Publicacions Matemàtiques, Vol 35 (1991), 323–332.

CONTINUITY OF THE VISIBILITY FUNCTION

ANA FORTE CUNTO

Abstract

G. Beer defined the visibility function of a set S and proved its continuity in the interior of S. It is proved here that the visibility function of a planar Jordan domain S is continuous precisely at the cone points of the boundary of S.

1. Notations and basic definitions

Unless otherwise stated, all the points considered here are included in the Euclidean plane E_2. The complement, interior, closure, boundary and convex hull of a set S are denoted by $\complement S$, $\operatorname{int} S$, $\operatorname{cl} S$, $\operatorname{bd} S$ and $\operatorname{conv} S$, respectively. The open segment joining x and y is denoted $(x\ y)$. The substitution of one or both parentheses by square ones indicates the adjunction of the corresponding endpoints. The ray issuing from x and going through y is denoted $R(x \to y)$, while $R(yx \to)$ is the ray issuing from x and going in the opposite direction to that of $R(x \to y)$. Rays are always closed. We say that x *sees* y *via* S if $[x\ y] \subset S$. The *star of* x *in* S is the set $\operatorname{st}(x, S)$ of all the points of S that see x via S. A *star-center* of S is a point $x \in S$ such that $\operatorname{st}(x, S) = S$. The *convex kernel* of S is the set $\ker S$ of all the star-centers of S. S is *starshaped* if $\ker S \neq \phi$. A *Jordan domain* is a compact connected set of E_2 whose boundary is homeomorphic to the unit circle. The open and closed disks of center x and radius δ will be denoted $U(x; \delta)$ and $B(x; \delta)$, respectively.

If $y \in \operatorname{bd} S$ and $x \in \operatorname{st}(y, S)$ we say that the ray $R(x \to y)$ is *inward through* y if there exists $t \in R(xy \to)$ such that $(y\ t) \subset \operatorname{int} S$. Otherwise we say that $R(x \to y)$ is *outward through* y. The *inner stem* of y with respect to S is the set $\operatorname{ins}(y, S)$ formed by y and all the points of $\operatorname{st}(y, S)$ that issue outward rays through y.

A point $x \in S$ is a *point of local convexity* if there exists $\varepsilon > 0$ such that $S \cap B(x, \varepsilon)$ is convex. Otherwise, x is a *point of local nonconvexity*. We remark that the distinction is significant only for boundary points, since every interior point is trivially of local convexity. The set of all points of local convexity of S and that of all points of local nonconvexity are denoted $\operatorname{lc} S$ and $\operatorname{lnc} S$, respectively. It is easy to see that $\operatorname{lc} S$ is open and $\operatorname{lnc} S$ is closed in the relative topology of $\operatorname{bd} S$. An *obstruction zone* is a connected component of $\operatorname{lnc} S$.

A point $x \in \text{bd } S$ is a *flat point* if $x \in \text{lc } S \cap \text{lc } CS$. The set of all such points is denoted flp S. $x \in \text{bd } S$ is an *inflection point* (and the set of all inflection points is denoted ifp S) if either $x \in \text{lnc } S \cap \text{lnc } CS$ or $x \in \text{lnc } S \cap \text{lc } CS \cap \text{cl(flp } S)$. An arc $\Gamma \subset \text{bd } S$ *keeps the sense of curvature* if either $\Gamma \subset \text{lc } S$ or $\Gamma \subset \text{lnc } S$.

If S is a closed set with nonempty interior and $x \in S$, then the *set of critical visibility of x in S* is the set

$$\text{cv}(x, S) = \text{int } S \cap \text{bd st}(x, S).$$

Each point of this set is a *point of critical visibility of x in S*. The point $x \in S$ is *clearly visible from y via S* if there exists $\varepsilon > 0$ such that $B(x, \epsilon) \cap S \subset \text{st}(y, S)$.

