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A UNIQUENESS THEOREM FOR INVARIANTLY HARMONIC FUNCTIONS IN THE UNIT BALL OF $\mathbb{C}^n$

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Abstract

We prove a boundary uniqueness theorem for harmonic functions with respect to Bergman metric in the unit ball of $\mathbb{C}^n$ and give an application to a Runge type approximation theorem for such functions.

Let B be the unit ball in $\mathbb{C}^n$ and S its boundary. The invariant laplacian Δ in B is the Laplacian associated to the Bergman metric and it is given in coordinates by

$$\Delta = (1 - |z|^2) \sum_{i,j=1}^{n} (\delta_{ij} - \bar{z}_i z_j) \bar{D}_i D_j.$$

The term invariant comes from the fact that it commutes with all automorphisms Ψ of B: $\Delta(u \circ \Psi) = \Delta u \circ \Psi$. Correspondingly, those functions $u \in C^2(B)$ annihilated by Δ are called *invariantly harmonic* or M-harmonic (see [**4**, chapter 4] for the more relevant properties of these functions).

The aim of this note is to give a boundary uniqueness theorem for M-harmonic functions and an application to a Runge type approximation problem.

1. The uniqueness theorem essentially states that S, though Δ completely degenerates there, is non-characteristic for a certain Cauchy problem:

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Theorem. *Let U be a ball centered at $\zeta \in S$, and let $u \in C^\infty(U \cap \bar{B})$ satisfy $\Delta u = 0$ in U. Then from*

$$u = \frac{\partial^n u}{\partial r^n} = 0 \text{ on } U \cap S$$

it follows that all derivatives of u are zero on $U \cap S$ (hence $u \equiv 0$ if it is real-analytic accross S).

The proof will show in fact that u and $\frac{\partial^n u}{\partial r^n}$ determine all the others derivatives of u on S. The statement suggests that the following Cauchy-Kowalevski type theorem is probably true: if f, g are real-analytic functions defined on $U \cap S$, there is another ball $V \subset U$ containing ζ and a real-analytic function u in V such that $\Delta u = 0$ in $V \cap B$ and $u = f$, $\frac{\partial^n u}{\partial r^n} = g$ on $V \cap S$.

Proof: Let $\Delta_0 = \sum_{i,j=1}^n (\delta_{ij} - \bar{z}_i z_j)\bar{D}_i D_j$ and let $R = \sum_1^n z_j D_j$ be the radial (holomorphic) derivative; write $R = N + iT$, then $N = r\frac{\partial}{\partial r}$ and T is a real tangent field to S. We will show that u alone determines $N^j u$, $j \le n-1$ and that u, $N^n u$ determine all derivatives at points of S. A computation shows that

$$\Delta_0 N - N\Delta_0 = \Delta_0 + N^2 + T^2$$

$$\Delta_0 T - T\Delta_0 = 0. \tag{1}$$

$$TN - NT = 0. \tag{2}$$

From this it easily follows by induction on k that

$$\Delta_0 N^k = P_k(N)\Delta_0 + kN^{k+1} + R_k(N,T) \tag{3}$$

where $P_k(x)$ is a monic polynomial of degree k and $R_k(x,y)$ is of degree $\le k+1$ in x, y, but of degree $\le k$ in x.

Next, the following *normal-tangential decomposition* in **[2]** is needed

$$\Delta_0 = \frac{1}{|z|^2}\left\{(1-|z|^2)R\bar{R} + \Lambda + (n-1)N\right\}.$$

