Publicacions Matemàtiques, Vol 36 (1992), 359–367.

NON-COMMUTATIVE SEPARABILITY AND GROUP ACTIONS

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Dedicated to the memory of Pere Menal

Abstract

We give conditions for the skew group ring $S * G$ to be strongly separable and H-separable over the ring S. In particular we show that the H-separability is equivalent to S being central Galois extension. We also look into the H-separability of the ring S over the fixed subring R under a faithful action of a group G. We show that such a chain: $S * G$ H-separable over S and S H-separable over R cannot occur, and that the centralizer of R in S is an Azumaya algebra in the presence of a central element of trace one.

In [**A**] we introduced the concept of subring–Galois extensions as a generalization of central Galois extensions and give a generalization of the correspondence theorem given by DeMeyer in [**D**] and Szeto in [**SM**]. Similar correspondence theorems were given by Sugano in [**S**] using H–separability. Separability for non-commutative rings was introduced by Hirata, and the notions of H–separability and "strong" separability were introduced by Hirata in [**HI**] and MacMahon and Mewborn in [**MM**] respectively. Strong separability is a weaker notion than H–separability, but both are special cases of the general notion of separability of ring extensions.

In the case of group actions we present here conditions for strong and H–separability of skew group rings and in particular we show that the skew group ring $S * G$ is H–separable over S if and only if S is a central Galois extension. Furthermore, in this case $S*G$ is a $Z(S)$–Galois

*Partially supported by a grant from the Faculty Development Fund of the University of Michigan–Flint and a fellowship from the Centre de Recerca Matematica, Barcelona, Spain.

extension (in the terminology of [**A**]), allowing us to express $S = Z(S)R$ and $S*G = Z(S)I$ where I is the algebra of G-central functions. We then study the separability of $C_S(R)$ over its fixed subring and give conditions for S to be $C_S(R)$–Galois.

All rings here are associative and have a unity element 1. $Z(R)$ will denote the center of a ring R, and $C_A(B)$ will denote the "centralizer of B in A", i.e. the elements of the ring A which commute with all the elements of the subring B of A.

1. Definitions and Notations

Let B be a subring of a ring A with 1.

The extension $B \subset A$ is called *separable* (or A is *separable* over B) if any of the following equivalent conditions is satisfied:

1) The multiplication map $\mu : A \otimes_B A \to A$ splits as an $(A - A)$-bimodule map.
2) There exists an element $e \in A \otimes_B A$ (called a separability element), such that $\alpha e = e \alpha$ for all $\alpha \in A$ and $\mu(e) = 1$.

The ring A is said to be *strongly separable* over B if $A \otimes_B A \cong K \oplus L$ as $(A - A)$-bimodules, where $\mathrm{Hom}_{A,A}(K, A) = 0$ and $L \oplus H \cong A^n$ for some $(A - A)$-bimodules K, L, H and some positive integer n. In case $K = 0$ we say that A is *H-separable* over B. Strongly separable extensions are separable but the converse is false, see [**MM**].

There is an equivalent definition for this kinds of separability in terms of the natural $(A - A)$-bimodule map $\varphi : A \otimes_B A \to \mathrm{Hom}(\Delta_c, A_c)$ where $\varphi(a \otimes b)(x) = axb$, C is the center of A and Δ is the centralizer of B in $A, C_A(B)$. The ring A is *strongly separable* over B if and only if Δ_c is finitely generated projective C-module and φ is an split epimorphism. Similarly, A is *H-separable* over B if and only if Δ_c is finitely generated projective C-module and φ is an isomorphism. For details see [**HI**] and [**MM**].

Now let's consider group actions. Let S be a ring with 1, let G be a finite group acting faithfully as automorphisms of S and let $R = S^G$ be the fixed ring under G. Writing $g(r) = {}^g r$, the skew group ring $S * G$ is the free left S–module with basis the elements of G and multiplication given by the rule $gs = {}^g s g$ for all $s \in S$ and $g \in G$. Denote by π the element $\sum\limits_{g \in G} g \in S * G$. The action of G on S is said to be G Galois if S is finitely generated projective right R–module and the natural map $\phi : S * G \to \mathrm{End}_R S$ given by $\phi(rg)(x) = r\,({}^g x)$ is a ring isomorphism; or equivalently, there exist elements a_i, b_i (called a *G-Galois* basis) such that $\sum_i a_i\, {}^g b_i = 1$ if $g = 1$ and the sum is 0 if $g \neq 1$ (i.e., $S\pi S = S * G$). The "trace map", $tr : S \to R$ is given by $tr(x) = \sum_{g \in G} {}^g x$ which is an $(R - R)$-bimodule homomorphism.