2. Statement of the problem

In [**1**] G. Beer defines the *visibility function* as the one that assigns to each point x of a fixed measurable set S in the Euclidean space $\mathbb{E}_n$, the Lebesgue outer measure of $\text{st}(x, S)$. We shall denote it $v(x)$.

In [**1**], [**2**] and [**3**] several theorems about the continuity of $v(x)$ in open sets, or in the interior of the sets considered, are demonstrated. The purpose of the present paper is to study the behavior of the visibility function in the boundary of a Jordan domain S. The study is restricted to this case in order to avoid difficulties as those presented in the examples of [**2**] and [**3**]. In this case, the theorems shown in [**1**] assure the continuity of $v(x)$ in int S.

Furthermore, the boundary curve must have finitely many inflection points in the smooth case, and finitely many angular points in the nonsmooth case. This will prevent the existence of *singular points* (i.e. points of accumulation of inflection points or angular points). The study of the star of a singular point seems almost unmanageable to this author. A Jordan domain without singular points will be a *regular Jordan domain*.

We make a local study of the star's measure in a point $x \in \text{bd } S$ using the *domain of good behavior* N, that is a neighborhood of x having the following characteristics:

i) its center will be x;
ii) N includes neither inflection points nor angular points of bd S except possibly x itself.

Clearly ii) assures that each of the two subarcs Γ_1 and Γ_2 of $N \cap \text{bd } S$, having x as one endpoint, keeps the sense of curvature. We generalize this local results using the fact that the stars are fans spanned by S. The definition of this concept is given below.

3. Auxiliary results

Lemma 3.1. *Let S be a closed set of the plane, $\{x; y; z\} \subset S$ such that $[x\, y] \cup [y\, z] \subset S$. Let $T = \text{conv}\{x; y; z\}$ have at most one point $w \in \text{lnc}\, S$ such that $w \in (x\, z)$. Then $T \subset S$.*

Proof: A slight variation in the proof of Corollary 2 of [6] yields this lemma that is, in the same spirit of Valentine's result, an useful consequence of Tietze's theorem on local convexity. ■

Lemma 3.2. *Let S be a regular Jordan domain and $x_0 \in$ bd S. There exists a domain of good behavior $N = B(x_0, \delta)$. Furthermore, $A = [B(x_0, \delta) \sim \{x_0\}] \bigcap$ bd S consists of two connected arcs ending at x_0, such that each of them keeps the sense of curvature.*

Proof: Define

$$K = \{z \in \text{bd}\, S | z \text{ is an inflection point or an angular point}\}$$
$$\delta = d(x_0, K \sim \{x_0\})$$

From the inexistence of singular points it follows that $\delta > 0$. Let Γ be the connected component of $[\text{bd}\, S \cap B(x_0, \delta/2)]$ that includes x_0. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ where each of these subarcs ends at x_0 and keeps the sense of curvature. Define $B_n = B(x_0, \delta/2n)$ and Γ_{1n}, Γ_{2n} as the connected components of $\Gamma_1 \cap B_n$ and $\Gamma_2 \cap B_n$, respectively, that include x_0. Let $\Delta_n = \Gamma_n \sim [\Gamma_{1n} \cup \Gamma_{2n}]$.

Owing to the simplicity of bd S, there exists a positive integer m such that $B_m \cap \Delta_m = \phi$. The ball $N = B_m$ satisfies the thesis. ■

In the sequel, the domain of good behavior with respect to x_0 will be denoted by N. Let $x_0 \in$ bd S and L_α be a line through x_0. The *maximal segment* determined by L_α in S is the connected component I_α of $L_\alpha \cap S$ that includes x_0. The union of all those maximal segments is the *fan in x_0 spanned by S*. The *angular amplitude* of a fan $S(aa\, S)$ will be the normalized Lebesgue measure (in bd N of the radial projection of S from x_0 over bd N. S will be an *angular connected fan* if that projection is connected in the relative topology of bd N.

Lemma 3.3. $\text{ins}(x_0, S)$ *and* $\text{st}(x_0, S)$ *are fans in* x_0.

