LOCAL COHOMOLOGY IN CLASSICAL RINGS*

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In memoriam of Prof. Pere Menal i Brufal

Abstract

The aim of this paper is to establish the close connection between prime ideals and torsion theories in a non necessarily commutative noetherian ring. We introduce a new definition of support of a module and characterize some kinds of torsion theories in terms of prime ideals. Using the machinery introduced before, we prove a version of the Mayer-Vietoris Theorem for local cohomology and establish a relationship between the classical dimension and the vanishing of the groups of local cohomology on a classical ring.

In this paper we show the relationship between prime ideals and torsion theories on a left noetherian, non necessarily commutative, ring R. The techniques we use are based on the prime ideals associated to a left R-module M and on its support, which we will define here in Section 2. All the rings in this paper are left noetherian.

In Section 1, we provide the interaction between the associated prime ideals of a left R-module M and if it is torsion or torsionfree, we give a characterization of symmetric torsion theories which is useful to characterize stable and symmetric torsion theories. Recall that a torsion theory σ is symmetric if for every $a \in L(\sigma)$, there is a two-sided ideal $b \subseteq a$, and $\sigma$ is stable is the class of all $\sigma$-torsion left R-modules is closed under taking essential extensions. In Section 2, we introduce the support of a left R-module and give its basic properties. We apply, in Section 3, the technique introduced before to the local cohomology. First we prove a version of the Mayer-Vietoris Theorem to

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local cohomology, and secondly, we establish the relationship between
the classical dimension of a left \( R \)-module and the dominant dimension
relative to a torsion theory \( \sigma \).

Let us recall some results on associated prime ideals and tertiary de-
composition of modules on a left noetherian ring. These results are
available in the literature, but we include them here for completeness.

Let \( R \) be a left noetherian ring, a left \( R \)-module \( M \) is prime if
\( \text{Ann}_R(N) = \text{Ann}_R(L) \) for every submodule \( L \) of \( N \). Let \( M \) be a left
\( R \)-module, a prime ideal \( p \) of \( R \) is associated to \( M \) if there is a prime
submodule \( N \) of \( M \) such that \( p = \text{Ann}_R(N) \). The set of all prime ideals
associated to \( M \) is called \( \text{Ass}_R(M) \). Every maximal element in the
set of all the annihilators of non zero submodules of \( M \) is a prime ideal
associated to \( M \), so if \( M \) is a non zero left \( R \)-module, then \( \text{Ass}_R(M) \neq \emptyset \).

Let \( M \) be a non zero left \( R \)-module, and \( p \) a prime ideal, we say \( M \)
is \( p \)-cotertiary if \( \text{Ass}_R(M) = \{p\} \). A proper submodule \( N \) of \( M \) is \( p \)-tertiary in \( M \) if \( M/N \) is \( p \)-cotertiary. The left \( R \)-module \( M \) (resp. a
submodule \( N \) of \( M \)) is cotertiary (resp. tertiary) if it is \( p \)-cotertiary
(resp. \( p \)-tertiary) for some prime ideal \( p \).

Lemma 0.1. Let \( M \) be a left \( R \)-module. The following statements are
equivalent:

1. \( M \) is \( p \)-cotertiary.
2. \( \text{Ann}_M(p) \) is essential in \( M \) and \( p \) contains all the two-sided ideals
   that annihilate some non zero submodule of \( M \).

Proof: 1 \( \Rightarrow \) 2). If \( M \) is \( p \)-cotertiary, then \( \text{Ass}_R(M) = \{p\} \), let \( H \subseteq M \)
be a non-zero submodule such that \( H \cap \text{Ann}_M(p) = 0 \), so \( \text{Ass}_R(H) \subseteq \text{Ass}_R(M) = \{p\} \), and \( \text{Ass}_R(H) = \{p\} \), which is a contradiction.

