LINEARIZATION AND EXPLICIT SOLUTIONS OF THE MINIMAL SURFACE EQUATION

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Abstract

We show that the apparatus of support functions, usually used in convex surfaces theory, leads to the linear equation $\Delta h + 2h = 0$ describing locally germs of minimal surfaces. Here Δ is the Laplace-Beltrami operator on the standard two-dimensional sphere. It explains the existence of the sum operation of minimal surfaces, introduced recently. In 4-dimensional space the equation $\Delta h + 2h = 0$ becomes inequality wherever the Gauss curvature of a minimal hypersurface is nonzero.

0. Introduction

Recently great progress was achieved in the investigation and construction of examples of minimal surfaces in $\mathbb{R}^3$ [1]–[3]. The Gauss map usually plays a significant role and its singularities in a sense control topology if the surface is complete [4]. It was also noticed [5], [6] that there exists a "sum" operation $M_1 + M_2$ for two minimal surfaces M_1, M_2. It may seem to be strange, for the usual form of the minimal surface equation is essentially nonlinear. True, given a conformal minimal map $\mathbb{R}^2 \supset U \xrightarrow{x} M \subset \mathbb{R}^3$ we have a *linear* equation $\Delta x = 0$ [7]. However, the condition of conformality is nonlinear itself.

In this paper, we show that apparatus of support functions usually used in convex surface theory leads to the linear and completely integrable equation of minimal surfaces in $\mathbb{R}^3$. We are able to write down an explicit formula describing locally *all* minimal surfaces with nonvanishing curvature which is quite different from the Weierstrass description. We hope our method will be useful in global problems, too. It automatically implies the existence of the sum operation.

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1. The main equation

Let M be a smooth oriented hypersurface in $\mathbb{R}^N$ and $G : M \to S^{N-1}$ be its Gauss map. Then [7] G is a local diffeomorphism wherever the Gauss curvature of M is nonzero. From now on, we assume that this condition holds at every point of M. Then G becomes a covering over its image $G(M)$. Let $U \subset G(M)$ be a simply-connected proper domain, then $G^{-1}(U)$ is a disjoint union of open V_i, $i \in I$, and $G|_{V_i} : V_i \to U$ is a diffeomorphism which we call G_i. We supply S^{N-1} with the canonical metric of curvature 1.

Definition. By support function $h_i : U \to \mathbb{R}$ we mean

$$(1) \qquad h_i(n) = \left(G_i^{-1}(n), n \right) .$$

Lemma 1. *The function $h_i(n)$ determines $G_i^{-1}(n)$ in the following way:*

$$(2) \qquad G_i^{-1}(n) = h_i(n)n + \operatorname{grad} h_i(n) ,$$

where $\operatorname{grad} h_i(n)$ is computed in terms of the metric of S^{N-1} and looked at as a vector in $\mathbb{R}^N$.

Proof: Let $G_i^{-1}(n) = x$. For $y \in V_i$ we have by (1)

$$(3) \qquad h_i\left(G_i(y) \right) = (y, G_i(y)) .$$

If $X \in T_x S^{N-1}$ then, differentiating (3) along X we obtain

$$(\operatorname{grad} h_i(G_i(x)) , \ G_{i_*}X) = (X, G_i(x)) + (x, G_{i_*}X) .$$

But $(X, G_i(x)) = 0$ by the definition of the Gauss map, so

$$(4) \qquad ((x - \operatorname{grad} h_i(G_i(x))) , \ G_{i_*}X) = 0 .$$

By the nondegeneracy condition, G_{i_*} maps isomorphically $T_x M$ onto $T_{G_i(x)} S^{N-1}$. The latter space coincides with $T_x M$ as a subspace of $\mathbb{R}^N$ so, for some $\mu \in \mathbb{R}$, we have

$$x - \operatorname{grad} h_i\left(G_i(x) \right) = \mu G_i(x) ,$$

or

$$G_i^{-1}(n) = \mu n + \operatorname{grad} h_i(n)$$

Taking scalar product with n and accounting (1) and $(\operatorname{grad} h_i(n), n) = 0$ we obtain (2). ∎

Lemma 2. *Let $A(x)$ be the second fundamental operator in $T_x M$. Then*

(5) $$A(x) = (h_i(n)E + \operatorname{Hess} h_i(n))^{-1} \ .$$

Here $n = G_i(x)$, E is the identity operator in $T_n S^{N-1} = T_x M$ and $\operatorname{Hess} h_i(n)$ is the Hessian operator [8] on the sphere S^{N-1}.

