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TORSION UNITS IN GROUP RINGS

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Abstract

Let $U(RG)$ be the unit group of the group ring RG. In this paper we study group rings RG whose support elements of every torsion unit are torsion, where R is either the ring of integers $\mathbb{Z}$ or a field K.

Let R be a commutative ring with identity, G be a group and $U(RG)$ be the group of units of the group ring RG. Denote by $T(G)$, the set of torsions elements of G. It is proved in [**2**], that if $T(U(\mathbb{Z}G))$ is a subgroup, then $T(U(\mathbb{Z}G)) = \pm T(G)$. In this note we study group rings RG whose support of every torsion unit is in $T(G)$.

Theorem 1. *Let R be an integral domain, F be its quotient field and G be a non torsion group. If the support of every torsion unit of RG is in $T(G)$, then $T(G)$ is a subgroup with every subgroup of $T(G)$ normal in G and every idempotent of $FT(G)$ central in FG.*

Proof: Let $t \in T(G)$ be of order n and let $x \in G \backslash T(G)$. Then $\alpha = t + (1-t)x(1+t+\cdots+t^{n-1}) \in U(RG)$ and $\alpha^n = 1$.

Since $\operatorname{supp}(\alpha) \subseteq T(G)$; $x = txt^k$ for some $k = 1, 2, \ldots, n-1$, thus $x^{-1}tx = t^x \in \langle t \rangle$. +3 if $x \in G \backslash T(G)$, then $x \in N_G(\langle t \rangle)$, where $N_G(\langle t \rangle)$ is the normalizer of $\langle t \rangle$ in G.

If $y \in T(G)$ and $x \in G \backslash T(G)$, then $x \in N_G(\langle y \rangle)$. Since $N_G(\langle y \rangle)/C_G(y)$ is finite, so $x^m \in C_G(y)$ for some positive integer m. Now $(xy)^m = x^m y^{x^{m-1}} y^{x^{m-2}} \ldots y^x y$ and as $y^x \in \langle x \rangle$, so $(xy)^{mk} = x^{mk}$, where k is the order of y. Hence xy is of infinite order. Thus $xy \in N_G(\langle t \rangle)$ and so $y \in N_G(\langle t \rangle)$. Hence $\langle t \rangle$ is normal in G for every $t \in T(G)$.

Let e be an idempotent in $FT(G)$ and $x \in G\backslash T(G)$. There exists $r \in R$ such that $rex(1-e) \in RT(G)$ and so $1 + rex(1-e) \in U(RG)$ with $(1+rex(1-e))^{-1} = 1 - rex(1-e)$. Now for any $t \in T(G)$,

$$\beta = (1 - rex(1-e))t(1 + rex(1-e)) \in T(URG) \text{ and}$$
$$\beta = t + x\delta - x^2\Theta, \text{ where } \delta = r(t^x e^x(1-e) - e^x(1-e)t) \text{ and}$$
$$\Theta = r^2 e^{x^2}(1-e^x)(te)^x(1-e),\ \delta,\ \Theta \in RT(G).$$

Since $\beta \in TU(RG)$ and $T(U(RG)) \subseteq (RT(G))$, so $\text{supp}(\beta) \subseteq T(G)$. Thus, $\delta = 0$ and $\Theta = 0$. Now $x\delta = 0$ implies that $ex(1-e)t = tex(1-e)$. Hence $ex(1-e)$ commutes with every element of $RT(G)$.

Thus $ex(1-e) = 0$.

Similarly, we have $(1-e)xe = 0$. Thus it follows that $e^x = e$ for every $x \in G\backslash T(G)$.

Now if $y \in T(G)$ and $x \in G\backslash T(G)$, then xy is also of infinite order. So $e^y = (e^x)^y = e^{xy} = e$. Thus e is central in RG. This proves the result. ■

By virtue of the above theorem the problem thus reduces to determine RG such that $T(U(RG)) \subseteq U(RT(G))$. We now assume that R is either the ring of integers $\mathbb{Z}$ or a field K.

For the integral group rings $\mathbb{Z}G$, we have the following situation.

Theorem 2. *Let G be a nontorsion group such that $T(G)$ is a subgroup and that $G/T(G)$ be right ordered. Then, the following conditions are equivalent:*

(1) $TU(\mathbb{Z}G) \subseteq U(\mathbb{Z}T(G))$
(2) *$T(G)$ is either abelian or a Hamiltonian group such that if $T(G)$ is nonabelian, $\alpha \in T(G)$, of odd order n, then the multiplicative order of 2 in $\mathbb{Z}_n$ is an odd number*
(3) $U(\mathbb{Z}G) = U(\mathbb{Z}T(G))G$.

