$ and $g(x)^{p-1}||g||_p^{r-p} = t^r[f(x) - g(x)]^{q-1}||f - g||_q^{r-q}$ a.e. on* supp f , *or $r = 1$ and $g = 0$.*

iii) *If $p = 1$, $a > 0$ and $g \ne f$ then $a < \infty$ and $f(x) - g(x) = constant = \lambda \ge 0$, a.e. on $[0,a]$, where a and λ verify*

$$\left(\int_0^a f - a\lambda\right)^{r-1} = t^r \lambda^{q-1}\left[a\lambda^q + \int_a^\infty f^q\right]^{(r/q)-1} \qquad (*)$$

Furthermore, if $r = 1$, then $a < t^{q'}$ (q' is the conjugate exponent $\frac{1}{q} + \frac{1}{q'} = 1$) and $\lambda = [t^{q'} - a]^{-1/q}||f\chi_{[a,\infty)}||_q$

iv) *If $g = 0$, then $r = 1$. Moreover, if $p = 1$ then $f \in L^q \cap L^\infty$ and $t^{q'/q}||f||_\infty \le ||f||_q$.*

v) *If $g = f$ $(r = 1)$ and $p = 1$ then length* supp $f = b \le t^{q'}$.

Proof: i) Assume that $m\{x; f(x) = g(x)\} > 0$. Then there exists $n \in \mathbb{N}$ such that $B_n = \{x \in \text{supp } f; 1/n < f(x) = g(x)\}$ and $m(B_n) > 0$. Let φ be the function defined by $\varphi(\delta) = \Phi(g\chi_{B_n^c} + (f - \delta)\chi_{B_n})$ for $0 \le \delta \le 1/n$. As $\varphi(0) = \Phi(g)$ and $\varphi'(0^+) < 0$ we necessarily have that $r > 1$ and $f = g$.

ii) Let $A_n = \{x \in \text{supp } f; g(x) = 0, f(x) > 1/n\}$ and let $\varphi(\delta) = \Phi(g + \delta\chi_{A_n})$ for $0 \le \delta \le 1/n, n \in \mathbb{N}$. If $r > 1$ or $f \ne g$ the same reasons as before imply that $m(A_n) = 0$, and so, $0 < g$ a.e. on supp f. Consider now any measurable set A contained in supp f, with $0 \le m(A) < \infty$. The function $\varphi(\delta) = \Phi(g + \delta\chi_A)$ has its minimum in $\delta = 0$, hence $\varphi'(0) = 0$ (note that $p > 1$). Thus

$$\int_A ||g||_p^{r-p}|g|^{p-1} - t^r||f - g||_q^{r-q}|f - g|^{q-1} = 0$$

and then ii) follows since A is arbitrary.

iii) Let A be any compact interval contained in $[0,a)$. Let φ be the function defined by $\varphi(\delta) = \Phi(g + \delta\chi_A)$ for $|\delta| < \inf\{f(x); x \in A\}$. It is clear that $\varphi'(0)$ does exists and actually $\varphi'(0) = 0$. Therefore we obtain

$$\int_A ||g||_1^{r-1} - t^r||f - g||_q^{r-q}|f - g|^{q-1} = 0.$$

Since this expression is true for any compact interval contained in $[0,a)$ we deduce that $f - g$ = constant , a.e.$x \in [0,a]$. If $\lambda = f(x) - g(x)$, then

$g = (f - \lambda)\chi_{[o,a]}$ and consequently a and λ have to verify the equation $(*)$. Note that $a < \infty$ as the function $f - g \in L^q$. If $r = 1$ we easily compute that

$$1 = t\left[a + \int_a^\infty \left(\frac{f}{\lambda}\right)^q\right]^{(1/q)-1}$$

and then $(t^{q'} - a)\lambda^q = \int_a^\infty f^q$ which implies iii).

iv) If $g = 0$ is an extremal solution, then $f \in L^q$. We repeat the preceding arguments by considering now the function $\varphi(\delta) = \Phi(\delta\chi_{[c,d]})$ where $[c,d] \subsetneq$ supp f and $0 < \delta < f(d)$. Since φ has the minimum in $\delta = 0, \varphi'(0+) \geq 0$ and hence

$$0 \leq \lim_{\delta \to 0^+} r\delta^{r-1}(d-c)^{r/p} - rt^r \int_c^d f^{q-1}||f||_q^{r-q}$$

what implies that $r = 1$. If $p = 1$ we therefore obtain $1 \geq t||f||_q^{r-q} f(x)$ a.e. on supp f.

v) If $g = f (r = 1)$ similar arguments to those appearing in iv) show v). This concludes the proof of the lemma. ■

Next we will study the case $p > 1$.

II.2 In view the preceding lemma we have to consider the class of functions g verifying: i) $g \in L^p, f - g \in L^q$,

ii) $0 < g < f$ and

$$t^r||f - g||_q^{r-q}[f(x) - g(x)]^{q-1} = ||g||_p^{r-p} g(x)^{p-1}$$

a.e. on supp f.

Let $\mathcal{A}$ denote the class of functions satisfying i) and ii). It is very simple to check that for them

$$\Phi(g) = t^r||f - g||_q^{r-q} \int f(f-g)^{q-1} = ||g||_p^{r-p} \int fg^{p-1}.$$

Using the strictly decreasing function h defined by $h(y) = \dfrac{(f(x) - y)^{q-1}}{y^{p-1}}$ for $0 < y < f(x)$, we see that the following facts are true : i) If $g \in \mathcal{A}$, g is non increasing. ii) $\mathcal{A}$ is totally ordered. Indeed, let g_1, g_2 be two elements in $\mathcal{A}$ and write $M_i = ||g_i||_p^{r-p}||f - g_i||_q^{q-r}$, $i = 1, 2$. It is clear that $M_1 = M_2$ (respectively $M_1 < M_2$) implies $g_1(x) = g_2(x)$ a.e.$x \in$ supp$f$ (respectively $g_1(x) > g_2(x)$ a.e. $x \in$ supp f). Furthermore, for $q \geq r$, the inequality $g_1 > g_2$ yields $M_1 > M_2$. Hence the set $\mathcal{A}$ has only one element which is necessarily the unique extremal solution of $K_r(t; f)$.

Let now assume $p \leq r < q$. If $\overline{g}$ denotes the inf $\mathcal{A}$ we are going to prove that $\overline{g} = \min \mathcal{A}$ and that $\overline{g}$ is the unique extremal solution of $K_r(t; f)$. Indeed, we realize that for any two elements of $\mathcal{A}$, $g_1 < g_2$ we have $\Phi(g_1) < \Phi(g_2)$.