Let T be a G–stable subring of S (that is ${}^g t \in T$ for all $t \in T$, $g \in G$), we say that S is a *T–Galois extension of R* if the action of G on T is G–Galois. For details and properties, see [**A**]. If X is a subset of S, let $I(X) = \{g \in G /\ {}^g x = x \quad \forall x \in X\}$ be the *"inertia group"* of X, ($I(X)$ is always a subgroup of G).

2. Separability and skew group rings

In [**MS**, theorems 2.2 and 2.3] it is shown that if S is a simple ring, G a finite outer group of automorphisms of S and $F = I(Z(S))$, then $S * G$ is H-separable over $S * F$ and $S * G$ is H-separable over S if and only if F is trivial. But in this case $S * G$ is simple and hence the action of G on S is G-Galois. We'll give a general result relating G-Galois actions with strong and H-separability.

Let $D = C_{S*G}(S)$ and $C = Z(S * G)$. The action of G on S induces a faithful action of G on $S * G$ via conjugation, ${}^g\alpha = g\alpha g^{-1}$ for $\alpha \in S * G$; and G also acts on D. Let M be the inertia group of D, thus G/M acts faithfully on D by ${}^{\overline{h}}\alpha = {}^g\alpha$ for any $g \in \overline{h}$.

Lemma 2.1. $D^G = D^{G/M} = C$.

Proof: The first equality is obvious since M is the inertia group of D. Now let $\alpha \in D^G$, then $\alpha g = g\alpha \quad \forall g \in G$ and by definition of D, $s\alpha = \alpha s \quad \forall s \in S$; hence $\alpha \in C$. Conversely, if $\alpha \in C$, $\alpha g = g\alpha \quad \forall g \in G$ and hence $\alpha \in D^G$, (is clear that $C \subseteq D$). ∎

Theorem 2.2. *Let M be the inertia group of $D = C_{S*G}(S)$ and let C be the center of $S * G$. Assumme there is a central element w in S with $tr_M(w) = 1$. If D is G/M-Galois over C, then $S * G$ is strongly separable over S.*

Proof: Let $\varphi : S * G \otimes_S S * G \to \mathrm{Hom}(D_C, S * G_C)$ be the natural $(S * G - S * G)$-bimodule map, and let $\{a_i, b_i\}$ be a G/M-Galois basis for D over C; then define the maps f_i by $f_i(x) = tr_{G/M}(b_i x)$, thus $f_i \in \mathrm{Hom}(D_C, C_C)$ and $\{a_i, f_i\}$ form a dual projective basis for D over C.

First we show that $\{f_i\}$ is a basis for $\mathrm{Hom}(D_C, S * G_C)$ as $(S * G - S * G)$-bimodule. For, let $\alpha \in D$, $\quad f \in \mathrm{Hom}(D_C, S * G_C)$, then $f(\alpha) = f\left(\sum_i a_i f_i(\alpha)\right) = \sum_i f(a_i)f_i(\alpha) = \sum_i f_i(\alpha)f(a_i)$; thus $f = \sum f(a_i)f_i = \sum f_i f(a_i)$. Now we prove that φ is an epimorphism. Note that $\varphi(g\otimes g^{-1})(\alpha) = g\alpha g^{-1}$, thus $\varphi(g\otimes g^{-1})$ acts as $\overline{g}\in G/M$ on D and $\varphi(g\otimes g^{-1}) = \varphi(h\otimes h^{-1})$ whenever $\overline{g} = \overline{h}$ in $G/M(*)$. Choose $\{h_1,\dots,h_p\}$ a transversal of M in G, then

$$\begin{aligned} f_j(x) = tr_{G/M}(b_j x) &= \sum_{h_i} {}^{h_i}(b_j x) = \sum_{h_i} {}^{h_i}b_j {}^{h_i}x \\ &= \sum_i {}^{h_i}b_j\varphi(h_i\otimes h_i^{-1})(x) = \sum_i \varphi(h_i b_j\otimes h_i^{-1})(x). \end{aligned}$$

