UNITARY SUBGROUP OF INTEGRAL GROUP RINGS

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Abstract

Let $A$ be a finite abelian group and $G = A \rtimes \langle b \rangle$, $b^2 = 1$, $a^b = a^{-1}$, $\forall a \in A$. We find generators up to finite index of the unitary subgroup of $\mathbb{Z}G$. In fact, the generators are the bicyclic units. For an arbitrary group $G$, let $B_2(\mathbb{Z}G)$ denote the group generated by the bicyclic units. We classify groups $G$ such that $B_2(\mathbb{Z}G)$ is unitary.

Let $\mathbb{Z}G$ be the integral group ring of an arbitrary group $G$ and let $f : G \to U(\mathbb{Z}) = \{\pm 1\}$ be an orientation homomorphism. For each $x = \sum_{g \in G} a_g g$, we put $x^f = \sum_{g \in G} a_g f(g) g^{-1}$. In particular, if $f$ is trivial, $x^f$ coincides with the standard $x^*$. Let $U(\mathbb{Z}G)$ be the group of units of $\mathbb{Z}G$. Then $u \in U(\mathbb{Z}G)$ is called $f$-unitary if $u^{-1} = u^f$ or $u^{-1} = -u^f$. All $f$-unitary elements of $U(\mathbb{Z}G)$ form a subgroup $U_f(\mathbb{Z}G)$ containing $G \times U(\mathbb{Z})$. We refer to $U_f(\mathbb{Z}G)$ as the $f$-unitary subgroup of $U(\mathbb{Z}G)$. Interest in the group $U_f(\mathbb{Z}G)$ arose in algebraic topology and unitary $K$-theory [4].

We are interested in the constructive description of $U_f(\mathbb{Z}G)$. If $G$ is finite cyclic, then Bovdi [1] gave a linearly independent set of generators for a torsion free subgroup of finite index in $U_f(\mathbb{Z}G)$. This was extended to finite abelian groups by Hoechsmann-Sehgal in [3]. We give generators up to finite index of $U_f(\mathbb{Z}G)$ if $G$ is a finite dihedral group. In fact, the generators consist of the bicyclic units. The subgroup $B_2(\mathbb{Z}G)$ of $U(\mathbb{Z}G)$ generated by all the bicyclic units of $\mathbb{Z}G$ plays an important role in the study of $U(\mathbb{Z}G)$ (see [5], [6]). In Theorem 2, we characterize groups $G$ for which $B_2(\mathbb{Z}G)$ is unitary.

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**2. \( U_f(ZG) \) for dihedral groups**

First, we recall some definitions. For an element \( a \in G \) of finite order \( n \) write \( \tilde{a} = 1 + a + \cdots + a^{n-1} \). Denote by \( t(G) \) the set of all torsion elements of \( G \). If \( a, b \in G, \quad o(a) < \infty \), then

\[
u_{a,b} = 1 + (1 - a)b\tilde{a}
\]

has an inverse \( u_{a,b}^{-1} = 1 - (1 - a)b\tilde{a} \). Moreover, \( u_{a,b} = 1 \) if and only if \( b \) normalizes \( \langle a \rangle \). The elements \( u_{a,b}, \quad a, b \in G \) are called *bicyclic units* of \( ZG \) and the group generated by them is denoted by \( B_2(ZG) \). We recall [5] that by \( B_1(ZG) \) is understood the group generated by the Bass cyclic units of \( ZG \). It is known [5] that if \( G \) is a finite dihedral group and \( Z \) is the centre of \( U(ZG) \) then \( \langle (Z, B_2(ZG)) \rangle \) (equivalently, \( \langle B_1(ZG), B_2(ZG) \rangle \)) is of finite index in \( U(ZG) \). We prove

**Theorem 1.** Let \( G \) be the dihedral group

\[D_{2n} = \langle a^n = 1 = b^2 | a^b = a^{-1} \rangle.\]

Suppose \( f \) is an orientation homomorphism of \( G \) with kernel \( \langle a \rangle \). Then the index \( \langle U_f(ZG) : B_2(ZG) \rangle \) is finite.