Let $\overline{g}$ be defined by $\overline{g}(x) = \inf\{g(x); g \in \mathcal{A}\} \in L^p$ (L^p is order continuous, see [5]). Define $\alpha = \inf_{g\in\mathcal{A}} \|g\|_p$ and let $(g_n)_n$ be a sequence in $\mathcal{A}$ such that $\|g_1\|_p \leq \ldots \leq \|g_n\|_p \leq \ldots \to \alpha$. Therefore $g_1 \geq g_2 \geq \ldots \geq g_n \geq \ldots \geq 0$. Since $t^r\|f-g_n\|_q^r \leq \Phi(g_n) \leq \Phi(g_1)$ we have that $g_0 = \lim g_n \in L^p$ and $f - g_0 \in L^q$. By passing to the limit in II.2. it is easily checked that $g_0 \in \mathcal{A}$, too. Eventually we conclude that $\overline{g} = g_0 \in \mathcal{A}$. ■

We summarize all these facts in the following theorem

II.3 Theorem. *The case $p > 1$.*

i) *If $r \geq q$ then the class $\mathcal{A}$ has only one element which is the unique extremal solution of $K_r(t;f)$.*

ii) *If $p \leq r < q$ then the least element of $\mathcal{A}$ is the unique extremal solution of $K_r(t;f)$.*

iii) *If $1 < r < p$ we know that the unique extremal solution of $K_r(t;f)$ is an element of $\mathcal{A}$.*

iv) *If $1 = r < p$ the solutions verify the equation*

$$t[f(x)-g(x)]^{q-1}\|g\|_p^{p-1} = g(x)^{p-1}\|f-g\|_q^{q-1} \qquad (*)$$

Now g could be equal to 0 or f. Furthemore the solution is unique except if $f = \lambda\chi_{[0,a]}$. In this case $t = a^{1/p-1/q}$ and $K_1(t;f) = \lambda a^{1/p}$.

Moreover in the four cases, if g is an extremal solution we have

$$K_r(t;f) = \|g\|_p^{1-(p/r)}\left(\int fg^{p-1}\right)^{1/r} = t\|f-g\|_q^{1-(q/r)}\left(\int f(f-g)^{q-1}\right)^{1/r}.$$

Proof: We only have to prove the last part of iv). If $g_1 \neq g_2$ are extremal solution of $K_1(t;f)$ then all the points of the segment $[g_1, g_2]$ are extremal solutions. So we may suppose $g_1 \neq 0 \neq g_2$ and $f - g_1 \neq 0 \neq f - g_2$. Since L^p and L^q are strictly convex spaces the same reasons appearing in the lemma I.4 say that $g_1 = ag_2$, $f - g_1 = b(f - g_2)$ for positive a, b . Therefore $f = cg_2$, for some $c > 1$, and thus $tg_2(x)^{q-p} = \|g_2\|_q^{q-1}\|g_2\|_p^{p-1}$ a.e. on supp f. Hence $f = \lambda\chi_{[0,a]}$, $\lambda \geq 0$ and $t = a^{1/p-1/q}$. ■

II.4 Let us now consider the case $p = 1$. We shall denote by b =length of supp $f \leq \infty$. As it appears in lemma II.1., the extremal solutions of $K_r(t;f)$, g, have the following expression $g = (f-\lambda)\chi_{[o,a]}$ with $0 \leq a < \infty$. If $a > 0$ then λ verifies the equation II.1-iii)-(*). We define the function $H(a,\lambda)$ by

$$H(a,\lambda) = \left(\int_o^a f - a\lambda\right)^{r-1} - t^r\lambda^{q-1}\left[a\lambda^q + \int_a^\infty f^q\right]^{(r/q)-1} \qquad (r > 1)$$

$$H(a,\lambda) = 1 - t\lambda^{q-1}\left[a\lambda^q + \int_a^\infty f^q\right]^{(1/q)-1} \qquad (r = 1)$$

for $0 < a \leq b$, $0 \leq \lambda \leq \frac{1}{a}\int_0^a f$, $r > 1$ and $0 < \lambda < \infty$, $r = 1$. If a is a fixed positive number the function $H(a,\lambda)$ is strictly decreasing in the variable λ and $H(a,0) > 0$. On the other hand, $H(a, \frac{1}{a}\int_0^a f) < 0$ in the case $r > 1$, and for $r = 1$ $\lim_{\lambda\to\infty} H(a,\lambda) < 0$ if and only if $a < t^{q'}$. Hence, the equation $H(a,\lambda) = 0$ has only one solution $\lambda = \lambda_a$ for any $0 < a \leq b$ (case $r > 1$) and for any $0 < a < \min\{t^{q'}, b\}$ (case $r = 1$).

When $a = b$ and $r = 1$ the corresponding equation $H(a,\lambda) = 0$ is then $1 = tb^{1/q-1}$ and so, this equation has solution if and only if $t^{q'} = b$. Furthermore, if g is an extremal solution of the functional $K_1(t; f)$ with $a = b$, then $b < \infty$, $t^{q'} = b$, $K_1(t;f) = ||f||_1$ (all the functions $f - \lambda$ are extremal solutions of the functional for $0 \leq \lambda \leq f(b^-)$) and $H(b, f(b)) = 0$. In order to determinate the possible extremal solutions g we only have to check the value of a equal to the length of supp g. The main tool for computing this value a is the following lemma

II.5 Lemma. *Let g be an extremal solution of $K_r(t;f)$ and let $[0,a] =$ supp g, $a > 0$. Then $f(a^+) \leq \lambda_a \leq f(a^-)$.*

Proof: Since $g = (f - \lambda_a)\chi_{[o,a]} \geq 0$ we find $\lambda_a \leq f(a^-)$. If $\lambda_a < f(a^+)$, we would have $\lambda_a < f(x)$ for all $x \in (a,c)$. By considering the auxiliary function $\varphi(\delta) = \Phi(g + \delta\chi_{[a,c]})$ defined for $0 < \delta < \lambda_a$, we easily would check that

$$\varphi'(0^+) = rt^r||f - g||_q^{r-q} \int_a^c \left(\lambda_a^{q-1} - f^{q-1}\right) < 0$$

and, consequently, g would not be extremal solution for the functional $K_r(t;f)$ which contradicts our assumption. Hence the lemma follows. ■

II.6 Remark. As an immediate consequence of Lemma II.5. we have to study the set $I = \{x \in (0,b]; H(x, f(x^-)) \leq 0 \leq H(x, f(x^+))\}$, because the length of the support of the possible non null solutions belongs to I. Then we define the function $F(x) = H(x, f(x))$ defined for $0 < x \leq b$. The following properties will allow us to compute easily the extremal solutions of $K_r(t;f)$.