Proof: Both sets are starshaped and x_0 is a star-center for each of them. ■

Lemma 3.4. $I = N \cap \text{ins}(x_0, S)$ *is an angular connected fan.*

Proof: We consider two alternatives:

(i) Let $u \in I$, $v \in I$ and x_0 be not collinear with these points. There exist $u' \in R(ux_0 \rightarrow) \cap N$ and $v' \in R(vx_0 \rightarrow) \cap N$ such that $(x_0\, u'] \cap \text{int}\, S = \phi$

and $(x_0\ v'] \cap \operatorname{int} S = \phi$. Assume that $(x_0\ u')$ and $(x_0\ v')$ are both included in $N \cap \complement S$. Define

$$\beta = \min\{d(x_0,u);\ d(x_0,v);\ d(x_0,u');\ d(x_0,v')\}$$

$$\begin{array}{ll} B' = B(x_0,\beta/2) & ,\quad B'_1 = \operatorname{bd} B', \\ u_1 \in [x_0\ u] \cap B'_1 & ,\quad v_1 \in [x_0\ v] \cap B'_1, \\ u'_1 \in [x_0\ u'] \cap B'_1 & ,\quad v'_1 \in [x_0\ v'] \cap B'_1. \end{array}$$

Clearly $(x_0\ u'_1] \subset \complement S$, and $(x_0\ v'_1] \subset \complement S$. Furthermore, condition (i) implies that $v_1 \notin L(x_0\ u'_1)$, hence $[v_1\ u'_1] \cap \operatorname{bd} S \neq \phi$. An analogous argument shows that $[u_1\ v'_1] \cap \operatorname{bd} S \neq \phi$. Let $q_1 \in [v_1\ u'_1] \cap \operatorname{bd} S$ and $p_1 \in [u_1\ v'_1] \cap \operatorname{bd} S$. Hence $p_1 \neq x_0$ and $q_1 \neq x_0$ by condition (i), and bd S cannot cross neither $(x_0\ u'_1)$ nor $(x_0\ v'_1)$ since these segments are entirely included in $\complement S$. A similar argument shows that bd S cannot cross neither $(x_0\ u_1)$ nor $(x_0\ v_1)$. Since N is the domain of good behaviour of x_0, only two subarcs of bd S (call them Γ_1 and Γ_2), both having x_0 as one extreme, are included in N. Define two circular sectors of B':

$$S_1 = (v_1\ x_0\ u'_1) \text{ and } S_2 = (u_1\ x_0\ v'_1)$$

such that $q_1 \in S_1$ and $p_1 \in S_2$. It follows that

(1) $$\Gamma_1 \subset S_1,\ \Gamma_2 \subset S_2$$

and

(2) $$\operatorname{int}\ (\operatorname{conv}\ \{u_1;\ v_1;\ x_0\}) \cap \operatorname{bd} S = \phi$$

(3) $$\operatorname{int}\ (\operatorname{conv}\ \{u'_1;\ v'_1;\ x_0\}) \cap \operatorname{bd} S = \phi$$

If $z \in [u_1\ v_1]$, it follows from (2) and Lemma 3.1 that $[z\ x_0] \subset S$ and $z \in \operatorname{st}(x_0,S)$. Define now $z' \in (u'_1\ v'_1) \cap R(zx_0 \rightarrow)$. Using (3) and Lemma 3.1 we obtain that $(x_0\ z') \subset \complement S$, and $z \in \operatorname{ins}(x_0,S)$. A very similar argument holds when one or both of the segments $[x_0\ u'_1]$, $[x_0\ v'_1]$ is included in $\operatorname{bd} S$. We conclude that I is convex, whence its radial projection from x_0 onto $\operatorname{bd} N$ must be connected in the relative topology of $\operatorname{bd} N$.

(ii) Let $u \in I$, $v \in I$, and $x_0 \in (u\ v)$.