We need the

**Proposition.** Let \( G \) be a group containing a subgroup \( A \) of index 2 and an element \( b \) such that \( G = \langle A, b \rangle \) and \( b^{-1}ab = a^{-1} \) for all \( a \in A \). Suppose that \( A^2 \neq 1 \). If \( f \) is an orientation homomorphism of \( G \) with kernel \( A \), then

1) the centre of \( U_f(ZG) \) coincides with \( t_2(A) \times \langle -1 \rangle \), where

\[t_2(A) = \{ a \in t(A) : a^2 = 1 \};\]

2) the centre of \( U(ZG) \) is the direct product of \( t_2(A) \times \langle -1 \rangle \) and a torsion free abelian group \( T \) such that \( U(ZA) = \langle -1 \rangle \times A \times T \) and \( x = x^* \) for all \( x \in T \).

**Proof:** Let \( x = x_1 + x_2b, \quad x_i \in ZA \) be a central unit in \( ZG \). Since \( G \) is a subgroup of \( U(ZG) \),

\[x = b^{-1}xb = x_1^* + x_2b \quad \text{and} \quad x = a^{-1}xa = x_1 + a^{-2}x_2b\]

for all \( x \in A \). Then \( x_i = x_i^* \) and

\[(1) \quad x_2(1 - a^2) = 0\]
for all \( a \in A \). We wish to prove that \( x_2 = 0 \).

Let us suppose that \( x_2 \neq 0 \). From (1) we obtain that \( A^2 \) is finite. Let \( \hat{A}^2 \) denote the sum of all elements of \( A^2 \). If \( H \) is a normal subgroup of \( G \), then denote by \( \Delta(G,H) \) the ideal of \( \mathbb{Z}G \) generated by elements of the form \( h - 1 \) with \( h \in H \). Clearly,

\[
\mathbb{Z}G/\Delta(G,H) \cong \mathbb{Z}(G/H).
\]

If \( \chi(y) \) is the sum of the coefficients of \( y \), then the element

\[
x + \Delta(G, A) = \chi(x_1) + \chi(x_2)b + \Delta(G, A)
\]

is trivial, because \( |G/A| = 2 \) [7, p. 46]. This implies that one of the numbers \( \chi(x_1) \) or \( \chi(x_2) \) equals \( \pm 1 \) and the other is zero. From (1) we obtain \( x_2 = z \hat{A}^2, \ z \in \mathbb{Z}A \). \( \chi(x_2) = \chi(z)|A^2| \), and this is possible only in the case when \( \chi(x_2) = 0 \).

Suppose \( A = A^2 \). Then \( x_2 = \gamma \sum_{a \in A} a \) for some \( \gamma \in \mathbb{Z} \). From the equality \( \chi(x_2) = \gamma |A| = 0 \) we obtain \( \gamma = 0 \) and \( x_2 = 0 \), which leads to a contradiction. Thus \( A \neq A^2 \). Write \( x_2 = (\sum \alpha_i c_i) \hat{A}^2 \) with \( \alpha_i \in \mathbb{Z} \) where \( c_i \)'s are a transversal of \( A^2 \) in \( A \). Then

\[
x_1 + x_2b + \Delta(G, A^2) = x_1 + \left( \sum_i \alpha_i c_i \right) \hat{A}^2b + \Delta(G, A^2)
\]

\[
= x_1 + \left( |A^2| \sum_i \alpha_i c_i \right) b + \Delta(G, A^2)
\]

is a unit in \( \mathbb{Z}(G/A^2) \). Since \( G/A^2 \) is an abelian group of exponent two, by Higman's theorem [7, p. 57], all units of \( \mathbb{Z}(G/A^2) \) are trivial. Obviously, \( \sum_i \alpha_i = 0 \) and if \( \alpha_i \neq 0 \) for some \( i \), then \( \alpha_i |A^2| \neq \pm 1 \). Thus, \( \alpha_i = 0 \) for all \( i \) and the equality \( x_2 = 0 \) is contradictory. Hence, \( x = x_1 \in U(\mathbb{Z}A) \) and \( x^* = x = x_1 = x_1 \). Clearly, if \( x \in U(\mathbb{Z}A) \) and \( x^* = x \), then \( x \) is a central unit of \( \mathbb{Z}G \).

It is well known (see [2]) that \( U(\mathbb{Z}t(A)) = \pm t(A) \times T \) and \( U(\mathbb{Z}A) = \pm A \times T \), where every element \( u \in T \) satisfies the condition \( u = u^* \). Therefore the centre of \( U(\mathbb{Z}G) \) is the direct product of subgroups \( \pm t_2(A) \) and \( T \). This is 2) of the Proposition.