2 \( \Rightarrow \) 1). Let \( p, q \in \text{Ass}_R(M) \), and \( N \subseteq M \) a prime submod-
ule such that \( q = \text{Ann}_R(N) \), if \( \text{Ann}_R(p) \) is essential in \( M \), then
\( 0 \neq N \cap \text{Ann}_M(p) \), and \( q \subseteq p \), so we have \( q = p \). \( \blacksquare \)

Let \( M \) be a non zero finitely generated left \( R \)-module. A tertiary
decomposition of a submodule \( N \) of \( M \) is a finite family of tertiary sub-
modules \( \{N_1, \ldots, N_r\} \) such that

1. \( N = N_1 \cap \cdots \cap N_r \).
2. The decomposition is irreducible.
3. If \( \text{Ass}_R(M/N_q) = \{p_i\} \), then \( p_i \neq p_j \) for \( i \neq j \).

Lesieur and Croisot proved in [5] that if \( M \) is a non zero finitely
generated \( R \)-module, then every submodule \( N \) of \( M \) has a tertiary de-
composition, and if \( N = N_1 \cap \cdots \cap N_r = L_1 \cap \cdots \cap L_s \) are two tertiary
decomposition of \( N \) in \( M \), then \( r = s \) and \( \{p_1, \ldots, p_r\} = \{q_1, \ldots, q_s\} \).
More results related to tertiary submodules can be found in Stenström's book [8].

1. Torsion theories and associated prime ideals

In this section we will study the relationship between torsion theories and prime ideals on left noetherian rings, we will consider mainly symmetric torsion theories.

Let $R$ be a ring and $\sigma$ a torsion theory in $R$-mod, we define

$$Z(\sigma) = \{ p \in \text{Spec}(R); R/p \in T_\sigma \}.$$  

and

$$K(\sigma) = \{ p \in \text{Spec}(R); R/p \in F_\sigma \}.$$  

In some cases $\{Z(\sigma), K(\sigma)\}$ is a partition of $\text{Spec}(R)$.

Lemma 1.1. [4] Let $R$ be a ring and $\sigma$ a torsion theory in $R$-mod, then either $p \in Z(\sigma)$ or $p \in K(\sigma)$.

Lemma 1.2. Let $R$ be a ring and $\sigma$ a torsion theory in $R$-mod, then for every $\sigma$-torsionfree left $R$-module $M$ we have $\text{Ass}_R(M) \subseteq K(\sigma)$.

Proof: If $q \in \text{Ass}_R(M)$ and $q \notin K(\sigma)$, then $q \in Z(\sigma)$; therefore there is $m \in M$ such that $q = \text{Ann}_R(Rm)$ and an epimorphism $R/q \to Rm$; since $R/q$ is $\sigma$-torsion, so $Rm$ is also $\sigma$-torsion, which is a contradiction, therefore it must be $\text{Ass}_R(M) \subseteq K(\sigma)$.

To prove the converse it is necessary to put conditions on the torsion theory, as we will see later.

Proposition 1.3. Let $R$ be a ring and $\sigma$ a torsion theory in $R$-mod, then the following statements are equivalent for any left $R$-module $M$:

1. $\sigma$ is symmetric.
2. If $M$ is $\sigma$-torsion, then $\text{Ass}_R(M) \subseteq Z(\sigma)$.
3. If $\text{Ass}_R \subseteq K(\sigma)$, then $M$ is $\sigma$-torsionfree.

Proof: 1 $\Rightarrow$ 2). Let $p \in \text{Ass}_R(M)$, then there is $N \subseteq M$ such that $p = \text{Ann}_R(N)$, we can assume $N$ is cyclic and generated by an element $n$, therefore $p$ is the biggest two-sided ideal contained in $\text{Ann}_R(n) \in \mathcal{L}(\sigma)$, so $p \in \mathcal{L}(\sigma)$.

2 $\Rightarrow$ 1). Let $a \in \mathcal{L}(\sigma)$, we consider a chain

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = R/a,$$
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such that $M_{i+1}/M_i$ is $p_i$-cotertorial and $p_i(M_{i+1}/M_i) = 0$. We have that every $M_{i+1}/M_i$ is $\sigma$-torsion, therefore $p_i \in \text{Ass}_R(M_{i+1}/M_i) \subseteq Z(\sigma) \subseteq \mathcal{L}(\sigma)$, and so $p_1 \cdots p_n (R/a) = 0$, therefore $p_1 \cdots p_n \subseteq a$; finally, since the product of elements of $\mathcal{L}(\sigma)$ is also in $\mathcal{L}(\sigma)$, we have that $\sigma$ is symmetric.