Since $R(ux_0 \rightarrow)$ is outward, $x_0 \in (u\ v)$ and $[x_0\ v] \subset S$, it follows that $[x_0\ v] \subset \operatorname{bd} S$. The same argument proves that $[x_0\ u] \subset \operatorname{bd} S$, and as S is a Jordan domain, $I \cap B'$ must be a half-circle. Clearly, its projection from x_0 onto $\operatorname{bd} N$ must be connected. ■

Lemma 3.5. *$J = N \cap \operatorname{st}(x_0,S)$ is an angular connected fan.*

Proof: We consider three alternatives:

(1) Let $u \in J$, $v \in J$ and $u \in N \cap \operatorname{st}(v,S)$, while $x_0 \notin [u\ v]$. Let $A = \operatorname{conv}(\{u;\ v;\ x_0\})$. Then $\operatorname{int} A \cap \operatorname{bd} S = \phi$ since any crossing of $\operatorname{bd} S$ over

$[x_0\ u]$, $[x_0\ v]$ or $[u\ v]$ would ruin the conditions of visibility. If $z \in [u\ v]$, by Lemma 3.1 it follows that $z \in J$. Hence, J results convex, and its radial projection from x_0 over $\operatorname{bd} N$ must be connected.

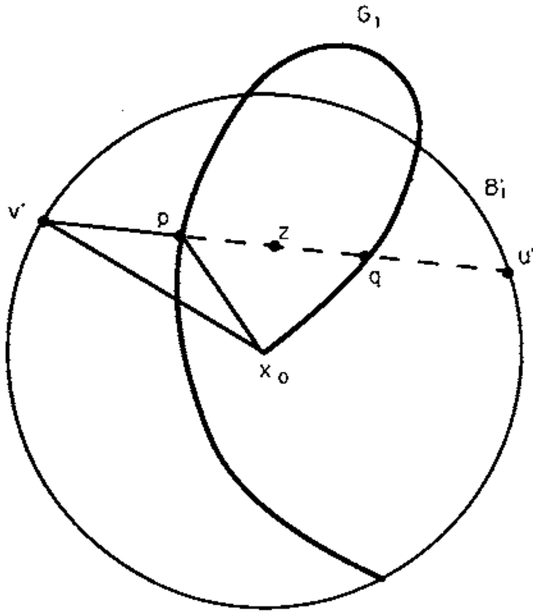
(2) Let $u \in J$, $v \in J$, $u \notin N \cap \operatorname{st}(v, S)$, while $x_0 \notin [u\ v]$. Define

$$\delta = \min\{d(u, x_0); d(v, x_0)\}, B' = B(x_0, \delta/2), B'_1 = \operatorname{bd} B',$$
$$u' \in [x_0\ u] \cap B'_1, v' \in [x_0\ v] \cap B'_1.$$

Clearly $u' \in J$, $v' \in J$, and we may assume that $u' \notin N \cap \operatorname{st}(v', S)$, since otherwise we would be in the situation of part (1). Let

$$z \in (u'\ v') \cap \complement S, p \in (v'\ z) \cap \operatorname{bd} S \cap N \cap \operatorname{st}(v', S),$$
$$q \in (u'\ z) \cap \operatorname{bd} S \cap N \cap \operatorname{st}(u', S).$$

The visibility conditions assure that $p \neq x_0$, $q \neq x_0$ and $p \neq q$. We intend to prove that p and q belong to different boundary arcs separated by x_0 in $\operatorname{bd} S \cap N$. It may happens that $p = v'$ or $q = u'$, but in this case one of the boundary arcs would be a segment, and since $x_0 \notin [p\ q]$, these points must belong to different arcs. Let us assume that $p \neq v'$ and $q \neq u'$, and suppose that p and q belong to the same boundary arc Γ_1. Without loss of generality assume