II.7 Lemma. *The following properties are true:*

i) *$F(.)$ is non decreasing. Furthermore if $F(x_1) = F(x_2)$ with $x_1 < x_2$ then f is constant in the closed interval $[x_1, x_2]$.*
ii) *$F(x^-) = H(x, f(x^-))$ and $F(x^+) = H(x, f(x^+))$.*
iii) *If $x_1 < x_2$, $H(x_1, f(x_1)) \leq H(x_2, f(x_2))$.*
iv) *I is either empty or an interval and the function f is constant on its interior. Furthermore $I = \{x; F(x^-) \leq 0 \leq F(x^+)\}$.*

Proof: i) Let $x_1 < x_2$ be two positive numbers. Since f is non decreasing we have that

$$x_2 f(x_2) - x_1 f(x_1) \leq f(x_2)(x_2 - x_1) \leq \int_{x_1}^{x_2} f$$

and therefore

$$\left[\int_o^{x_1} f - x_1 f(x_1)\right]^{r-1} \leq \left[\int_o^{x_2} f - x_2 f(x_2)\right]^{r-1}$$

If $r \geq q$ it is also clear that

$$x_1 f(x_1)^q + \int_{x_1}^{\infty} f^q \geq x_1 f(x_1)^q + (x_2 - x_1) f(x_2)^q + \int_{x_2}^{\infty} f^q \geq x_2 f(x_2)^q + \int_{x_2}^{\infty} f^q$$

Hence

$$f(x_1)^{q-1}\left[x_1 f(x_1)^q + \int_{x_1}^{\infty} f^q\right]^{(r/q)-1} \geq f(x_2)^{q-1}\left[x_2 f(x_2)^q + \int_{x_2}^{\infty} f^q\right]^{(r/q)-1}$$

In the case $1 \leq r < q$ we realize that

$$\begin{aligned}
& f(x_1)^{q-1}\left[x_1 f(x_1)^q + \int_{x_1}^{\infty} f^q\right]^{(r/q)-1} = \\
& f(x_1)^{r-1}\left[x_1 + \int_{x_1}^{\infty}\left(\frac{f}{f(x_1)}\right)^q\right]^{(r/q)-1} \geq \\
& f(x_2)^{r-1}\left[x_1 + \int_{x_1}^{\infty}\left(\frac{f}{f(x_1)}\right)^q\right]^{(r/q)-1} \geq \\
& f(x_2)^{r-1}\left[x_2 + \int_{x_2}^{\infty}\left(\frac{f}{f(x_1)}\right)^q\right]^{(r/q)-1} \geq \\
& f(x_2)^{r-1}\left[x_2 + \int_{x_2}^{\infty}\left(\frac{f}{f(x_2)}\right)^q\right]^{(r/q)-1}
\end{aligned}$$

and thus the first part of i) is proved. If $F(x_1) = F(x_2)$ and $r > 1$ we have, in particular, that

$$x_2 f(x_2) - x_1 f(x_1) = \int_{x_1}^{x_2} f \geq (x - x_1) f(x) + (x_2 - x) f(x_2)$$

for any x, $x_1 < x < x_2$. Then $f(x) = f(x_1) = f(x_2)$ and f is constant in $[x_1, x_2]$.

Suppose now that $r = 1$, then we find $\int_{x_1}^{x_2}\left(\frac{f}{f(x_1)}\right)^q = x_2 - x_1$ and therefore we have that f is also constant in $[x_1, x_2]$.

ii) This property is easily computed from the special expression defining the function $F(.)$.

iii) $H(x_1, f(x_1^+)) = \lim_{z \to x_1^+} F(z) \leq \lim_{z \to x_2^-} F(z) = H(x_2, f(x_2^-))$.

iv) Let x_1, x_2 be two different point of I, $x_1 < x_2$. By applying iii) we find that $H(z, f(z)) = 0$ for all $x_1 < z < x_2$. Then the set I is an interval and consequently f is constant in $\mathring{I}$. The last assertion in iv) is a consequence of iii). ■

II.8 Remark. If we denote by g_z the function $g_z = (f - \lambda_z)\chi_{[o,z]}$, where λ_z is the unique solution of $H(z,\lambda) = 0$, then $g_z = (f - f(z))\chi_{[o,z]}$ for all $z \in \mathring{I}$. Hence g_z is the same function and therefore $\Phi(g_z)$ is constant for all $z \in \mathring{I}$.

II.9 Theorem.

i) *If $F(0^+) \geq 0$ then $r = 1, g = 0$ is the unique extremal solution of the functional K_1. In this case $f \in L^q \cap L^\infty$, $\|f\|_q \leq \|f\|_\infty t^{q/q'}$ and $K_1(t;f) = t\|f\|_q$*

ii) *If $r = 1, \lim_{x\to 0^+} F(x) \leq 0$ then $b \leq t^{q'}$ and $g = f$ is extremal solution of the functional. Now $K_1(t;f) = \|f\|_1$*

iii) *If either $r = 1$ and $\lim_{x\to 0^+} < 0$ or $r > 1$ and $\lim_{x\to b^-} F(x) > 0$ there exists an unique extremal solution g for the functional $K_r(t;f)$. This function g is defined by $g = (f - \lambda)\chi_{[o,a]}$ where $a = \inf I$ $(I \neq \emptyset)$. The number λ is the unique solution of $H(a,\lambda) = 0$ and $f(a^+) \leq \lambda \leq f(a^-)$. The expression for $K_r(t;f)$ is*

$$K_r(t;f) = \left[\left(\int_0^a f - a\lambda\right)^r + t^r\left(a\lambda^q + \int_a^\infty f^q\right)^{r/q}\right]^{1/r}$$

Proof: This is a consequence of the preceding results, namely, II.1-iii),iv), v), II.4, II.7 and II.8. ■

Remark. i) When $r = 1$ and $F(x) < 0$ for all $x < b$ then $b < t^{q'}$ and f is the unique extremal solution of the functional. If $F(x) = 0$ for all $x \in [a,b]$, then the function $g = (f - \lambda)\chi_{[0,a]}$ is also extremal solution for the functional (λ is the corresponding solution of the equation $H(a,\lambda) = 0$).

ii) The fact that the extremal solutions for the K_r-functional between L^1 and L^q are horizontal slicings of f could be extended to the more general case of the functional $K_r(t; f, L^1, L)$, $1 < r$, where L is a strictly convex rearrangement invariant space on $[0,\infty)$. Indeed, if g is an extremal solution, g and $f - g$ are non increasing (see I.2, I.4 and I.5). Then $\{x; f(x) - g(x) > 0\} = [0,a)$. If $c < a$ we have $f(x) - g(x) > \epsilon$ on $[0,\epsilon)$ for some $\epsilon > 0$. For every point $x \in [0,\epsilon)$ there exists an interval J_x such that x belongs to J_x and g is constant on J_x. If this were not true let

$$g_\epsilon = g\chi_{[x-\epsilon,x+\epsilon]^c} + \left(\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} g\right)\chi_{[x-\epsilon,x+\epsilon]}$$

Since L is strictly convex $g \neq g_\epsilon$ would imply $\Phi(g_\epsilon) < \Phi(g)$ what contradicts our assumption on g. Eventually an argument of compactness implies that the function g is constant on $[0,b)$, for all $b < a$ and so, on $[0,a)$.