Proof: Denote for a moment $F_i = G_i^{-1}$ on U. By (2) we have $F_i(n) = h_i(n)n + \operatorname{grad} h_i(n)$. Let $Z \in T_n S^{N-1}$. By the definition of the Hessian operator, $\operatorname{Hess} h_i(n)(Z) = (\nabla_Z \operatorname{grad} h_i)(n)$, where ∇ is the Levi-Civita connection on the sphere. For any vector field v on the sphere we have [8] $\nabla_Z v = v'_Z + (v, Z)n$, where v'_Z denotes usual differentiation in $\mathbb{R}^N$. So by (2),

$$(\operatorname{Hess} h_i(n))(Z) = F_{i_*}Z - (\operatorname{grad} h_i(n), Z)n - h_i(n)Z + (\operatorname{grad} h_i(n), Z)n$$
$$= F_{i_*}Z - h_i(n)Z \ ,$$

or $F_{i_*}Z = h_i(n)Z + (\operatorname{Hess} h_i(n))(Z)$. Let $Y = F_{i_*}Z$, then $Z = G_*Y$. By definition of A, $A(x)Y = G_{i_*}Y = Z$, which proves (5). ∎

Theorem 1. *Suppose $N = 3$ and M is minimal. Then for any proper simply-connected domain $U \subset G(M)$ and any branch h_i we have*

(6) $$\Delta h_i + 2h_i = 0 \ ,$$

where Δ is the Laplace-Beltrami operator on the sphere S^2. Conversely, if h is a solution of (6) in an open $U \subset S^2$ then the formula $x(n) = h(n)n + \operatorname{grad} h(n)$ determines a smooth map from U to $\mathbb{R}^3$, which is either a constant or a conformal and minimal immersion outside a locally finite set of isolated singularities (branch points).

Proof: M is minimal iff $\operatorname{tr} A(x) = 0$ everywhere. For an invertible operator A in 2-space we have $\operatorname{tr} A^{-1} = \frac{\operatorname{tr} A}{\det A}$ so by (5), $\operatorname{tr} A(x) = 0$ is equivalent to $0 = \operatorname{tr}(h_i(n)E + \operatorname{Hess} h_i(n)) = 2h_i + \Delta h_i$. This proves the first statement of the theorem. Now suppose h yields (6). Denote $F(n) = h(n)n + \operatorname{grad} h$. From the proof of the Lemma 2 we know that $F_*(n) = h(n)E + \operatorname{Hess} h(n)$. In particular, it means that $F_*(n)$ maps $T_n S^2$ to itself and is symmetric in $T_n S^2$. Further, by (6), $\operatorname{tr} F_*(n) = 0$. Note that any symmetric operator with the zero trace in 2-space is represented by a matrix $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ in any orthonormal basis and is thus conformal, so F is conformal, and for $n \in U$ either rank $F_*(n) = 2$ or $F_*(n) = 0$. Denote by Z the set of points where $F_* = 0$. As (6) is

elliptic, $F(n)$ is analytic along with $h(n)$, so if F is non-constant, then Z is nowhere dense. Outside Z, $F(n)$ is a conformal immersion and we have just shown that $T_{F(n)}F(U) = T_n S^2$, so, for the Gauss map we have $G(F(n)) = n$. Hence the support function $\tilde{h}$ of $F(U \sim Z)$ is $\tilde{h}(n) = (h(n)n + \operatorname{grad} h(n), n) = h(n)$. From the first part of the theorem we see that $F|_{U \sim Z}$ is minimal. It follows that F is harmonic in $U \sim Z$, but F is analytic in U and Z is nowhere dense, hence F is harmonic everywhere in U. Locally in conformal coordinates (x, y) we have $n \in Z \Leftrightarrow \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$, hence $\mathcal{F}'_z(n) = 0$ where $z = x + iy$ and $\mathcal{F}$ is holomorphic and $\operatorname{Re} \mathcal{F} = F$, so Z is locally finite. $\blacksquare$

Theorem 2. *Suppose $N = 4$ and M is minimal. Then for any proper simply-connected $U \subset G(M)$ and any branch $h_i, \Delta h_i(n) + 2h_i(n)$ doesn't change sign in U.*

Proof: Let $\lambda_i(x)$, $1 \leq i \leq 3$, be the principal curvatures of M in x. Then $\lambda_1 + \lambda_2 + \lambda_3 = 0$ by minimality condition. Suppose $\Delta h_i(n) + 2h_i(n) = 0$. somewhere in U and let $x = G_i^{-1}(n)$. As $\operatorname{tr} A^{-1}(x) = \frac{\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3}{\det A(x)}$ we obtain by (5) that $\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 0$. This implies $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = (\lambda_1 + \lambda_2 + \lambda_3)^2 - 2(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) = 0$ in x which is impossible by nondegeneracy condition. $\blacksquare$