Therefore $f_j\in \mathrm{Im}(\varphi)$ and hence φ is epic. Notice that the expression of f_j above is independent of the choice of the transversal of M in G by $(*)$. It is only left to show that φ splits as $(S*G-S*G)$-bimodule homomorphism. Let M be given by the set $\{m_1,\dots,m_q\}$ and let $l_k = \sum_{i,j} h_i m_j w b_k\otimes(h_i m_j)^{-1}\in S*G\otimes_S S*G$. Then

$$\varphi(l_k) = \sum_{i,j} {}^{h_i m_j}w^{\overline{h_i}\overline{m_j}} b_k\varphi(h_i m_j\otimes(h_i m_j)^{-1}) \quad \text{and by} \quad (*)$$

$$= \sum_{i,j} {}^{h_i m_j}w^{\overline{h_i}} b_k\varphi(h_i\otimes h_i^{-1}) = \sum_i\left(\sum_j {}^{h_i m_j}w\right)\varphi\left(h_i b_k\otimes h_i^{-1}\right) = f_k.$$

Hence we may define the map $\psi: \mathrm{Hom}(D_C, S*G_C)\to S*G\otimes_S S*G$ by linearity with $\psi(f_k) = l_k$. To show that ψ is an $(S*G-S*G)$-bimodule map, we need to show $\alpha l_k = l_k\alpha$ for all $\alpha\in S*G$. Let $r\in S$, since $b_k\in D$ and w is central in S we have:

$$\begin{aligned} rl_k &= \sum_{i,j} r(h_i m_j)wb_k\otimes(h_i m_j)^{-1} \\ &= \sum_{i,j}(h_i m_j)^{(h_i m_j)^{-1}}rwb_k\otimes(h_i m_j)^{-1} \\ &= \sum_{i,j} h_i m_j wb_k\otimes {}^{(h_i m_j)^{-1}}r(h_i m_j)^{-1} \\ &= \sum_{i,j} h_i m_j wb_k\otimes(h_i m_j)^{-1}r = l_k r, \end{aligned}$$

and if $g\in G$, we have:

$$gl_k = \sum_{i,j} gh_i m_j wb_k\otimes(h_i m_j)^{-1} = \sum_{i,j}(gh_i)m_j wb_k\otimes((gh_i)m_j)^{-1}g\ ,$$

but $\{gh_i\}$ is another transversal of M in G, hence by $(*)$ $gl_k = l_k g$ and therefore ψ is an $(S*G - S*G)$-bimodule map. We then have

$$\varphi(\psi(f)) = \varphi\left(\psi\left(\sum_k f(a_k)f_k\right)\right) = \varphi\left(\sum_k f(a_k)l_k\right)$$
$$= \sum_k f(a_k)\varphi(l_k) = \sum_k f(a_k)f_k = f.$$

and so ψ splits φ. ■

Now we want to show an equivalent condition for the skew group ring $S*G$ to be H-separable over S. We start by giving some notation and some neccesary conditions assuming all the notation as in theorem 2.2.

For every $g \in G$ define $\phi_g = \{r \in S/\ r\,{}^g s = sr \quad \forall s \in S\}$. If $\phi_g \neq 0$ g is said to be *ω-inner*, and if $\phi_g = 0$ for every $g \neq 1$ G is said to be *ω-outer*. It is not difficult to see that $D = \sum_{g\in G} \phi_g g$.

For the proof of the main theorem we will need a result that appears in [**A**], and we reproduce here for completeness.

Proposition 2.3. ([**A**, prop. 3.3]) *Assume $S*G$ is H-separable over S. Then G is ω–outer and $D = Z(S)$.*

Proof: Since $S*G \cong \sum_{g\in G}^{\oplus}(S\otimes g)$ as S-S–bimodules, $C_{S*G}(D) = S$ by [**S**, proposition 1.3]. Hence $Z(D) = C_{S*G}(D)\cap D \subseteq S$ and therefore $C \subseteq Z(D) \subseteq Z(S)$. Now let $r_g \in \phi_g$, so $x = r_g g \in D$, and hence $tr_{G/M}(x) = \sum_{\overline{h}\in G/M} hr_g gh^{-1} = \sum_{\overline{h}\in G/M} {}^h r_g hgh^{-1} \in C \subseteq S$. Thus ${}^h r_g = 0$ if $hgh^{-1} \neq 1$, this is if $g \neq 1$ and so $r_g = 0$ if $g \neq 1$. Therefore $\phi_g = 0$ if $g \neq 1$, and so G is ω–outer. By the comment above $D = \phi_1 \cdot 1$, so $D = Z(S)$. ■