III. The case $r = \infty$

It is straightforward to establish that $K_\infty(t;f) = \lim_{r\to\infty} K_r(t;f)$. This expression would allow us to obtain the equation for the functional K_∞ by passing to the limit in the corresponding extremal solutions for $K_r(t;f)$. We prefer to compute directly the solution by using again the calculus of variations. We already know that if g is the extremal solution for K_∞ then $K_\infty(t;f) = \|g\|_p = t\|f-g\|_q > 0$.

The following lemma is the crucial tool for determining the solution when $p > 1$.

III.1 Lemma. *Let $p > 1$, and let g be an extremal solution of the functional $K_\infty(t;f)$. Then $0 < g(x) < f(x)$ and*

$$(f(x)-g(x))^{q-1} = g(x)^p \|g\|_p^p \int (f-g)^{q-1} g$$

a.e. x on supp f.

Proof: Let A and B two disjoint measurable sets of positive and finite measure, contained in supp g. The equation

$$\|g + \delta\chi_A + \mu\chi_B\|_p = t\|f - (g + \delta\chi_A + \mu\chi_B)\|_q$$

defines δ as an implicit function $\delta = \delta(\mu)$ in a neighbourhood of $\delta = \mu = 0$; we therefore have

$$\delta'(0) = -\frac{\int_B \left[\|g\|_p^{1-p} g^{p-1} + t\|f-g\|_q^{1-q}(f-g)^{q-1}\right]}{\int_A \left[\|g\|_p^{1-p} g^{p-1} + t\|f-g|_q^{1-q}(f-g)^{q-1}\right]}$$

Consider now the function $\varphi(\mu) = \|g + \delta\chi_A + \mu\chi_B\|_p$. Since this function is $C^{(1}$ and has a minimum in $\mu = 0$ we find

$$0 = \varphi'(0) = \|g\|_p^{1-p}\left[\delta'(0)\int_A g^{p-1} + \int_B g^{p-1}\right].$$

By an easy computation we obtain

$$\frac{\int_A (f-g)^{q-1}}{\int_A g^{p-1}} = \frac{\int_B (f-g)^{q-1}}{\int_B g^{p-1}}.$$

As this equality is true for any couple of disjoint measurable sets in supp g, we deduce that there exists a constant λ such that $(f-g)^{q-1} = \lambda g^{p-1}$, a.e. on supp g. If we would take $A \subseteq$ supp g and $B \subseteq$ (supp $f \setminus$ supp g) and repeat all the arguments we would get that

$$\int_A (f-g)^{q-1}\int_B g^{p-1} = \int_A g^{p-1}\int_B (f-g)^{q-1} = 0.$$

Therefore $f = g$ a.e. on B which implies that supp $g =$ supp f, and that concludes the lemma. ■

We can settle the case $p > 1$

III.2 Theorem. *There exists a unique extremal solution of the corresponding functional $K_\infty(t;f)$. This function is the unique element which verifies :* $0 < g < f$,

$$\frac{(f-g)^{q-1}}{g^{p-1}} = \|g\|_p^{-p} \int g(f-g)^{p-1}$$

a.e.x on supp f and $\|g\|_p = t\|f-g\|_q$.

Proof: Let $\mathcal{A}$ be the class of functions in L^p verifying: $0 < g < f$,

$$\frac{(f-g)^{q-1}}{g^{p-1}} = \|g\|_p^{-p} \int g(f-g)^{p-1}$$

a.e.x on supp f and $\|g\|_p = t\|f-g\|_q$. The same arguments appearing in II.2. prove that $\mathcal{A}$ is a totally ordered set in L^p. But if $g_1 \leq g_2$ are two elements in $\mathcal{A}$, we have $\|g_1\|_p \leq \|g_2\|_p = t\|f-g_2\|_p \leq t\|f-g_1\|_q = \|g_1\|_p$ and then $g_1 = g_2$. This proves the theorem. ■

Next we will study the case $p = 1$. For that we need the following

III.3 Lemma. *Let g be an extremal solution of $K_\infty(t;f)$, $p = 1$. If* supp *$g = [0,a]$ then $0 < a < \infty$, $f - g = \lambda$ a.e. on $[0,a]$, where λ is the unique solution of the equation*

$$\int_0^a f - a\lambda = t\left[a\lambda^q + \int_a^\infty f^q\right]^{1/q}$$

and $f(a^+) \leq \lambda f(a^-)$.

Proof: We begin by repeating the same arguments which appear in the proof of lemma III.1. We have to take A and B in $[0,a]$ far enough from a, in order to ensure that the corresponding auxiliary functions we use are defined in a neighbourhood of $\mu = 0$. We obtain that $f - g = \lambda$ a.e. x on $[0,a]$ (we do not know yet if $\lambda > 0$; in any case it is clear that $0 < a < \infty$). Since $g = (f-\lambda)\chi_{[o,a]}$ and $\|g\|_1 = t\|f-g\|_p$, the constant λ has to verify the equation before stated that obviously has only one solution.