2. Applications

We turn to applications of our result. Let M_1, M_2 be two minimal surfaces in $\mathbb{R}^3$ such that $G(M_1) \cap G(M_2)$ has a nonempty interior in S^2. In [5] and [6] their sum $M_1 + M_2$ is defined by parametrization $x(n) = G_1^{-1}(n) + G_2^{-1}(n)$. Let $h_i^{(1)}$ and $h_j^{(2)}$ be two branches of support functions of M_1 and M_2 respectively. Then by (2) we have $x(n) = h(n)n + \operatorname{grad} h(n)$. where $h = h_i^{(1)} + h_j^{(2)}$. Next, both $h_i^{(1)}$ and $h_j^{(2)}$ yield (6) which is linear, hence h yields (6), too. Theorem 1 implies thus the minimality of $M_1 + M_2$. Moreover, given a minimal M and any Killing vector field Z in S^2 we can define the *derivative surface* M'_Z by

$$(7) \qquad\qquad h(M'_Z) = h'_Z \, ,$$

which is also minimal.

Example. Let M_1 be a catenoid defined in euclidean coordinates (x, y, z) by the equation $x^2 + y^2 = (ch\, z)^2$. Then direct computations show that

(8) $$h_1(n) = h(n_x, n_y, n_1) = 1 - n_Z \arctan h\, n_Z\,,$$

and $G(M_1) = S^2 \sim \{\pm p\}$ where $p = (0, 0, 1)$. Let $g_i \in SO(3)$, $i = 2, \ldots, m$, $g_1 = id$ be rotations such that the sets $\{\pm g_i p\}$ are pairwise disjoint. Let $h = \sum h_1 \circ g_i$, and $V = S^2 \sim \bigcup_i \{\pm g_i p\}$. Then we have the following

Proposition 1. *The surface $\psi : n \longmapsto h(n)n + \operatorname{grad} h$ is a complete minimal surface in $\mathbb{R}^3$ with only a finite number of branch points, and its Gauss image omits precisely $2m$-point set $\bigcup_i \{\pm g_i p\}$ (compare [5]).*

Proof: First we note that in some neighbourhood of p, ψ cannot have branch points. Indeed, suppose $\det (h(n)E + \operatorname{Hess} h(n)) = 0$, then for some vector $X \in T_n S^2, \|X\| = 1$, $(h_1(n)E + \operatorname{Hess} h_1(n))X = -\sum_{i \geq 2} (h_i(n)E + \operatorname{Hess} h_i(n)) X$. We know that for $i \geq 2$ $h_i(n)$ does not have singularities near p, so in some neighbourhood of p and for some constant C we would have

$$\| (h_1(n)E + \operatorname{Hess} h_1(n)) X\| \leq C\|X\|\,.$$

Denote $(h_1(n)E + \operatorname{Hess} h_1(n)) X = Y$, then $\|(h_1(n)E + \operatorname{Hess} h_1(n))^{-1} Y\| \geq C^{-1}\|Y\|$. Actually $(h_1(n)E + \operatorname{Hess} h_1(n))^{-1}$ is the second fundamental operator of the catenoid M_1, as we saw in Lemma 2, therefore, its eigenvalues are $\pm\sqrt{-K(n)}$, where $K(n)$ is the Gaussian curvature at $G^{-1}(n)$. As it is well-known (and easy to verify) that K is decaying to zero at infinity, the above inequality is impossible. Of course, the same is true about all the singular points $\pm g_i p$, hence, being locally finite, the set of branch points should be finite. Next, as the metric of catenoid is complete, we have $\int_\gamma \| (h_1(n)E + \operatorname{Hess} h_1(n))\, \dot\gamma(t)\| = \infty$ for any curve $\gamma : [0, \infty) \to S^2$ such that $\lim_{t \to \infty} \gamma(t) = p$. Hence the same arguments show that this is true for h instead of h_1 and finally, ψ is complete. ∎

Now consider the Enneper surface [7] $\varepsilon : \mathbb{R}^2 \to \mathbb{R}^3$. The composition $G \circ \varepsilon$ with the Gauss map coincides with the inverse stereographic projection $\pi^{-1} : \mathbb{R}^2 \to S^2 \sim \{p\}$, so $K \neq 0$ and the support function h is defined in $S^2 \sim \{p\}$. Straightforward computations show that $K \xrightarrow[x \to \infty]{} -0$ on ε, hence we can apply the same construction to obtain

Proposition 2. *For any given finite set $E \subset S^2$ there exists a complete minimal surface in $\mathbb{R}^3$ with only a finite number of branch points whose Gauss image omits precisely the set E.*

The conjecture of Meeks [4] states that for every $k > 1$ there exists an embedded minimal surface homeomorphic to a compact manifold punctured in k points. The problem of Osserman [7], solved by Fujimoto [4], asks whether the statement of our Proposition 2 holds for some smooth complete minimal surface (without branch points).