that $q \in \operatorname{arc}(x_0, p) \subset \Gamma_1$. By the definition of p, it is not the last point of Γ_1 in B'. Let $A = \operatorname{conv}(\{v'; p; x_0\})$. We observe the position of Γ_1 with respect to A. If $\Gamma_1 \cap \operatorname{int} A = \phi$, either a double point or a forbidden change in the sense of curvature would appear in $\operatorname{bd} S \cap N$. If $\Gamma_1 \cap \operatorname{int} A \neq \phi$, it would imply a double point, a contradiction of the visibility conditions or a forbidden change of curvature. Hence, p and q must belong to different boundary arcs. Let $S_1 = (v'\, x_0\, u')$ the circular sector of B' that includes z; $S_2 = B' \cap \complement S_1$. Then $\operatorname{bd} S \cap B' \subset S_1$. Define

$$z' \in R(zx_0 \to) \cap B'_1.$$

Then $z' \in J$, and $z' \in \operatorname{int} S_2$. Using part (1) we obtain that

$$[u'\, z'] \cup [z'\, v'] \subset J,$$

and the radial projection of this union over $\operatorname{bd} N$ is connected.

(3) Let $u \in J$, $v \in J$ but $x_0 \in [u\, v]$.

Let B', B'_1, u' and v' be as in (2). Let $z \in \complement S \cap B'$, and take

$$p \in [u'\, z] \cap \operatorname{st}(u', S) \cap \operatorname{bd} S$$
$$q \in [v'\, z] \cap \operatorname{st}(v', S) \cap \operatorname{bd} S$$

Since z, u and v are not collinear, it follows that $p \neq q$. Using that $z' = R(zx_0 \to) \cap B'_1$, and the same arguments as in (2) we obtain

$$[u'\, z'] \cup [z'\, v'] \subset J. \quad \blacksquare$$

Lemma 3.6. *Let $\{x_n [n \in \mathbb{N}\}$ be a sequence of points in S such that* $\lim x_n = x_0$. *Then*

$$\operatorname{ins}(x_0, S) \subset \left[\bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \operatorname{st}(x_j, S)\right] \cup \{x_0\} \cup Q = \left[\lim_{x_n \to x_0} \operatorname{st}(x_n, S)\right] \cup \{x_0\} \cup Q,$$

where Q is included in the union of a finite number of maximal segments with respect to x_0 and has null measure.

Proof: Let $p \in \operatorname{ins}(x_0, S)$, $p \neq x_0$. We consider two alternatives:

a) If x_0 is clearly visible from p, there exists a neighborhood $\mathcal{U}(x_0)$ such that $\mathcal{U}(x_0) \cap S \subset \operatorname{st}(p, S)$. Then if $x_n \to x_0$, $x_n \in \mathcal{U}(x_0) \cap S\, \forall n > n_0$ and $x_n \in \operatorname{st}(p, S)\, \forall n > n_0$. Hence

$$p \in \bigcap_{j=n_0}^{\infty} \operatorname{st}(x_j, S) \subset \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \operatorname{st}(x_j, S).$$

b) If x_0 is not clearly visible from p, it follows from Theorem 2.1 of [**5**] that $(p\, x_0) \cap \operatorname{lnc} S \neq \phi$. Let $z \in (p\, x_0) \cap \operatorname{lnc} S$. If z is a smooth point of $\operatorname{bd} S$, there exists an obstruction zone $\Gamma \subset \operatorname{lnc} S$ that includes z. Then $\operatorname{conv} \Gamma$ is supported by, at most, two rays issuing from x_0. Since there are finitely many obstruction zones, and for each of them there are at most two maximal segments of critical visibility, the family of such segments is finite. If z is not a smooth point of $\operatorname{bd} S$, a similar argument, based in the definition of regular Jordan domain, assures that the family of segments of critical visibility is finite. $\blacksquare$

Lemma 3.7. *Let F be an angular connected fan in x_0 and $m(F)$ its planar Lebesgue measure. Then $m(F) > 0$ if and only if $aa(F) > 0$.*