It is easy to see that $0 \leq \lambda \leq f(a^-)$. The more delicate part is to prove the inequality $\lambda \geq f(a^+)$, which actually implies $\lambda > 0$ (indeed, $\lambda = 0$ would say that supp f = supp g , $f = g = 0$). Suppose that $0 \leq \lambda \leq f(a^+)$. We may assume $\lambda < f(x)$ for all $x \in [a, a+\epsilon]$. We denote by $I_1 = [0,\epsilon]$ and by $I_2 = [a, a+\epsilon]$ (ϵ is small enough so that $I_1 \subseteq [0,a]$). The equation

$$\|g + \mu\chi_{I_1} + \delta\chi_{I_2}\|_1 = t\|f - (g + \mu\chi_{I_1} + \delta\chi_{I_2})\|_q$$

defines δ as an implicit function of μ , $\delta = \delta(\mu)$, whenever $\mu \in [\mu_0, 0]$ and $\delta(\mu) \in [0,\delta_0]$; besides $\delta \in C^{(1}[\mu_0, 0]$. (A special form of the classical implicit function theorem has to be applied for the case when the conditions are required in the boundary of the domain. The proof may be adapted from the classical one). Now

$$\delta'(0^-) = -\frac{\epsilon\left(1 + t\|f-g\|_q^{1-q}\lambda^{q-1}\right)}{\epsilon + t\|f-g\|_q^{1-q}\int_a^{a+\epsilon} f^{q+1}}$$

The function $\varphi(\mu) = \|g + \mu\chi_{I_1} + \delta\chi_{I_2}\|_1$ verifies $\varphi'(0^-) > 0$. Hence g would not be extremal solution of the functional and thus the lemma is proved. ∎

Consider the function

$$H(a,\lambda) = \int_0^a f - a\lambda - t\left[a\lambda^q + \int_a^\infty f^q\right]^{1/q}$$

for $a > 0$, $0 \le \lambda \le \frac{1}{a}\int_0^a f$. The situation now is similar to that appearing when $r < \infty$. The following facts could be deduced in the same way. Fixed $a > 0$, $H(a,.)$ is a strictly decreasing function. Clearly $H(a, \frac{1}{a}\int_0^a f) < 0$. If a is the length of supp of g, extremal solution, $H(a,0) > 0$ and so, $a > a_0$ where a_0 is the unique positive real such that $\|f\chi_{[0,a_0]}\|_1 = t\|f\chi_{[a_0,\infty]}\|_q$.

In general $a < b$ implies that $H(a, f(a^-)) \le H(a, F(a^+)) \le H(b, f(b^-))$. Hence we define the non empty interval $I = \{x \in (a_0,\infty); H(x, f(x^-)) \le 0 \le H(x, f(x^+))\}$. It is easily checked that $\inf I = \min I > a_0$. Eventually we get the solution as follows

III.4 Theorem. *The case $p = 1$. There exists a unique extremal solution of the functional $K_\infty(t;f)$. The function g is defined by $g = (f-\lambda)\chi_{[0,a]}$ where $a_0 < a = \min I < \infty$ and λ is the solution of $H(a,\lambda) = 0$. Moreover $f(a^+) \le \lambda \le f(a^-)$ and*

$$K_\infty(t;f) = \int_0^a f - a\lambda = t\left[a\lambda^q + \int_a^\infty f^q\right]^{1/q}.$$

Appendix

In this part we introduce the functionals $K_{r,s}$ and $\mathcal{K}_{r,s}$ and we compute them for any function $f \in L^p + L^q$. Next we compare the monotonicity relations associated to these functionals (see definitions below) and eventually, we shall give a new characterization of the $(1,\mathcal{K}_{p,q})$-monotonicity in terms of an interpolation theorem.

Recall that a Banach space X is said to be an *intermediate space* between L^p and L^q if X is continuously embedded between L^p and L^q,i.e. $L^p \cap L^q \hookrightarrow X \hookrightarrow L^p + L^q$. An intermediate space X is said to be an *interpolation space* with respect to (L^p, L^q) if any linear operator $T \in \mathcal{L}(L^p) \cap \mathcal{L}(L^q)$ is also bounded on X ($\mathcal{L}(A)$ denotes the space of bounded linear operators on the Banach space A, see **[2]** and **[3]** for more information).

Next r, s will be two fixed real $1 \le r, s < \infty$ and $t > 0$.

A.1. Definitions. *i) Let f be a non negative non increasing function in $L^p + L^q$*

$$K_{r,s}(t;f;L^p,L^q) = K_{r,s}(t;f) = \inf(\|g\|_p^r + t\|h\|_q^s)$$

where the infimum is defined over all possible decompositions of $f = g + h$, $g \in L^p$ and $h \in L^q$. The functional $\mathcal{K}_{r,s}$ is defined in a similar way but considering only disjointly supported functions

$$\mathcal{K}_{r,s}(t;f;L^p,L^q) = \mathcal{K}_{r,s}(t;f) = \inf_{\substack{f=g+h \\ g\wedge h=0}} (\|g\|_p^r + t\|h\|_q^s).$$

ii) An intermediate space X is $(C, K_{r,s})$-monotone (respec. $(C,\mathcal{K}_{r,s})$-monotone) if given $f \in X$, $g \in L^p+L^q$ such that $K_{r,s}(t;f) \geq K_{r,s}(t;g)$ (respec. $\mathcal{K}_{r,s}(t;f) \geq \mathcal{K}_{r,s}(t;g)$) for all $t > 0$, then $g \in X$ and $\|g\|_X \leq C\|f\|_X$.

If $r = p$, $s = q$ we have the $K_{p,q}$-functional used by Sparr (cf. [11]) which is nothing but the L-functional appearing in [10]. We begin by stating the results corresponding to the $K_{r,s}$-functional and we will not prove the theorems because the proofs are similar to the previous ones.

A function $g \in L^p$ is *an extremal solution* for $K_{r,s}(t;f)$ if $K_{r,s}(t;f) = \|g\|_p^r + t\|f-g\|_q^s$.

We consider first the case $p > 1$. Let $\mathcal{A}$ be the class of the functions g which verify: i) $g \in L^p$, $f - g \in L^q$, ii) For almost every $x \in \operatorname{supp} f, 0 < g < f$ and

$$rg(x)^{p-1}\|f-g\|_q^{q-s} = ts(f(x)-g(x))^{q-1}\|g\|_p^{p-r}.$$

$\mathcal{A}$ is a totally ordered set.

A.2. Theorem.

i) *If $s \geq q$ and $r \geq p$, $\mathcal{A}$ has only one element which is the unique extremal solution of the functional $K_{r,s}(t;f)$*

ii) *If $s \geq r \geq p$, the function $g = \min \mathcal{A}$ is the unique extremal solution of $K_{r,s}(t;f)$*

iii) *In the other cases there exists an extremal solution g of $K_{r,s}(t;f)$, such that $g \in \mathcal{A}$. This extremal solution is unique except when $r = s = 1$, $f = \lambda\chi_{[0,b]}$ and $t = b^{1/p-1/q}$.*

In the three cases

$$\begin{aligned} K_{r,s}(t;f) &= \frac{r}{s}\ \|g\|_p^{r-p} \int g^{p-1}[f - (1-\frac{s}{r})g] \\ &= t\|f-g\|_q^{s-q} \int (f-g)^{q-1}[f-(1-\frac{s}{r})g] \end{aligned}$$

where g is the extremal solution of the functional.