Theorem 2.4. *Let D, M, C, S, G and w as in theorem 2.2. D is G-Galois over C and M is trivial if and only if $S*G$ is H-separable over S.*

Proof: (⇒) Assume the same notation as in the proof of theorem 2.2; so now we have $l_k = \sum_{i,j} h_i w b_k \otimes (h_i)^{-1} = \sum_i h_i b_k \otimes h_i^{-1}$, and hence

$$\psi\cdot\varphi(1\otimes 1) = \sum_k \varphi(1\otimes 1)(a_k)l_k = \sum_k a_k \sum_i h_i b_k \otimes h_i^{-1}$$
$$= \sum_i \left(\sum_k a_k{}^{h_i} b_k\right) h_i \otimes h_i^{-1} = 1\otimes 1.$$

Thus $\psi \cdot \varphi = \mathrm{id}_{S*G \otimes_S S*G}$ and φ is an isomorphism.

($\Leftarrow$) Assume $m \in M$ and $\alpha \in D$, then $\varphi(m \otimes m^{-1})(\alpha) = m\alpha m^{-1} = \alpha = \varphi(1 \otimes 1)(\alpha)$, but φ is an isomorphism, hence $M = 1$. Now we will show D is G-Galois over C. By proposition 2.3 D is commutative, and by [**S**, proposition 1.3] D is a separable C–algebra. Assume that there exists a non zero idempotent $e \in D$ and a pair $h \neq g \in G$ such that ${}^g x e = {}^h x e$ for all $x \in D$. If we let $e' = {}^{g^{-1}} e$, we have $e' \neq 0$ and $xe' = {}^{g^{-1}h} x e' = e' \, {}^{g^{-1}h} x$. But G is ω–outer, hence $g^{-1}h = 1$, thus $g = h$, a contradiction. Therefore D is G–Galois over C by [**DI**, proposition III. 1.2]. ■

If S is a simple ring and G is outer, then $Z(S)$ is a field, and hence G/M is G/M-Galois over $Z(S)$ where $M = I(Z(S))$. Therefore applying the previous theorems we obtain an improvement of [**MS**, Theorem 2.3 and Theorem 2.2,ii)] :

Corollary 2.5. *Let S be a simple ring and G be outer.*

i) *If $\exists\, w \in Z(S)$ such that $tr_M(w) = 1$, then $S * G$ is strongly separable over S.*
ii) *$S * G$ is H-separable over S if and only if $M = 1$.*

We can see now a relationship between H-separability and T-Galois extensions in the following corollaries:

Corollary 2.6. *$S * G$ is H-separable over S if and only if S is a central Galois extension of R.*

Proof: ($\Leftarrow$) $\exists\, a_i, b_i \in Z(S)$ such that $\sum a_i \pi_G b_i = 1$, but $Z(S) \subseteq C_{S*G}(S) = D$ and D is G-invariant, hence D is G-Galois over $D^G = C$ and by theorem 2.4 $S * G$ is H-separable over S.

($\Rightarrow$) Obvious from the theorem 2.4 and proposition 2.3. ■

The case of commutative rings is now determined:

Corollary 2.7. *Let S be a commutative ring. $S * G$ is H–separable over S if and only if S is G–Galois over R.*

Consider again the action of G on $S * G$ by conjugation. It follows that the centralizer of G in $S * G$ is precisely equal to the fixed ring $(S * G)^G = I$, which in the language of C^*–algebras is called the algebra of G–central functions, (see [**OP**]). Hence we obtain:

Proposition 2.8. *Let $S * G$ be H–separable over S. Then $S * G$ is a $Z(S)$–Galois extension of I and therefore $S * G = Z(S)I$.*

3. H–separability and fixed ring

Now we study some neccesary conditions for the ring S to be H–separable over the fixed ring R. The centralizer of R in S will be denoted by E and all the notation from Section 2 will be assumed.