Proof: Taking x_0 as the origin, the area of F can be easily computed by a positive radial function $r(\theta)$ that depends on the argument θ, whose range of variation is the projection of F from x_0 over $\operatorname{bd} N$. Let m and n be the endpoints of this projection. If m, n, and x_0 are not collinear, then let $A = \operatorname{conv}(\{m; x_0; n\})$, and α be the angle formed by $R(x_0 \to n)$ and $R(x_0 \to m)$ exterior to A. Then, it is clear that $aa(F) = 2\pi - \alpha$. Hence,

$$m(F) = \int_{\alpha}^{2\pi} \int_{0}^{r(\theta)} r \, dr \, d\theta \tag{1}$$

Both implications of the thesis follow readily. Otherwise, if m, n and x_0 are in the same line L, take $t \in F \sim L$, and $A = \operatorname{conv}(\{m, x_0, n, t\})$. A very similar argument holds and (1) is valid with $\alpha = \pi$. ■

Lemma 3.8. *Let M and L be two connected fans with vertex x_0, $M \subset L$. Then $m(M) < m(L)$ if and only if $aa(M) < aa(L)$.*

Proof: The set $F = (L \sim M) \cup \{x_0\}$ is a fan in x_0 whose projection from x_0 over $\operatorname{bd} N$ is not necessarily connected, but has at most two connected components. Hence $F = F_1 \cup F_2$ and $L = M \cup F_1 \cup F_2$. Both implications of the thesis can be obtained from equality (1) of Lemma 3.7 and the additivity of Lebesgue measure. ■

Lemma 3.9. $m(\operatorname{ins}(x, S) \cap N) < m(\operatorname{st}(x, S) \cap N)$ *implies that* $m(\operatorname{ins}(x, S)) < m(\operatorname{st}(x, S))$.

Proof: Let us assume the existence of a system of polar coordinates centered at x and similar to the one described in Lemma 3.7. If α and β are the angular coordinates of the endpoints of the radial projection of $\operatorname{ins}(x, S)$ over $\operatorname{bd} N$, and α', β' are the corresponding coordinates for $\operatorname{st}(x, S)$, then $aa(\operatorname{ins}(x, S)) = \beta - \alpha$, $aa(\operatorname{st}(x, S)) = \beta' - \alpha'$. Let ε be the radius of N. From Lemma 3.8 it follows that $\alpha' \leq \alpha \leq \beta \leq \beta'$. Using the notation of Lemma 3.7 it results that

$$m(\operatorname{st}(x, S)) = m(\operatorname{st}(x, S) \cap N) + \int_{\alpha'}^{\beta'} \int_{\varepsilon}^{r(\theta)} r \, dr \, d\theta$$

and

$$m(\operatorname{ins}(x, S)) = m(\operatorname{ins}(x, S) \cap N) + \int_{\alpha}^{\beta} \int_{\varepsilon}^{r(\theta)} r \, dr \, d\theta$$

and the strict inequality of the thesis follows readily. ■

4. The main theorem and its corollaries

Theorem 4.1. *Let S be a regular Jordan domain and $x_0 \in \operatorname{bd} S$. Then, the following statements are equivalent:*
(i) $m(\operatorname{st}(x_0, S)) = m(\operatorname{ins}(x_0, S))$
(ii) v is continuous in x_0.

Proof: (i) $\Rightarrow$ (ii). Assume that v is discontinuous in x_0. Owing to the upper semicontinuity of v (Beer, [1], there must exist a sequence $\{x_n\}$ in S such that $x_n \to x_0$ but $\lim_{n\to\infty} v(x_n) < v(x_0)$. Hence,

$$\lim_{n\to\infty} m(\operatorname{st}(x_n, S)) < m(\operatorname{st}(x_0, S)) \tag{1}$$

From Lemma 3.6 it follows that

$$\operatorname{ins}(x_0, S) \subset \left[\bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \operatorname{st}(x_j, S)\right] \bigcup \{x_0\} \bigcup Q \tag{2}$$

where Q has null measure. From (2) it follows that

$$m(\operatorname{ins}(x_0, S)) \leq m \left[\bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} \operatorname{st}(x_j, S)\right] + m(\{x_0\}) + m(Q) =$$
$$\lim_{n\to\infty} m \left[\bigcap_{j=n}^{\infty} \operatorname{st}(x_j, S)\right] \leq \lim_{n\to\infty} m(\operatorname{st}(x_n, S)).$$

From this inequality and (1) we obtain a contradiction of (i).