Concluding Remarks.

1. Our main equation (6) admits separation of variables. Fix $x_0 \in S^2$, then in spherical polar coordinates near (r, φ) near x_0 the spherical metric becomes $dr^2 + \sin^2 r \, d\varphi^2$ and the Laplace-Beltrami operator becomes $\Delta f = f_r' + \frac{\cos r}{\sin r} f_r' + \frac{1}{\sin^2 r} f_\varphi''$. Substituting $f(r, \varphi) = \frac{1}{\sqrt{\sin r}} g(r, \varphi)$ we obtain that $\Delta f + 2f = 0$ is equivalent to $g_r'' + g \left(\frac{1}{4\sin^2 r} + \frac{9}{4} \right) + \frac{g_\varphi''}{\sin^2 r} = 0$. By Fourier methods one finds

$$g(r, \varphi) = \sum_{m=-\infty}^{\infty} C_m(r) e^{im\varphi}$$

   where $C_m(r)$ satisfy

$$C_m'' + C_m \left((\frac{1}{4} - m^2)\frac{1}{\sin^2 r} + \frac{9}{4} \right) = 0 \ .$$

2. A rather surprising phenomenon follows from our description. Namely, if $\Delta h + 2h = 0$ in an open $U \subset S^2$, then the "Monge-Ampere" $\psi = \det(hE + \operatorname{Hess} h)$ satisfies some second order PDE. Indeed, we know from Theorem 1 that the surface M parametrized by $F(n) : n \mapsto h(n)n + \operatorname{grad} h(n)$ is minimal and its curvature at $F(n)$ is $\psi^{-1}(n)$. Let's pull back on S^2 the minimal surface's metric. We will obtain $\widehat{g} = \mu(n)g_0$, because the Gauss map $F^{-1}(n)$ is conformal (here g_0 is the spherical metric). To compute $\mu(n)$, we note that $F^* ds = \pm\psi(n) ds_0$, where ds, ds_0 are the area elements on M, S^2 respectively. Hence, $g = |\psi(n)|g_0$. Therefore, the curvature of $-\psi(n)g_0$ is $\psi^{-1}(n)$ (compare with Ricci-Curbastro Theorem, [10]). This is equivalent to some PDE.

3. Suppose M is a complete minimal surface of finite total curvature. Then by the theorem of Osserman [7] the Gauss map $G : M \to S^2$ extends to a holomorphic map $\widetilde{G}$ of the completion $\widetilde{M}$, and

$\widetilde{M} \sim M$ is finite, say $\widetilde{M} \sim M = \{p_1, \ldots p_m\}$. Let $\mathcal{L} \subset \widetilde{M}$ be the finite set of branch points of $\widetilde{G}$, say $\mathcal{L} = \{q_1 \ldots q_k\}$. Then $N = \widetilde{M} \sim ((\widetilde{M} \sim M) \cup \mathcal{L})$ (is the finite covering of $S^2 \sim \widetilde{G}((\widetilde{M} \sim M) \cup \mathcal{L})$.) Next, the support function $h(n)$ becomes single-valued on N and we see that *every complete minimal surface of finite total curvature determines a solution of the equation* $\Delta h + 2h = 0$ *in a finite covering of the standard sphere punctured in a finite number of points.*

4. Consider the flat metric $g = dx^2 + dy^2 - dz^2$ in $\mathbb{R}^{2,1}$. If for a surface $M \subset \mathbb{R}^{2,1}$, $g|_M$ is positively defined, then there exists a correctly defined Gauss map G from M to the hyperboloid $S : x^2 + y^2 - z^2 = -1$. It is well-known that $g \mid S$ is the standard hyperbolic metric. Just as before we can define a support function $h(n)$. Formula (2) in this case reads $G_i^{-1}(n) = -h_i(n) + \operatorname{grad} h_i(n)$. Formula 5 becomes $A(x) = (-h_i(n)E + \operatorname{Hess} h_i(n))^{-1}$ and (6) becomes $\Delta h_i(n) - 2h_i(n) = 0$ for minimal surfaces M with time-like normals.

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