Suppose that \( x = x_1 + x_2b \) is a central unit in \( U_f(\mathbb{Z}G) \). Since \( G \) is a subgroup of \( U_f(\mathbb{Z}G) \), \( x \) is central in \( U(\mathbb{Z}G) \). It follows that \( x = x_1 \) and \( xx^* = x_1x_1^* = x_1^2 = \pm 1 \). Therefore, by Higman's theorem \( x_1 = \pm a \) where \( a \in t_2(A) \). This completes the proof of the Proposition. \( \blacksquare \)
Proof of Theorem 1: Let $G$ be the dihedral of order $2n$ given by $G = \langle a^n = 1 = b^2, a^b = a^{-1} \rangle$. If $n = 2$, then the theorem is trivial. So we may apply the last Proposition. Let $Z$ be the centre of $U(ZG)$. Then we know that $(U(ZG) : \langle B_2(ZG), Z \rangle) < \infty$. We have seen in the Proposition above that $Z_1$, the centre of $U_f(ZG)$, is finite and $Z_1 < Z$. It suffices to prove, therefore, that $B_2(ZG)$ is unitary. If $u_{x,y} \neq 0$, then $\sigma(x) = 2$ and

$$u_{x,y} = 1 + (1 - x)y(1 + x).$$

Now, $y = a'x^\epsilon$, $\epsilon = 0$ or 1. Since $x(1 + x) = 1 + x$ we have, in any case,

$$u_{x,y} = 1 + (1 - x)a'(1 + x).$$

Then $u_{x,y} = 1 + (1 + x)f(a')f(1 - x)f = 1 + (1 - x)a^{-4}(1 + x)$. Therefore, $u_{x,y} u_{x,y}^* = 1 + (1 - x)(a^4 + a^{-4})(1 + x) = 1$ as $(a^4 + a^{-4})$ is central. This completes the proof of the theorem. □

Remark. The last theorem holds for nonabelian groups $G = \langle A, b \rangle$ where $A$ is finite abelian and $b^2 = 1$, $a^b = a^{-1}$ for all $a \in A$. If $A$ is an elementary 2-group, then so is $G$ and there is nothing to prove. Suppose $A^2 
eq 1$. The nonlinear irreducible representations $\rho$ of $G$ are induced from those of $A$ and $\rho(ZG) = \rho(D)$ for some dihedral subgroup $D$ of $A$. The result follows.

3. Unitarity of the subgroup $B_2(ZG)$

Theorem 2. Let $G = \langle A, b \rangle$ where $A$ is the kernel of the nontrivial orientation homomorphism $f : G \to U(\mathbb{Z})$. The subgroup $B_2(ZG)$ is nontrivial and $f$-unitary if and only if $G$ is non-Hamiltonian in which an element $b \neq 1$ of finite order can be chosen such that one of the following conditions is fulfilled:

1) $A$ is an abelian group, the order of the element $b$ divides 4 and $ab^{-1} = a^{-1}$ for all $a \in A$;
2) $A$ is a Hamiltonian 2-group, $G$ is the semidirect product of $A$ and $\langle b | b^2 = 1 \rangle$, and every subgroup of $A$ is normal in $G$;
3) $A$ is a Hamiltonian 2-group and $G$ is the direct product of a Hamiltonian 2-subgroup of $A$ and a cyclic group $(b)$ of order 4;
4) $t(A)$ is an abelian group, every subgroup of $t(A)$ is normal in $G$ and $bab^{-1} = a^{-1}b^ai$ for all $a \in A$, where the integer $i$ depends on $a$.

We need the following.
Lemma. Suppose that $G$ has a subgroup $A$ of index 2 with $G = (A, b)$ and $o(b) < \infty$. Suppose further that $A \neq N_A((b))$ and

1) $t(A)$ is abelian and all subgroups of $t(A)$ are normal in $A$;
2) $bgb^{-1} = g^{-1}$ for all $g \in A \setminus N_A((b))$.

Then $bab^{-1} = a^{-1}$ for all $a \in A$ and $b^4 = 1$.

Proof: Let $c \in N_A((b))$. Choose $a \in A \setminus N_A((b))$. At first, suppose $c$ has finite order. Then by (2) we have

$$a^{-1}bcb^{-1} = b(ac)b^{-1} = c^{-1}a^{-1}.$$ 

If $a \in t(A)$, then by (1) we have $bcb^{-1} = c^{-1}$. If $a$ has infinite order, there exists an integer $n$ such that $a^n c = ca^n$, since $(c)$ is normal in $A$. By hypothesis, $a^n c \notin N((b))$ and thus

$$a^{-n}bcb^{-1} = b(a^n c)b^{-1} = c^{-1}a^{-n}.$$ 

It follows that $bcb^{-1} = c^{-1}$ as desired. Now it is enough to prove that $c$ cannot have infinite order. Suppose that $o(c) = \infty$ and $o(a) < \infty$. Then there is an $n$ such that $c^n a = ac^n$. Clearly, $ac^n \notin N((b))$. We have

$$a^{-1}bc^n b^{-1} = b(ac^n)b^{-1} = c^{-n}a^{-1}.$$ 

It follows that $bc^n b^{-1} = c^{-n}$. This is impossible because $c^n \in N((b))$.