Here Λ is the box-laplacian on S; its particular expression will not be needed, only the fact that it is a tangential operator. It follows that at points $w \in S$

$$(\Delta_0 v)(w) = (\Lambda v)(w) + (n-1)Nv(w). \tag{4}$$

Applying this to u we see that $Nu = 0$ on S. Assume by induction we have proved $N^{(k)}u = 0$ on S whenever $u \in C^\infty(U \cap \bar{B})$ is M-harmonic and zero on $U \cap S$, $k \leq n-1$. Then by (3) and (4)

$$(n-1)N^{(k+1)}u = \Delta_0 N^{(k)}u = kN^{k+1}u + R_k(N,T)u$$

on $U \cap S$. By (2),

$$R_k(N,T)u = \sum_{\substack{i \leq k \\ i+j \leq k+1}} N^i T^j u$$

and by (1), $T^j u$ is also M-harmonic and obviously zero on S. By the induction hypothesis, $R_k(N,T)u = 0$ on $U \cap S$. Then we conclude that $N^{(k+1)}u = 0$ if $k < n-1$. If $k = n-1$ we cannot conclude $N^{(n)}u = 0$ but it is clear that if this is known to hold, then the induction can continue and so $N^{(j)}u = 0$ for all j on $U \cap S$, which proves the theorem. ■

There is some connection of this result with a result from Folland [**1**] according to which an M-harmonic function u in the whole ball of class C^n up to the boundary must be in fact pluriharmonic.

2. As an application of the theorem we prove:

Theorem. *Let $K \subset B$ be a compact set such that $B \backslash K$ is connected. Then, every v satisfying $\Delta v = 0$ in a neighbourhood of K is the uniform limit on K of a sequence of M-harmonic functions u_n in B, continuous on $\bar{B}$.*

It must be pointed out that this result can be proved as well by combining a general result of [**3**] on analytic-hypoelliptic operators and [**4**, 5.5.4]. Our proof proceeds by duality and relies on some well-known facts that we proceed to recall.

There is a decomposition formula, valid at least for $u \in C^2(\bar{B})$,

$$u(z) = \int_S P(\zeta,z)u(\zeta)\,d\sigma(\zeta) + \int_B \Delta u(\zeta)G(\zeta,z)\,d\lambda(\zeta) \tag{5}$$

that corresponds to the Poisson-Green formula in Euclidean space. Here $d\sigma$ is the normalized Lebesgue measure on S, $d\lambda(\zeta) = (1-|\zeta|^2)^{-n-1}dV(\zeta)$ is the invariant measure, $P(\zeta,z)$ is the invariant Poisson (or Poisson-Szegö) kernel

$$P(\zeta,z) = \frac{(1-|z|^2)^n}{|1-\bar{\zeta}z|^{2n}},\ \zeta \in S,\ z \in B,$$

and $G(\zeta, z)$ is the Green function with pole at z,

$$G(\zeta, z) = G(\varphi_z(\zeta), 0) = c_n \int_{|\varphi_z(\zeta)|^2}^{1} \frac{(1-t)^{n-1}}{t^n}\, dt,$$

c_n being a constant and φ_z the automorphism of B, unique up to unitary transformations, that sends z to 0 and 0 to z. Moreover,

$$1 - |\varphi_z(\zeta)|^2 = \frac{(1-|z|^2)(1-|\zeta|^2)}{|1-\bar{\zeta} z|^2},$$

so that G is in fact symmetric. One way of obtaining (5) is to write precisely the Poisson-Green formula for $u \circ \varphi_z$ at 0 and change variables in the resulting integrals.

A second (and better) way of looking at (5) is through the Green identity in the Bergman metric for $A \subset B$

$$\int_A (u\Delta v - v\Delta u)\, d\lambda = \int_{\partial A} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS.$$

Here ν is the outward unitary normal (by the Bergmann) metric to ∂A and dS is the induced measure on ∂A. Formally, one obtains (5) by specializing to $A = B$, $v(\zeta) = G_z(\zeta) = G(\zeta, z)$ and checking that

$$\begin{aligned} &\Delta_\zeta G(\zeta, z)\, d\lambda(\zeta) = \delta_z, && z \in B \\ &G(\zeta, z) = 0 && \zeta \in S \\ &\frac{\partial}{\partial \nu_\zeta} G(\zeta, z)\, dS = P(\zeta, z)\, d\sigma && \zeta \in S,\ z \in B \end{aligned} \tag{6}$$

(in a rigorous way one should choose as A the ball of radious $r < 1$ and then make $r \to 1$).