2 $\Rightarrow$ 3). If $M$ is not $\sigma$-torsionfree, so $\sigma(M) \neq 0$, and $\emptyset \neq \text{Ass}_R(\sigma(M)) \subseteq \text{Ass}_R(M) \subseteq K(\sigma)$; on the other hand $\sigma(M)$ is $\sigma$-torsion, so $\text{Ass}_R(\sigma(M)) \subseteq Z(\sigma)$, which is a contradiction.

3 $\Rightarrow$ 2). Let $M$ be a $\sigma$-torsion $R$-module, if $\text{Ass}_R(M) \not\subseteq Z(\sigma)$, then there is some $q \in \text{Ass}_R(M)$, and $q \in K(\sigma)$. Therefore there is $0 \neq N \subseteq M$ such that $\text{Ass}_R(N) = \{q\}$, by the hypothesis we have $N \in \mathcal{F}_\sigma$, which is a contradiction. ■

As a consequence, for any left $R$-module $M$ and any symmetric torsion theory $\sigma$, we have $M$ is $\sigma$-torsionfree if, and only if, $\text{Ass}_R(M) \subseteq K(\sigma)$. The analogous result for $\sigma$-torsion modules will characterize stable and symmetric torsion theories. To reverse the condition (2) in Proposition 1.3 we need consider a new condition on $\sigma$.

Lemma 1.4. Let $R$ be a ring and $\sigma$ a stable torsion theory in $R$-mod; for any left $R$-module $M$, if $\text{Ass}_R(M) \subseteq Z(\sigma)$, then $M$ is $\sigma$-torsion.

Proof: We can assume $M$ is finitely generated, let us consider a tertorial decomposition of 0 in $M$,

$$0 = N_1 \cap \ldots \cap N_n,$$

with $N_i$, $p_i$-tertierary in $M$, $1 \leq i \leq n$, $\text{Ass}_R(M) = \{p_1, \ldots, p_n\}$ and $p_i \neq p_j$ if $i \neq j$; so there is a monomorphism $M \rightarrow \oplus_{i=1}^n M/N_i$. Let $X$ be a $p$-cotertorial left $R$-module with $p \in \mathcal{L}(\sigma)$, so $\text{Ann}_X(p)$ is essential in $X$, for every $x \in \text{Ann}_X(p)$ we have $x \in \sigma(X)$, then $\text{Ann}_X(p) \subseteq \sigma(X)$, and since $\sigma$ is stable, $X$ is $\sigma$-torsion. As a consequence every $M/N_i$ is $\sigma$-torsion and so $M$ is $\sigma$-torsion. ■

Lemma 1.5. Let $R$ be a ring and $\sigma$ a symmetric stable torsion theory in $R$-mod, then for every left $R$-module $M$ we have:

1. $\text{Ass}_R(\sigma(M)) = \text{Ass}_R(M) \cap Z(\sigma)$.
2. $\text{Ass}_R(M/\sigma(M)) = \text{Ass}_R(M) \cap K(\sigma)$.
3. $\text{Ass}_R(M) = \text{Ass}_R(\sigma(M)) \cup \text{Ass}_R(M/\sigma(M))$.

Proof: Let $\mathcal{P} = \text{Ass}_R(M) \cap Z(\sigma)$, then there is a submodule $N$ of $M$ such that $\text{Ass}_R(N) = \text{Ass}_R(M) \cap Z(\sigma)$ and $\text{Ass}_R(M/N) = \text{Ass}_R(M) \cap K(\sigma)$. Then we have $\text{Ass}_R(N) \subseteq Z(\sigma)$, so $N$ is $\sigma$-torsion,
and \( \text{Ass}_R(M/N) \subseteq \mathcal{K}(\sigma) \), so \( M/N \) is \( \sigma \)-torsionfree, it follows that \( N = \sigma(M) \).

This result can be used to provide a characterization of stable symmetric torsion theories in the following way.

**Proposition 1.6.** Let \( R \) be a ring and \( \sigma \) a symmetric torsion theory in \( R\text{-mod} \), then \( \sigma \) is stable if, and only if, \( \mathcal{T}_\sigma = \{ M; \text{Ass}_R(M) \subseteq \mathcal{Z}(\sigma) \} \).