Now let $o(c) = \infty$, $o(a) = \infty$. There exists an $n$ such that $bcb^n = c^n b$ and $a^{-1}c^n = bac^n b^{-1} = c^{-n}a^{-1}$. It follows that $[c^n, a^2] = 1$. Clearly, $a^2 c^n \notin N((b))$ and we get

$$a^{-2} c^n = ba^2 c^n b^{-1} = a^{-2}c^{-n}$$ 

which implies $c^{2n} = 1$, a contradiction. Since $b^2 \in A$, $bb^2 b^{-1} = b^{-2}$ and we have $b^4 = 1$, completing the proof of the lemma. □

Proof of Theorem 2:

"Necessity."

Suppose that $B_2(IZG)$ is nontrivial and $f$-unitary. Let us first prove that every finite subgroup $(a)$ of $A$ is normal in $G$. Let $n$ be the order of $(a)$. If $g \notin N_G((a))$, then $u_{a, g} = 1 + (1 - a)g\bar{a} \neq 1$. Then from the equality $u_{a, g}^{-1} = u_{a, g}^f$, we have

$$\bar{a}g^{-1}f(g)(1 - a^{-1}) = -(1 - a)g\bar{a}.$$ 

Multiplying by $\bar{a}$ we obtain $n(1 - a)g\bar{a} = 0$, which is impossible. Therefore, every subgroup of $t(A)$ is normal in $G$. Because $B_2(IZG) \neq 1$, $G \setminus A$
contains an element \( c \) of finite order with \( \langle c \rangle \) not normalized by \( A \). Then \( c^2 \in t(A) \) and \( c^2 \) is central in \( IG \). Clearly,

\[
u_{c,g} = 1 + (1 - c)g(1 + c)c^2
\]

and \( f(c) = -1 \). Since \( u_{c,g} \) is \( f \)-unitary, \( u_{c,g}u_{c,g}^f = 1 \) and it follows that

\[
(g + g^{-1}f(g))(1 + c)c^2 = c(g + g^{-1}f(g))(1 + c)c^2.
\]

Choose \( b \in G \setminus A \) such that \( b \) is a 2-element of least order and let \( g \in A \). In (1) taking \( c = b, g = bg^{-1}b^{1+2i} \) whenever \( g \notin N_A(\langle b \rangle) \). We obtain \( bgb^{-1} = g^{-1}b^{2i} \) for all \( g \in A \setminus N_A(\langle b \rangle) \) and

\[
(bg)^2 = (g^{-1}b^2g)^{i' + 1}.
\]

Clearly, \( bg \) is a 2-element in \( G \setminus A \) and \( i' \) is even, otherwise the order of \( bg \) is less than the order of \( b \), which is impossible. Therefore,

\[
(bgb^{-1})^2 = (g^{-1}b^{2i})^2 = g^{-1}b^{4i}
\]

for all \( g \in A \setminus N_A(\langle b \rangle) \).

a) Suppose that the order of \( b \) divides 4.

Then from (2) \( bgb^{-1} = g^{-1} \) for all \( g \in A \setminus N_A(\langle b \rangle) \).

If \( t(A) \) is abelian, then, by the Lemma, \( A \) is abelian and \( bab^{-1} = a^{-1} \) for all \( a \in A \). This is case 1) of the theorem.

If \( t(A) \) is nonabelian, then \( t(A) \) is a Hamiltonian group and

\[
t(A) = Q \times E \times T
\]

where \( Q \) is the quaternion group of order 8, \( E^2 = 1 \) and all elements of \( T \) are of odd order.

We wish to prove that \( A = t(A) \). Suppose that \( g \) is an element of infinite order of \( A \setminus N(\langle b \rangle) \). Then \( g^2 \in C_A(Q) \) and there exists an element \( w \) of order 4 of \( Q \) such that \( [b, w] = 1 \), because every subgroup of \( Q \) is normal in \( G \). Clearly, \( g^2w \notin N(\langle b \rangle) \) and by (2)

\[
wg^{-2} = bwg^2b^{-1} = bg^2wb^{-1} = w^{-1}g^{-2},
\]

which is impossible. Therefore, all elements of \( A \setminus N(\langle b \rangle) \) have finite orders.