We consider now the case $p = 1$. Let b be the length of support of f ($\leq \infty$). We define the function $F : (0, b) \to \mathbf{R}$ by

$$F(x) = r\left(\int_0^x f - xf(x)\right)^{r-1} - st\left[x + \int_x^b \left(\frac{f}{f(x)}\right)^q\right]^{s/q-1}$$

This function is non decreasing and constant in the intervals where f is constant.

A.3. Theorem.

i) *If $F(0^+) \geq 0$ then $r = 1$, $f \in L^p \cap L^\infty$ and $ts\|f\|_\infty \|f\|_q^{s-q} \leq 1$. Moreover $g = 0$ is the unique extremal solution of the functional and $K_{1,s}(t; f) = t\|f\|_q^s$.*

ii) *If $s = 1$ and $F(x) < 0$ for all $x < b$, then $f \in L^1$, $r\|f\|_1^{r-1} b^{1/q'} \leq t$, ($b < \infty$) and $g = f$ is the unique extremal solution of the functional. $K_{r,1}(t; f) = \|f\|_1^r$*

iii) *If $s = 1$ and $F(x) = 0$ for all $x \in [a, b]$, $a < b$, then $r = 1$, $t = b^{1/q'}$ and $g = f$ is extremal solution. Also the function $g = (f - f(a))\chi_{[0,a]}$ is extremal solution of the functional. Now $K_{1,1}(t; f) = \|f\|_1 = \|g\|_1 + t\|f - g\|_q$.*

iv) *In the other cases there exists only one extremal solution $g = (f - \lambda)\chi_{[0,a]}$, where $a = \sup\{x \in [0, b]; F(x^-) \leq 0\}$ and λ is the unique solution of the equation*

$$r\left(\int_0^a f - a\lambda\right)^{r-1} = st\left[a + \int_a^b \left(\frac{f}{\lambda}\right)^q\right]^{s/q-1}$$

Now

$$K_{r,s}(t; f) = r\left(\int_0^a f - a\lambda\right)^r + t\left[a\lambda^q + \int_a^b f^q\right]^{s/q}$$

Next we are going to compute the $\mathcal{K}_{r,s}$-functional we introduced before. By definition

$$\mathcal{K}_{r,s}(t; f) = \inf_A \|f\chi_A\|_p^r + t\|f\chi_{A^c}\|_q^s,$$

where the infimum runs over all subsets $A \subseteq \operatorname{supp} f$. The main result now is the following

A.4. Theorem. *If $f \in L^p + L^q$ the $\mathcal{K}_{r,s}$ functional has the following expression*

$$\mathcal{K}_{r,s}(t; f) = \min_{0 \leq x \leq \infty} \|f\chi_{[0,x]}\|_p^r + t\|f\chi_{[x,\infty)}\|_q^s.$$

Proof: We shall prove the theorem in three steps. To start with we shall establish the theorem for simple functions.

Step 1. Suppose that $f = \sum_{i=1}^{n} a_i \chi_{I_i}$ where $a_1 > \ldots > a_n > 0$, I_i's are pairwise disjoint intervals with length$(I_i) = m_i > 0$. It is quite clear that $\mathcal{K}_{r,s}(t; f) = \min_{x\in C} \varphi(x)$, being $x = (x_1, \ldots, x_n)$, $C = \prod_{i=1}^{n} [0, m_i]$ and φ the continuous function defined by

$$\varphi(x) = \left(\sum_{i=1}^{n} a_i^p x_i \right)^{r/p} + t \left(\sum_{i=1}^{n} a_i^q (m_i - x_i) \right)^{s/q}.$$

Let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$ a minimum of this function φ in C. If $\overline{x} \in \overset{\circ}{C}$ then $\frac{\partial \varphi}{\partial x_i}(\overline{x}) = 0$, for all $1 \le i \le n$. Easy computations would imply that $a_1 = \ldots = a_n$ which is a contradiction. Thus $\overline{x} \in \partial C$ and, for instance, we may assume that there exists j $(1 \le j \le n)$ such that $\overline{x}_j = \ell_j$ with $\ell_j = 0$ or $= m_j$. Now we consider the function

$$\psi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) = \varphi(x_1, \ldots, x_{j-1}, \ell_j, x_{j+1}, \ldots, x_n)$$

defined for $0 \le x_i \le m_i$, $i \ne j$. This function ψ attains its minimum in $(\overline{x}_1, \ldots, \overline{x}_{j-1}, \overline{x}_{j+1}, \ldots, \overline{x}_n)$. Repeating the argument we would obtain that there exists another coordinate j' such that $\overline{x}_{j'} = \ell_{j'}$ $(\ell_{j'} = 0$ or $m_{j'})$ and so on. Eventually we obtain that there is a natural number k, $1 \le k \le n$, such that $\overline{x} = (\ell_1, \ldots, \ell_{k-1}, \overline{x}_k, \ell_{k+1}, \ldots, \ell_n)$ where each $\ell_i = 0$ or m_i and $\overline{x}_k$ is equal to 0 or m_k or a solution of the equation in x: $\frac{\partial \varphi}{\partial x_k}(\ell_1, \ldots, \ell_{k-1}, x, \ell_{k+1}, \ldots, \ell_n) = 0$.

In order to precise the exact form of $\overline{x}$ we realize that if $\overline{x} = (0, \ldots, 0)$ or $\overline{x} = (m_1, \ldots, m_n)$ the proof of the Step 1 would be finished. So, we may suppose $\overline{x} \ne (0, \ldots, 0)$ and $\overline{x} \ne (m_1, \ldots, m_n)$. Let A be the well defined positive number

$$A = \frac{rq \left[\sum_{i=1}^{n} a_i^p \overline{x}_i \right]^{r/p-1}}{pst \left[\sum_{i=1}^{n} a_i^q (m_i - \overline{x}_i) \right]^{s/q-1}}$$

Let $\mathcal{A}$ be the set $\mathcal{A} = \{i; a_i^{q-p} \ge A\}$. The following lemma is the main tool in the proof of this step.