It is possible to characterize non necessarily symmetric torsion theories \( \sigma \) such that \( \text{Ass}_R(M) \subseteq \mathcal{Z}(\sigma) \) implies \( M \) is \( \sigma \)-torsion, like those torsion theories satisfying a property of the Artin-Rees type [6].

**Lemma 1.7.** Let \( R \) be a ring and \( \sigma \) a symmetric torsion theory in \( R\text{-mod} \), then

\[
\sigma = \bigwedge \{ \sigma_{R-p}; p \in \mathcal{K}(\sigma) \}.
\]

More generally, it is possible to associate to a set of prime ideals \( \mathcal{K} = \{ p_i; i \in I \} \) a symmetric torsion theory \( \sigma_{\mathcal{K}} \) defined by \( \sigma_{\mathcal{K}} = \bigwedge \{ \sigma_{R-p_i}; i \in I \} \). It is arise the following question: When is \( \mathcal{K}(\sigma_{\mathcal{K}}) = \mathcal{K} \)? we call a set \( \mathcal{K} \) of prime ideals is generically closed if for any pair of prime ideals \( p \subseteq q \) such that \( q \in \mathcal{K} \) we have \( p \in \mathcal{K} \). It is clear that for any symmetric torsion theory \( \sigma \) we have \( \mathcal{K}(\sigma) \) is generically closed. The next Proposition answer the above question.

**Proposition 1.8.** Let \( R \) be a ring, then there is a bijection between generically closed subset of \( \text{Spec}(R) \) and symmetric torsion theories \( \sigma \) in \( R\text{-mod} \).

**Proof:** Let \( \mathcal{K} \subseteq \text{Spec}(R) \), we define \( \sigma_{\mathcal{K}} = \bigwedge \{ \sigma_{R-p}; p \in \mathcal{K} \} \), then it is straightforward to show that \( \mathcal{K} = \mathcal{K}(\sigma_{\mathcal{K}}) \). Now the result follows from Lemma 1.7. ■

2. Torsion theories and the support of a module

Let \( M \) be a left \( R \)-module, we define the support of \( M \) as

\[
\text{Supp}_R(M) = \{ p \in \text{Spec}(R); M \text{ is not } \sigma_{R-p}\text{-torsion} \}.
\]

**Lemma 2.1.** Let \( 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \) a exact sequence of left \( R \)-modules, then

\[
\text{Supp}_R(M) = \text{Supp}_R(M') \cup \text{Supp}_R(M'').
\]
Proof: If $p \notin \text{Supp}_R(M)$, then $M$ is $\sigma_{R-p}$-torsion, so $M'$ and $M''$ are $\sigma_{R-p}$-torsion and $p \notin \text{Supp}_R(M') \cup \text{Supp}_R(M'')$. Conversely, if $p \notin \text{Supp}_R(M') \cup \text{Supp}_R(M'')$, then $M'$ and $M''$ are $\sigma_{R-p}$-torsion, so $M$ is $\sigma_{R-p}$-torsion and $p \notin \text{Supp}_R(M)$. 

**Proposition 2.2.** Let $a$ be a left ideal of $R$, and $\tilde{a}$ the bigger two-sided ideal contained in $a$, then

$$\text{Supp}_R(R/a) = V(\tilde{a}).$$

Proof: Let $p \notin \text{Supp}_R(R/a)$, then $R/a$ is $\sigma_{R-p}$-torsion and $a \in \mathcal{L}(\sigma_{R-p})$, so $\tilde{a} \subseteq p$, therefore $p \notin V(\tilde{a})$. The converse is obvious because all the implications are reversible. 

**Corollary 2.3.** Let $M$ be a finitely generated left $R$-module, then

$$\text{Supp}_R(M) = V(\text{Ann}_R(M)).$$

Proof: Let $M = Rm_1 + \ldots + Rm_n$, then we have the identities:

$$\text{Supp}(M) = \bigcup_{i=1}^{n} \text{Supp}(Rm_i) = \bigcup_{i=1}^{n} \text{Supp}(R/\text{Ann}_R(m_i)) =$$

$$= \bigcup_{i=1}^{n} V(\text{Ann}_R(m_i)) = \bigcup_{i=1}^{n} V(\text{Ann}_R(Rm_i)) =$$

$$= V(\bigcap_{i=1}^{n} \text{Ann}_R(Rm_i)) = V(\text{Ann}_R(M)).$$

A left noetherian ring $R$ is called left classical if all symmetric torsion theories are stable. For this kind of rings we can show a strong connection between associated prime ideals and prime ideals in the support of a left $R$-module.