Let \( g \) be an element of infinite order from \( N_A(\langle b \rangle) \) and let an \( a \in A \setminus N_A(\langle b \rangle) \). Clearly there exists \( n \) such that \( [g^n, a] = 1 \), because the finite
cyclic subgroup \( (a) \) is normal in \( G \). Then \( g^n a \in A \setminus N_A((b)) \) and \( g^n a \) is of infinite order, which leads to a contradiction. Therefore, \( t(A) = A \).

We claim that \( T = 1 \). Let \( v \) be an element of odd order from \( A \setminus N((b)) \). Obviously, there exists an element \( w \) of order 4 in \( Q \) such that \([b, w] = 1\), as every subgroup of \( Q \) is normal in \( G \). Thus \( vw \notin N((b)) \) and by (2)

\[
v^{-1}w = bwb^{-1} = w^{-1}v^{-1},
\]

which is impossible. Next, let \( v \) be an element of odd order from \( N_A((b)) \). Because \( (v) \triangleleft G \), \([v, b] = 1\). Clearly there is an element \( w \) of order 4 in \( Q \) such that \( b^{-1}wb = w^{-1} \) and \( vw \notin N_A((b)) \). Then

\[
w^{-1}v = bwb^{-1} = v^{-1}w^{-1},
\]

which is impossible. Hence, the structure of \( G \) is described in case 2) or 3) of the theorem.

b) Suppose that the order of \( b \) is \( 2^k \) \((k \geq 3)\).

Then by (2) \( b^2 \) belongs to the centre of \( t(A) \), because \( t(A) \) is abelian or Hamiltonian. Hence, \( t(A) \) is abelian and every subgroup of \( t(A) \) is normal in \( G \). Then from (2) \( bab^{-1} = a^{-1}b^{4j} \) for all \( a \in A \setminus N_A((b)) \). Denote by \( (b^{4r}) \) the subgroup generated by \( b^{4j} = abab^{-1} \), as \( a \) runs over \( A \setminus N_A((b)) \).

Put \( \tilde{G} = G/(b^{4r}) \), \( \tilde{A} = A/(b^{4r}) \) and \( \tilde{b} = b(b^{4r}) \). Then \( \tilde{G} \) satisfies the conditions of our Lemma and it follows that \( r = 1 \) and \( ba^{-1} = c^{-1} \) for all \( a \in \tilde{A} \). This is case 4) of the theorem.

"Sufficiency."

Let \( G \) satisfy one of the conditions 1)-4) of the theorem. Clearly, if a finite subgroup \( \langle c \rangle \) is not normal in \( G \), then \( c \in bA \), \( \langle c^2 \rangle = \langle b^2 \rangle \) and \( c^2 \) belongs to the centre of \( ZG \). Therefore,

\[
u_{c, g} = 1 + (1 - c)g(1 + c)c^2
\]

and

\[
u_{c, g} u_{c, g}^f = 1 + (1 - c)(g + g^{-1}f(g))(1 + c)c^2.
\]

Suppose that \( g \in A \). Then \( f(g) = 1 \) and \((g + g^{-1})c^2\) is a central element. This is obvious in cases 1), 2) and 3). Suppose that \( G \) satisfies the condition 4) of the theorem. Then \( \langle c^2 \rangle = \langle b^2 \rangle \), and \( bgb^{-1} = g^{-1}b^{4j} \) and \( G/(b^4) \) is abelian. Thus

\[
b(g + g^{-1})c^2b^{-1} = (g + g^{-1})c^2 = a^{-1}(g + g^{-1})c^2a
\]

and \((g + g^{-1})c^2\) is central in \( ZG \).
If \( g \in bA \), then \( g = ba \), \( f(g) = -1 \) and

\[ g^{-1} = a^{-1}b^{-1} = b^{-1}ab^2. \]

Clearly, \( g^{-1}\overline{c^2} = g\overline{c^2} \) and \((g + f(g)g^{-1})\overline{c^2} = 0\). Therefore, \( u_{c,d}u_{c,d}^f = 1 \) and the bicyclic units are \( f \)-unitary. Thus \( B_2(\mathbb{Z}G) \) is an \( f \)-unitary subgroup, proving the theorem \( \blacksquare \)

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