(ii) $\Rightarrow$ (i). Assume that $m(\operatorname{st}(x_0, S)) \neq m(\operatorname{ins}(x_0, S))$. It is clear that $m(\operatorname{st}(x_0, S)) - m(\operatorname{ins}(x_0, S)) > 0$, and that

$$D = [\operatorname{st}(x_0, S) \sim \operatorname{ins}(x_0, S)]$$

is a fan at x_0 having positive measure. From Lemma 3.8 it follows that $aa(\operatorname{st}(x_0, S)) > aa(\operatorname{ins}(x_0, S))$, whereas from Lemmas 3.3 and 3.4 we know that both $N \cap \operatorname{ins}(x_0, S)$ and $N \cap \operatorname{st}(x_0, S)$ are angularly connected. Then, the projection of D from x_0 over N_1 has at most two connected components, and at least one of them with positive measure. Hence, it is clear that $m(\operatorname{st}(x_0, S) \cap N) > m(\operatorname{ins}(x_0, S) \cap N)$, and that there exists a set $A \subset D \cap N$ such that $\operatorname{int} A \neq \phi$. Select $t \in \operatorname{int} A$ and $w \in \operatorname{ins}(x_0, S)$, and call $L(w, x_0)$ the line through w and x_0 and H^+, H^- the two open semiplanes in which this line divides $\mathbb{E}_2$. Since $t \notin L(w, x_0)$ we can assume that $t \in H^+$. There exists $\varepsilon > 0$ such that $B(t, \varepsilon) \subset A \subset D \cap N$. Since $R(t \to x_0)$ is inward, there exist a point $t' \in R(tx_0 \to) \cap H^- \cap S \cap N$ with $(t'x_0) \subset \operatorname{int} S$. Clearly $t' \notin \operatorname{ins}(x_0, S)$ and

t is not clearly visible form t'. Let $t_n = \frac{1}{n}t' + \frac{n-1}{n}x_0$. Hence $\forall n\ t_n \in (t'\ x_0)$ and $t_n \to x_0$. Furthermore, each of the t_n has the same visibility restrictions as t' with respect to t. Let $L(t, x_0)$ be the line through t and x_0, and U be the connected subset of S limited by $L(t, x_0)$ and not visible from t' via S. Furthermore, the points of U are not visible via S from each of the t_n. Call $\tilde{S} = S \sim U$ and let $\tilde{v}$ be the visibility function of $\tilde{S}$. Clearly it holds

$$\forall n\ \tilde{v}(t_n) = v(t_n),\ \tilde{v}(x_0) < v(x_0)$$

and from the upper semicontinuity of v we obtain

$$\lim_{n\to\infty} v(t_n) = \lim_{n\to\infty} \tilde{v}(t_n) \le \tilde{v}(x_0) < v(x_0)$$

that contradicts (ii). ■

We say that $x \in \operatorname{bd} S$ is a *cone point* if there exists a line L through x such that $\operatorname{st}(x, S)$ is included in one of the two closed semiplanes determined by L.

Lemma 4.2. *If S is a Jordan domain and x is a boundary point of S, there exists a line L through x that leaves* $\operatorname{ins}(x, S)$ *at one side of it.*

Proof: As lemma 3.4 states, $\operatorname{ins}(x, S) \cap N$ is an angular connected fan. If such a line does not exist, there must be a line through x that intersects the radial projection from x_0 of $\operatorname{ins}(x, S)$ on $\operatorname{bd} N$ in two points $\{t_1;\ t_2\}$, where at least one of them (say t_1) is not an endpoint of that projection. Hence there should exist $t_0 \in (x_0 t_1)$ such that $(x_0\ t_0)$ be included into $\operatorname{int} S$. But this should contradict the fact that $t_2 \in \operatorname{ins}(x, S)$. ■