**Lemma 2.4.** Let $R$ be a left classical ring and $M$ a left $R$-module, then for every prime ideal $p$ we have $p \in \text{Supp}_R(M)$ if, and only if, there is $q \in \text{Ass}_R(M)$ such that $q \subseteq p$.

Proof: Let $p \in \text{Supp}_R(M)$, then $M$ is not $\sigma_{R-p}$-torsion, so $\text{Ass}_R(M) \cap \mathcal{K}(\sigma_{R-p}) \neq \emptyset$ and there is $q \in \text{Ass}_R(M)$ such that $q \subseteq p$. The converse is easy.
Corollary 2.5. Let $R$ be a left classical ring and $M$ a left $R$-module, then $\text{Ass}_R(M) \subseteq \text{Supp}_R(M)$, and the two families have the same minimal elements.

Proposition 2.6. Let $R$ be a left classical ring and $\sigma$ a symmetric torsion theory in $R$-$\text{mod}$, then we have:

$$T_\sigma = \{M \in R$-$\text{mod}; \text{Supp}_R(M) \subseteq Z(\sigma)\}.$$

Proof: Since $R$ is a left classical ring, then $M$ is $\sigma$-torsion if, and only if, $\text{Ass}_R(M) \subseteq Z(\sigma)$. If $\text{Supp}_R(M) \subseteq Z(\sigma)$, therefore $\text{Ass}_R(M) \subseteq \text{Supp}_R(M) \subseteq Z(\sigma)$ and $M$ is $\sigma$-torsion. On the other hand, if $M$ is $\sigma$-torsion and $p \in \text{Supp}_R(M)$, then there is $q \in \text{Ass}_R(M) \subseteq Z(\sigma)$ such that $q \subseteq p$, so $p \in Z(\sigma)$ and $\text{Supp}_R(M) \subseteq Z(\sigma)$.

Proposition 2.7. Let $R$ be a left classical ring and $M$ a left $R$-module, then $\text{Supp}_R(M) = \text{Supp}_R(E(M))$.

Proof: We apply Lemma 2.4.

Corollary 2.8. Let $R$ be a left classical ring, $M$ a left $R$-module and

$$0 \to E_0(M) \to E_1(M) \to \cdots$$

a minimal injective resolution of $M$. Then for every $i \geq 0$ we have

$$\text{Supp}_R(E_i(M)) \subseteq \text{Supp}_R(M).$$

3. Local cohomology and Krull dimension

Let $R$ be a ring and $\sigma$ a torsion theory in $R$-$\text{mod}$, it is well known that $\sigma$ determines a left exact functor

$$\sigma : R$-$\text{mod} \to R$-$\text{mod},$$

if we derive on the right the functor $\sigma$, we have a sequence of functors $\{H_\sigma^n(-)\}_{n \geq 0}$, and for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of left $R$-modules, there is a long exact sequence

$$0 \to \sigma(M') \to \sigma(M) \to \sigma(M'') \to H_\sigma^1(M') \to \cdots$$

$$\cdots \to H_\sigma^{n-1}(M'') \to H_\sigma^n(M') \to H_\sigma^n(M) \to H_\sigma^n(M'') \to \cdots$$
Of course \( H^2_\sigma(M) \in T_\sigma \) for all left \( R \)-modules \( M \) and \( n \geq 0 \).

If \( \sigma \) is stable, then \( H^n_\sigma(M) = 0 \) for any \( \sigma \)-torsion left \( R \)-module \( M \) and \( n \geq 1 \), so \( H^n_\sigma(M) \cong H^n(M/\sigma(M)) \). One can also prove the main result: for any left \( R \)-module \( M \), we have an exact sequence

\[
0 \to \sigma(M) \to M \to Q_\sigma(M) \to H^1_\sigma(M) \to 0,
\]

where \( Q_\sigma(M) \) is the localization of \( M \) in the torsion theory \( \sigma \). In finishing this short summary on local cohomology, one can prove that \( H^{n+1}_\sigma \) is naturally isomorphic to \( R^nQ_\sigma \), the \( n \)-th right derived functor of \( Q_\sigma \).