Formula (5) implies the following facts:

(a) The general form of an M-harmonic function u in B, continuous on $\bar{B}$ is

$$u(z) = P[f](z) = \int_S P(\zeta, z) f(\zeta)\, d\sigma(\zeta)$$

with $f \in C(S)$; equivalently, $P[f]$ is the unique solution of the Dirichlet problem $\Delta u = 0$ in B, $u = f$ on S.

(b) For $u \in C^2(B)$ of compact support

$$u(z) = \int_B \Delta u(\zeta) G(\zeta, z)\, d\lambda(\zeta)$$

i.e. u coincides with the Green potential of its Laplacian. The simmetry of G then implies that for a measure μ with compact support in B, the Green potential $G\mu$

$$G\mu(z) = \int_B G(\zeta, z)\, d\mu(\zeta)$$

satisfies $(\Delta G\mu)\, d\lambda = d\mu$ in the weak sense, and in particular $\Delta G\mu = 0$ in the usual sense off the support of μ.

Finally, we will need a reformulation of (6) in terms of the Euclidean normal, which is

$$\text{(c)} \qquad P(\zeta, z) = c_n \frac{\partial^n}{\partial r^n} G(r\zeta, z)\big|_{r=1}, \ \zeta \in S, \ z \in B.$$

This is because what (6) really means is, as said before,

$$P(\zeta, z) = \lim_{r \to 1} \frac{\partial}{\partial \nu} G(r\zeta, z) \frac{dS}{d\sigma}.$$

Since $\nu = c_n(1-r)^n \frac{\partial}{\partial r}$ and $dS = c_n(1-r)^{1-2n}\, d\sigma$ (we denote by c_n all constants depending on n), (c) follows by L'Hopital's rule. Alternatively, (c) can be proved of course by direct computation.

Note that $|G(\zeta, z)| = 0(1-|\zeta|^2)^n$ for a fixed z. Hence $\frac{\partial^j}{\partial r^j} G(r\zeta, z) = 0$ for $\zeta \in S$, $j = 0, \ldots, n-1$.

Proof of the theorem: Let μ be a measure on K which is orthogonal to all M-harmonic functions in B, continuous on $\bar{B}$. By (a) above this is equivalent to

$$\text{(7)} \qquad \int_K P(\zeta, z)\, d\mu(z) = 0 \text{ for } \zeta \in S.$$

We consider the Green potential of μ

$$G(\mu)(w) = \int_K G(z, w)\, d\mu(z)$$

so that $\Delta G(\mu) = 0$ off K. Moreover $G(\mu)$ is still defined and is real analytic in a neighbourhood of $\bar{B}$, off K, because so is each φ_z for $z \in K$. Note that all such potentials satisfy

$$\frac{\partial^j}{\partial r^j} G(\mu)(\zeta) = 0, \ j = 0, \ldots, n-1, \ \zeta \in S.$$

By (c), (7) says that also

$$\frac{\partial^n}{\partial r^n} G(\mu)(\zeta) = 0, \ \zeta \in S.$$

Therefore, by the theorem in Section 1, all derivatives of $G(\mu)$ vanish at S and, since $B\backslash K$ is assumed to be connected, we conclude that $G(\mu)$ is identically zero in $B\backslash K$.

Let now v as in the statement; multiplying by a test function, we can assume that v is compactly supported in B. By (b),

$$v(z) = \int_B \Delta v(\zeta) G(\zeta, z)\, d\lambda(\zeta) = \int_{B\backslash K} \Delta v(\zeta) G(\zeta, z)\, d\lambda(\zeta).$$

Hence by Fubini's theorem

$$\begin{aligned}\int_K v(z)\, d\mu(z) &= \int_{B\backslash K} \Delta v(\zeta) \left\{ \int_K G(\zeta, z)\, d\mu(z) \right\} d\lambda(\zeta) = \\ &= \int_{B\backslash K} \Delta v(\zeta) G\mu(\zeta)\, d\lambda(\zeta) = 0\end{aligned}$$

which finishes the proof, by Hahn-Banach's theorem. ■

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