A.5. Lemma. *The set $\mathcal{A} \ne \emptyset$. If $k_0 = \max \mathcal{A}$ then*

$$\overline{x} = (m_1, \ldots, m_{k_0-1}, \overline{x}_{k_0}, 0, \ldots, 0),$$

where $\overline{x}_{k_0} = m_{k_0}$ *if* $a_{k_0}^{q-p} > A$ *and* $\dfrac{\partial\varphi}{\partial x_{k_0}}(\overline{x}) = 0$ *otherwise.*

Proof: We know that $\overline{x} = (\ell_1, ..., \ell_{k-1}, \overline{x}_k, \ell_{k+1}, ..., \ell_n)$ for some $k, 1 \le k \le n$. If $j \neq k$ let h be the real function defined by

$$h(y) = \varphi(\ell_1, ..., \overset{j)}{y}, ..., \overline{x}_k, ..., \ell_n).$$

As $\ell_j = 0$ (respectively $\ell_j = m_j$) implies that $h'(0^+) \ge 0$ (respec. $h'(m_j^-) \le 0$) it is easy to check that $\ell_j = 0$ (respec. $= m_j$) implies that $a_j^{q-p} \le A$ (respec. $a_j^{q-p} \ge A$). In the same way , if $\overline{x}_k = 0$ then $a_k^{q-p} \le A$, if $\overline{x}_k = m_k$ then $a_k^{q-p} \ge A$ and otherwise $a_k^{q-p} = A$.

As $\overline{x} \neq (0, ..., 0)$ the set $\mathcal{A} \neq \emptyset$. Hence if $j < k_0$ (respec. $j > k_0$) the corresponding j-th coordinate has to be m_j (respec. 0). Finally we only have to consider the different values of a_{k_0} with respect to A and this concludes the proof of the lemma. ■

By applying the lemma we have that $\mathcal{K}_{r,s}(t; f) = t\|f\|_q^s$ or $\mathcal{K}_{r,s}(t; f) = \|f\|_p^r$ or

$$\begin{aligned}\mathcal{K}_{r,s}(t; f) = \varphi(\overline{x}) &= \left(\sum_{i=1}^{k_0} a_i^p m_i + a_{k_0}^p \overline{x}_{k_0}\right)^{r/p} \\ &+ t\left(a_{k_0}^q (m_{k_0} - \overline{x}_{k_0}) + \sum_{k_0+1}^{n} a_i^q m_i\right)^{s/q} \\ &= \|f\chi_{[0,x]}\|_p^r + t\|f\chi_{[x,\infty)}\|_q^s\end{aligned}$$

where $x = m_1 + ... + m_{k_0-1} + \overline{x}_{k_0}$. In the three cases the Step 1 is proved.

Step 2. Suppose now that $f \in L^\infty$ and the length of the support of f is finite. We can approximate f by a non decreasing sequence of simple functions $(f_n)_n$ converging to f in the L^∞-norm. We apply the Step 1 to these simple functions f_n and we get that

$$\mathcal{K}_{r,s}(t; f_n) = \|f\chi_{[0,x_n]}\|_p^r + t\|f\chi_{[x_n,\infty)}\|_q^s$$

where $x_n \in \text{supp} f_n$. Since $\mathcal{K}_{r,s}(t, f_n) \le \mathcal{K}_{r,s}(t; f)$ by passing to a subsequence if necessary there exists $x = \lim x_n$ and thus we have

$$\lim \mathcal{K}_{r,s}(t; f_n) = \|f\chi_{[0,x]}\|_p^r + t\|f\chi_{[x,\infty)}\|_q^s \ge \mathcal{K}_{r,s}(t; f).$$

Hence

$$\mathcal{K}_{r,s}(t; f) = \|f\chi_{[0,x]}\|_p^r + t\|f\chi_{[x,\infty)}\|_q^s$$

for some $x \in$ supp f.

Step 3. If f is a general non increasing and non negative function in $L^p + L^q$ we approximate f by a sequence of truncations of f and we apply the ideas of the Step 2. ■

In order to determinate the point $x \in \operatorname{supp} f$ for which

$$\mathcal{K}_{r,s}(t;f) = \|f\chi_{[0,x]}\|_p^r + t\|f\chi_{[x,\infty)}\|_q^s$$

we consider the auxiliary function φ defined by

$$\varphi(x) = \left(\int_0^x f^p\right)^{r/p} + t\left(\int_x^\infty f^q\right)^{s/q}$$

for $0 < x < b =$ length of the support of f. This function φ is continuous and has derivative $\varphi'(x)$ almost everywhere. More precisely, there exist $\varphi'(x^+)$ and $\varphi'(x^-)$ and they are equal except at most in the discontinuity points of f. It is easy to check that

$$\varphi'(x^\pm) = \frac{ts}{q}\|f\chi_{[x,\infty)}\|_q^{s-q} f(x^\pm)^p[\Phi(x) - f(x^\pm)^{q-p}]$$

$0 < x < b$, where

$$\Phi(x) = \frac{rq}{pst}\frac{\|f\chi_{[0,x]}\|_p^{r-p}}{\|f\chi_{[x,\infty)}\|_q^{s-q}}$$

($\varphi'(x^\pm)$ denotes each one of the two hand side derivatives of φ, in the same way $f(x^\pm)$ represents the two lateral limits of f). Hence we have that if x_0 is a point of minimum for the function φ necessarily $x_0 = 0$, or $= b$ or $f(x_0^-)^{q-p} \geq \Phi(x_0) \geq f(x_0^+)^{q-p}$. In particular we have the following

A.6. Proposition.

i) *If either $r \geq p$ and $s > q$ or $r > p$ and $s \geq q$ there exists only one point $x_0 \in \operatorname{supp} f$ for which the minimum of φ is attained. Furthermore $x_0 = \sup\{x; \Phi(x) \leq f(x)^{q-p}\}$ and*

$$\mathcal{K}_{r,s}(t;f) = t\|f\chi_{[x_0,\infty)}\|_q^{s-q}\left[\frac{ps}{rq}\|f\chi_{[0,x_0]}\|_p^p\Phi(x) + \|f\chi_{[x_0,\infty)}\|_q^q\right]$$

ii) *If $r = p$ and $s = q$*

$$\mathcal{K}_{r,s}(t;f) = \int_0^\infty \min\{f(x)^p, tf(x)^q\}dx.$$

Proof: i) Under these hypotheses Φ is a strictely increasing function. Let

$$I = \{x \in (0,b); \Phi(x) \leq f(x)^{q-p}\}.$$

$I = \emptyset$ means that $f \in L^q \cap L^\infty$, $r = p$ and $\|f\|_\infty^{q-p} \leq \dfrac{q}{st}\|f\|_q^{q-s}$. Then $\varphi'(x^\pm) \geq 0$ for all x and then $\mathcal{K}_{r,s}(t;f) = t\|f\|_q^s$.

If $I = (0, b)$ then $b < \infty$ and $\varphi'(x^{\pm}) \leq 0$ for all $0 < x < b$. So $\mathcal{K}_{r,s}(t; f) = ||f||_p^r$.