**Proposition 3.1.** Let \( R \) be a ring and \( \sigma, \tau \) two symmetric and stable torsion theories in \( R\)-mod, then for every left \( R \)-module \( M \) there is an exact sequence

\[
\cdots \to H^i_{\sigma \wedge \tau}(M) \to H^i_\sigma(M) \oplus H^i_\tau(M) \to H^i_{\sigma \vee \tau}(M) \to H^{i+1}_{\sigma \wedge \tau}(M) \to \cdots
\]

**Proof:** Let \( a \in L(\sigma) \) and \( b \in L(\tau) \), then there is an exact sequence

\[
0 \to R/(a \cap b) \to R/a \oplus R/b \to R/(a + b) \to 0
\]

If we apply \( \text{Hom}_R(-, M) \), we have a long exact sequence

\[
0 \to \text{Hom}_R(R/(a + b), M) \to \text{Hom}_R(R/a \oplus R/b, M) \to \text{Hom}_R(R/(a \cap b), M) \to \text{Ext}^1_R(R/(a + b), M) \to \cdots
\]

Since direct limits are exact, there is an exact sequence

\[
0 \to \varprojlim \text{Hom}_R(R/(a + b), M) \to \varprojlim \text{Hom}_R(R/a \oplus R/b, M) \to \varprojlim \text{Hom}_R(R/(a \cap b), M) \to \lim \text{Ext}^1_R(R/(a + b), M) \to \cdots
\]

We know there are isomorphisms

\[
\varprojlim \text{Ext}^1_R(R/a, M) \cong \varprojlim \text{Ext}^1_R(R/a, M) \cong H^1_\sigma(M),
\]

and we will prove the isomorphisms

\[
\varprojlim \text{Ext}^1_R(R/(a \cap b), M) \cong H^1_{\sigma \vee \tau}(M).
\]

\( k \in L(\sigma \vee \tau) \) if, and only if, there are \( a \in L(\sigma) \) and \( b \in L(\tau) \) such that \( ab \subseteq k \). Since \( \sigma \) is stable it satisfies the Artin-Rees property, and there
is \( a_0 \in \mathcal{L}(\sigma) \) such that \( a_0 \cap b \subseteq ab \subseteq k \), thus \( \{a \cap b; a \in \mathcal{L}(\sigma), b \in \mathcal{L}(\tau)\} \) is a cofinal subset of \( \mathcal{L}(\sigma \vee \tau) \), therefore

\[
\lim_{a,b} \text{Ext}^i_R(R/a \cap b, M) \cong \lim_k \text{Ext}^i_R(R/k, M) \cong H^i_{\sigma \vee \tau}(M).
\]

Finally we can prove

\[
\lim_{a,b} \text{Ext}^i_R(R/(a + b), M) \cong H^i_{\sigma \wedge \tau}(M).
\]

If \( a \in \mathcal{L}(\sigma) \) and \( b \in \mathcal{L}(\tau) \), then \( a + b \in \mathcal{L}(\sigma \wedge \tau) \) and \( \{a + b; a \in \mathcal{L}(\sigma), b \in \mathcal{L}(\tau)\} \) is a cofinal subset of \( \mathcal{L}(\sigma \wedge \tau) \), so

\[
\lim_{a,b} \text{Ext}^i_R(R/(a + b), M) \cong \lim_k \text{Ext}^i_R(R/k, M) \cong H^i_{\sigma \wedge \tau}(M).
\]

Using the local cohomology, it is possible to establish a kind of dimension, the so called \( \sigma \)-dominant dimension. We say \( M \) has \( \sigma \)-dominant dimension greater or equal to \( n \) if \( H^i_\sigma(M) = 0 \) for all \( 0 \leq i < n \), or equivalently, the first \( n \) terms in a injective minimal resolution of \( M \) are \( \sigma \)-torsionfree.