Theorem 4.3. *The points of continuity of the visibility function of S on the boundary of S are precisely the cone points of that boundary.*

Proof: Let x be a cone point of $\operatorname{bd} S$, and L a line through x that divides $\mathbb{E}_2$ into two open semiplanes H^+ and H^- such that $\operatorname{st}(x, S) \subset \operatorname{cl}(H^+)$. Take $u \in \operatorname{st}(x, S) \sim L$, and let $u' \in R(ux \to)$ such that $(x\ u'] \subset H^-$. Since $u' \notin \operatorname{st}(x, S)$, we have two alternatives:

(a) $(x\ u') \cap S = \phi$.

(b) $(x\ u') \cap S \neq \phi$ and $(x\ u') \cap \complement S \neq \phi$.

Each of this alternatives produces easily a point $t \in (x\ u']$ such that $(x\ t) \cap \operatorname{int} S = \phi$, whence $u \in \operatorname{ins}(x, S)$. We have shown that

$$\operatorname{st}(x, S) \sim \operatorname{ins}(x, S) \subset L \cap S$$

and the last set has null measure. Hence x satisfies hypothesis (i) of Theorem 4.1 and v is continuous at x.

Conversely, assume that v is continuous at x. From Lemma 4.2 there exists a line L that produces a closed semiplane H^+ including $\operatorname{ins}(x, S)$. Repeating arguments used in the second part of Theorem 4.1 we obtain that the set $D = \operatorname{st}(x, S) \sim \operatorname{ins}(x, S)$ is a fan having one or two connected components and has null measure. It follows that $D \subset L$, whence $\operatorname{st}(x, S) \subset H^+$ and x is a cone point. ■

Theorem 4.4. *A smooth point of* bd S *is a cone point.*

Proof: For flat points, the theorem is almost trivial. In the case of inflection and concave points (i.e. points of $\operatorname{lnc} S \cap \operatorname{lc} \complement S$), the thesis follows from Lemma 4.2 and Theorems 2.4 and 2.5, respectively, from [**4**]. If x is a convex point (that is $x \in \operatorname{lc} S \cap \operatorname{lnc} \complement S$), let Γ be an arc of $\operatorname{bd} S$ such that $x \in \Gamma$ and $\Gamma \subset \operatorname{lc} S \cap \operatorname{lnc} \complement S$. Let $K = \operatorname{conv} \Gamma$ and $T(x)$ be the tangent line to K through x. Call H^+ and H^- the two open halfplanes determined by $T(x)$, where $K \subset \operatorname{cl} H^+$. Hence, $w \in H^-$ implies $[w\, x] \cap \complement S \neq \phi$, since $T(x)$ is supporting. Then $\operatorname{st}(x, S) \subset \operatorname{cl} H^+$ and x is a cone point. ■

Corollary 4.5. *The visibility function is continuous at smooth points.*

Proof: Inmediate from 4.3 and 4.4. ■

References

1. BEER, G., The index of convexity and the visibility function, *Pacific J.* **44** (1973), 59–67.
2. BEER, G., The continuity of the visibility function on a starshaped set, *Canadian J.* **24** (1972), 989–992.
3. BEER, G., Continuity properties of the visibility function, *Michigan Math. J.* **20** (1973), 297–302.
4. FORTE CUNTO, A. AND TORANZOS, F., Visibility inside a smooth Jordan domain, submitted for publication.
5. TORANZOS, F.A., Critical visibility and outward rays, *J. of Geometry* **33** (1988), 155–167.
6. VALENTINE, F.A., Local convexity and L_n sets, *Proceedings of the Amer. Math. Soc.* **16** (1965), 1305–1310.

Departamento de Matemática
Universidad de Buenos Aires
Pabellón 1 - Ciudad Universitaria
1428 Buenos Aires
ARGENTINA

Primera versió rebuda el 17 d'Octubre de 1989,
darrera versió rebuda el 22 de Març de 1990