Let X be a G invariant subset of S. It can be easily seen that $C_S(X)$ is a G–invariant subring of S and thus G acts on it. Furthermore we have that $(C_S(X))^G = C_R(X)$. Hence, if we take $X = R$ we get the following relation: $E^G = Z(R) \subseteq Z(E)$. On the other hand it is obvious that $Z(S) \subseteq Z(E)$.

Proposition 3.1. *Let S be H–separable over R. Then:*

1) *G is ω–inner.*
2) *$R = C_S(E)$*
3) *$E^G = Z(R) = Z(E)$*

Proof: 1) Recall that $\phi_g = \{r \in S/\ r\,{}^g s = sr \quad \forall s \in S\}$. Consider the $(S-S)$-bimodule Sg. Then $Eg = C_{Sg}(R)$ and $\phi_g g = C_{Sg}(S)$, therefore we get $Eg = E \otimes_{Z(S)} \phi_g g$ and hence $\phi_g \neq 0$.

2) It is clear that $R \subseteq C_S(E)$. Now, let $r \in C_S(E)$ and let $g \in G$. We can see g as an element of $\mathrm{Hom}_{R-R}(S,S)$ which is isomorphic to $E \otimes_{Z(S)} E$ by [**H2**, proposition 4.7]. Thus there exists elements $d_i, e_i \in E$ such that ${}^g x = \sum_i d_i x e_i$ for all $x \in S$, and therefore ${}^g r = \sum_i d_i r e_i = r \sum_i d_i e_i = r$; so $r \in R$.

3) By the comments above, it is only neccesary to show the second equality. But, by part 2) we have: $Z(R) = R \cap C_S(R) = R \cap E = C_S(E) \cap E = Z(E)$. ∎

Remark. Note that in proposition 2.3 we showed that if the skew group ring $S * G$ is H–separable over the base ring S, then the action of G must be ω–outer. Here we obtain the opposite condition, if the ring S is H–separable over the fixed ring R, the action of G must be ω–inner. Therefore we cannot have a "chain" of H–separabble extensions in faithful group actions.

Proposition 3.2. *Let S be H–separable over the fixed ring R and assume there exists a central element in S of trace one. Then E is separable over $Z(S)$ and H–separable over E^G (so E is an Azumaya algebra).*

Proof: The existence of a central element of trace 1 makes the trace map $tr : S \to R$ split as a $(R-R)$-bimodule map. Hence R is a direct summand of S as $(R-R)$-bimodules and by [**S**, proposition 1.3] E is separable over $Z(S)$. Furthermore, since $Z(S) \subseteq Z(E)$, the theorem of Azumaya for separable extension over commutative rings implies that E is separable over its center $Z(E)$ and $Z(E)$ is separable over $Z(S)$. Therefore, E is H–separable over $Z(E)$, which by proposition 3.1 is equal to the fixed subring E^G. ■

The action of G on S induces an action on E, but we need to consider the inertia subgroup $K = I(E)$. In this way G/K acts faithfully on E. We now describe conditions for E to be a Galois extension of E^G.

Proposition 3.3. *$g \in K$ if and only if $\phi_g \subseteq Z(E)$.*

Proof: Since $\phi_g \subseteq E$ the neccesary condition is obvious. Now let $\alpha \in \phi_g \subseteq Z(E)$; then $\alpha({}^g x - x) = 0$ for all $x \in E$ and therefore ${}^g x = x$ for all $x \in E$. ■

Theorem 3.4. *Let S be H–separable over R and assume there is a central element of trace 1. S is an E–Galois extension of R if and only if $C = E^G$ and K is trivial.*

Proof: ($\Rightarrow$) By definition of E-Galois extension, K is trivial and the action of G on E is G-Galois, moreover by proposition 3.2 E is H–separable over E^G. Furthermore, by **[S2]**, $E = \sum_g \phi_g$ is a direct sum and $\phi_g = Cx_g$, thus proposition 3.3 implies that $Z(E) = C$, so proposition 3.1 gives us the result.

($\Leftarrow$) Since K is trivial and the fixed elements in E coincide exactly with the central elements we have that the sum $\sum_g \phi_g$ is direct; moreover in this case $E = C_E(E^G)$ and $E^G = Z(E)$ giving us $C_E(E^G)$ equal to the direct sum of the correspondent ϕ_g'. Thus by **[S2**, theorem 1.2] the action of G on E is G–Galois. ■

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Rebut el 25 de Novembre de 1991