Otherwise $\varphi'(x^{\pm}) \leq 0$ for all $x \in I$ and $\varphi'(x^{\pm}) \geq 0$ if $x \notin I$. Hence i) holds.

ii) Now $\Phi(x) = \frac{1}{t}$ and this part of the proposition is easily checked. ■

Remark. We note that our $\mathcal{K}_{p,q}$ functional is exactly the corresponding $\mathcal{K}_{p,q}$ functional used by Sparr in [11, definition (3.1)]

Later on we shall prove that for $r, s \geq 1$ all the $K_{r,s}$-monotonicities are equivalent in any intermediate space between L^p and L^q and that a weaker result is true for the correspondig $\mathcal{K}_{r,s}$-monotonicities.

A.7. Proposition. *Let X be an intermediate space between L^p and L^q and $r, s \geq 1$. The following assertions are true: i) X is $(1, K_{r,s})$-monotone $\Leftrightarrow$ X is $(1, K)$-monotone. ii) If $r \geq p$ and $s \geq q$, X is $(1, \mathcal{K}_{r,s})$-monotone $\Leftrightarrow$ X is $(1, \mathcal{K}_{p,q})$-monotone.*

Proof: i) Set E the functional defined for $f \in L^p + L^q$ and $\lambda > 0$ by $E(\lambda; f) = \inf_{||g||_p \leq \lambda} ||f - g||_q$. It is clear that

$$K_{r,s}(t; f) = \inf_{\lambda} \left[\lambda + tE(\lambda^{1/r}; f)^s\right] \qquad (A.7.1)$$

and

$$E(\lambda; f) = \sup_t \left[\frac{K_{r,s}(t; f)}{t} - \frac{\lambda^r}{t}\right]^{1/s} \qquad (A.7.2)$$

(see [11], lemma 3.3). Then $K_{r,s}(t; f) \geq K_{r,s}(t; g)$ for all $t > 0$ if and only if $E(\lambda; f) \geq E(\lambda; g)$ for all $\lambda > 0$ and hence if and only if $K_{1,1}(t; f) \geq K_{1,1}(t; g)$ for all $t > 0$.

ii) The proof of this part is similar to the previous one by using a suitable modification of the functional E, namely $\mathcal{E}$, defined by

$$\mathcal{E}(\lambda; f) = \inf_{||f\chi_{[0,x]}||_p \leq \lambda} ||f\chi_{[x,\infty)}||_q$$

Now the corresponding similar expressions (A.7.1) and (A.7.2) for the functionals $\mathcal{K}_{r,s}$ and $\mathcal{E}$ occur. Actually, the only thing we have to compute is that given $\lambda > 0$ with $\mathcal{E}(\lambda; f) > 0$ there exists $t > 0$ such that $\mathcal{K}_{r,s}(t; f) = \lambda^r + t\mathcal{E}(\lambda; f)^s$. It is clear that there exists only one point y, $0 < y <$ length supp f, such that $||f\chi_{[0,y]}||_p = \lambda$ and $||f\chi_{[y,\infty)}||_q = \mathcal{E}(\lambda; f)$. Since $r \geq p$ and $s \geq q$ we take $t = \dfrac{rq\lambda^{r-p}}{ps\mathcal{E}(\lambda; f)^{s-q}} f(y)^{p-q}$ and we apply Proposition A.6. ■

Remark. For $r, s \geq 1$ it is clear that if X is $(1, \mathcal{K}_{r,s})$-monotone $\Rightarrow$ X is $(1, \mathcal{K}_{p,q})$-monotone.

Now we recall that a *lattice homomorphism* is a linear bounded operator between Banach lattices which maps disjointly supported functions into disjointly supported functions.

A.8. Theorem. *Let X be an intermediate space between L^p and L^q. X is $(1,\mathcal{K}_{p,q})$-monotone if and only if the following interpolation result is true: If $T \in \mathcal{L}(L^p) \cap \mathcal{L}(L^q)$ is a lattice homomorphism then $T \in \mathcal{L}(X)$ and*

$$\|T\|_{X\to X} \leq \max\{\|T\|_{L^p\to L^p}, \|T\|_{L^q\to L^q}\}.$$

Proof: First of all we remark that both assertions imply that X is a rearrangement invariant function space. Suppose that X is $(1,\mathcal{K}_{p,q})$-monotone and that T is a lattice homomorphism such that

$$\max\{\|T\|_{L^p\to L^p}, \|T\|_{L^q\to L^q}\} \leq 1.$$

If $f \in X$, since T is a lattice homomorphism we have

$$\begin{aligned}\mathcal{K}_{p,q}(t;Tf) &= \inf_{\substack{g+h=Tf\\ g\wedge h=0}} \|g\|_p^p + t\|h\|_q^q \leq \inf_{\substack{g+h=f\\ g\wedge h=0}} \|g\|_p^p + t\|h\|_q^q \\ &= \mathcal{K}_{p,q}(t;f)\end{aligned}$$

for all $t > 0$. Thus $Tf \in X$ and $\|Tf\|_X \leq \|f\|_X$.

On the converse hand, let $f \in X$ and $g \in L^p + L^q$ such that for all $t > 0$ $\mathcal{K}_{p,q}(t;f) \geq \mathcal{K}_{p,q}(t;g)$. Applying the lemma 4.2 of **[11]** we obtain that for each $\epsilon > 0$ there is a lattice homomorphism $T_\epsilon \in \mathcal{L}(L^p) \cap \mathcal{L}(L^q)$ which verifies $T_\epsilon(f) = g$. Then by using the hypotheses we have $g \in X$ and

$$\|g\|_X \leq \|T\|_{X\to X}\|f\|_X \leq (1+\epsilon)\|f\|_X$$

Since this is true for all $\epsilon > 0$ thus the proof of the theorem is complete. ■

As a consequence of this result we can establish the following characterization of interpolation spaces with respect to the couple (L^p, L^q); this corollary is in essence an interpolation result

A.9. Corollary. *Let X be an intermediate space between L^p and L^q. The following statements are equivalent: i) X is an interpolation space, ii) X is an interpolation space for lattice homomorphisms.*

Proof: We only have to show ii) $\Rightarrow$ i). We define a new equivalent norm $|||.|||$ on X in the following way

$$|||f||| = \sup \|Tf\|$$

where the infimum runs over all possible lattice homomorphisms $T \in \mathcal{L}(L^p) \cap \mathcal{L}(L^q)$ such that $\max\{\|T\|_{L^p\to L^p}, \|T\|_{L^q\to L^q}\} \leq 1$. It is clear that $(X, |||\cdot|||)$ is an exact interpolation space for lattice homomorphisms. Thus $(X, |||.|||)$ have to be $(1,\mathcal{K}_{p,q})$-monotone. Hence the result follows by applying the theorem 5.2 of **[11]**. ■

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