Proof: (1) implies (2). If $T(G)$ is non abelian, then by Theorem 1, $T(G) = A \times E \times K_8$, where A is abelian with every element of odd order, E is elementary abelian 2-group and K_8 is the Quaternion group of order 8.

Let $a \in A$ be of oder n. Then by [4, II.2.6]

$$\mathbb{Q}(\langle a\rangle) \times K_8) = (\mathbb{Q}\langle a\rangle)K_8 = \oplus \sum_{d|n} \mathbb{Q}(\xi_d)K_8.$$

Also $\mathbb{Q}(\xi_n)K_8 \cong \mathbb{Q}(\xi_n) \oplus \mathbb{Q}(\xi_n) \oplus \mathbb{Q}(\xi_n) \oplus \mathbb{Q}(\xi_n) \oplus S$, where S is either a division ring or $M_2(\mathbb{Q}(\xi_n))$. By Theorem 1, every idempotent of $\mathbb{Q}T(G)$

is central in $\mathbb{Q}(G)$, so $\mathbb{Q}(\xi_n)K_8$ has no noncentral idempotents. Thus S is a division ring and therefore, $\mathbb{Q}(\xi_n)K_8$ has no nonzero nilpotent elements. By [**4**, VI.1.13] $a^2+b^2+c^2=0$ has no nonzero solution in $\mathbb{Q}(\xi_n)$ and by [**4**, VI.1.15], this happens provided the multiplicative order of 2 modulo n is odd.

(2) implies (3) is by [1].

(3) implies (1), follows from an easy observation that $U(\mathbb{Z}G)/U(\mathbb{Z}T(G))\cong G/T(G)$. ■

Finally for group algebras we have the following theorem. Here $K*G$ denotes the crossed product of G over K.

Theorem 3. *Let K be a field of characteristic $p>0$, G be a non torsion group such that $T(G)$ is a subgroup and $G/T(G)$ be right ordered. Further let G be such that for every finitely generated subgroup H of G, $T(H)$ is finite. Then $T(U(KG))\subseteq U(KT(G))$ if and only if $T(G)$ is abelian group having no p-elements and every idempotent of $KT(G)$ is central in KG.*

Proof: Suppose that $T(U(KG))\subseteq U(KT(G))$. Then by Theorem 1, every subgroup of $T(G)$ is normal in G with every idempotent of $KT(G)$ central in KG.

If char $K=p>0$ and $t\in T(G)$ with $o(t)=p$, then as $\langle t\rangle$ is normal in G, so $|G:C_G(t)|<\infty$. Since G is non torsion, there exists an element x of infinite order in $C_G(t)$. Then $(1+x(1-t))^p=1$ and $1+x(1-t)\notin KT(G)$. Hence $T(G)$ has no p-elements.

Finally if $T(G)$ is non abelian, then the Quaternion group, $K_8\subseteq T(G)$ and $p\neq 2$ as $T(G)$ has no p-elements. So

$$\mathbb{Z}_pK_8\cong\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus\mathbb{Z}_p\oplus M_2(\mathbb{Z}_p),$$

contains a non central idempotent. Hence $T(G)$ is abelian.

For the converse, we may assume that G is finitely generated and so $T(G)$ is finite. Now

$$KT(G)=\oplus\sum_{i=1}^{m}F_i,\ \text{a direct sum of fields,}$$

since $T(G)$ is finite abelian and p does not divide $|T(G)|$.

It is given that every idempotent of $KT(G)$ is central in KG. Hence

$$KG=KT(G)*G/T(G)=\oplus\sum_{i=1}^{m}F_i*G/T(G)$$

and so

$$U(KG) \cong \operatorname*{Dr}_{i=1}^{m} U(F_i * G/T(G)).$$

Since $G/T(G)$ is right ordered, by [**4**, VI.1.6] $U(F_i * G/T(G))$ has only trivial units and so

$$T(U(KG)) \subseteq \operatorname*{Dr}_{i=1}^{m} T(U(F_i * G/T(G))) = \operatorname*{Dr}_{i=1}^{m} T(U(F_i)) \subseteq U(K(T(G)).$$

This proves the theorem. ■

By Theorem 3 and [**3**] we have

Corollary 4. *Let K be a field of characteristic p and G be nontorsion nilpotent or FC-group having no p-elements. Then $T(U(KG)) \subseteq U(KT(G))$ if and only if $T(U(KG))$ is a subgroup.*

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