Another dimension can be defined in (classical) left noetherian rings is the so called classical dimension. Let \( M \) be a left \( R \)-module, we define the classical dimension of \( M \) to be greater or equal to \( n \), \( \text{cl-dim}(M) \geq n \) if in \( \text{Supp}(M) \) there is a strictly ascending chain of prime ideals

\[
P_0 \subset P_1 \subset \cdots \subset P_n.
\]

And \( M \) has exactly classical dimension \( n \) if \( \text{cl-dim}(M) \geq n \) and \( \text{cl-dim}(M) \neq n - 1 \). In this way, we return to the definition of Krull dimension on commutative rings. It would be interesting to study how the classical dimension gives information on the structure of left noetherian rings. We will prove a theorem relating the classical dimension with the vanishing of some groups of local cohomology, and therefore with the \( \sigma \)-dominant dimension.

**Theorem 3.2.** Let \( R \) be a left classical ring, in which \( \sigma_{R-p} \) is perfect for every prime ideal \( p \), \( M \) a left \( R \)-module and \( \sigma \) a symmetric torsion theory. If \( \text{cl-dim}(M) = n \), then \( H^i_\sigma(M) = 0 \) for every \( i > n \).

**Proof:** Since every left \( R \)-module is a direct limit of finitely generated submodules, we can assume \( M \) is finitely generated, and since \( R \) is left noetherian, there is a finite chain of submodules of \( M \)

\[
0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M
\]
such that

1. Every $M_i$ is tertiary in $M_{i+1}$.
2. $M_{i+1}/M_i$ is annihilated by its associated prime ideal.

Then to prove the result we can reduce to the case in which $\mathcal{M}$ is $p$-coterinary and $\sigma$-torsionfree (annihilated by its associated prime ideal).

If $n=0$, then in any injective resolution of $\mathcal{M}$,

$$0 \to \mathcal{M} \to E_0(\mathcal{M}) \to E_1(\mathcal{M}) \to \cdots,$$

we have $\text{Ass}_R(\mathcal{E}_i(\mathcal{M})) \subseteq \text{Supp}_R(\mathcal{E}_i(\mathcal{M})) \subseteq \text{Supp}_R(\mathcal{M}) = \{p\}$, and $\mathcal{E}_i(\mathcal{M}) \in \mathcal{F}(\sigma)$ for all $i \geq 0$, and so $H^i_\sigma(\mathcal{M}) = 0$ for all $i > 0$. Let us consider $n > 0$, we assume the result is true for every left $R$-module $N$ such that $cl - dim(N) = m < n$. Since $\text{Ass}_R(\mathcal{M}) = \{p\} \subseteq K(\sigma)$, then $\mathcal{M}$ is $\sigma_{R-p}$-torsionfree, we have then an exact sequence

$$0 \to \mathcal{M} \to Q_{R-p}(\mathcal{M}) \to Q_{R-p}(\mathcal{M})/\mathcal{M} \to 0.$$

Because $\mathcal{M}$ is essential in $Q_{R-p}(\mathcal{M})$, it follows

$$\text{Ass}_R(Q_{R-p}(\mathcal{M})) = \text{Ass}_R(\mathcal{M}) = \{p\}.$$ 

If we apply now the Corollary 2.3 and Lemma 2.7. we have

$$\text{Supp}_R(\mathcal{M}) = V(p) = \text{Supp}_R(Q_{R\setminus p}(\mathcal{M})), $$

since $p \notin \text{Supp}_R(Q_{R-p}(\mathcal{M})/\mathcal{M})$, then

$$\text{Supp}_R(Q_{R-p}(\mathcal{M})/\mathcal{M}) \subseteq \text{Supp}_R(Q_{R-p}(\mathcal{M})) \setminus \{p\} = \text{Supp}_R(\mathcal{M}) \setminus \{p\},$$

and $m = cl - dim(Q_{R-p}(\mathcal{M})/\mathcal{M}) < cl - dim(\mathcal{M}) = n$. By induction we have

$$H^i_\sigma(Q_{R-p}(\mathcal{M})/\mathcal{M}) = 0 \text{ for } i > m.$$ 

We consider a minimal injective resolution of $Q_{R-p}(\mathcal{M})$,

$$0 \to Q_{R-p}(\mathcal{M}) \to E_0(Q_{R-p}(\mathcal{M})) \to E_1(Q_{R-p}(\mathcal{M})) \to \cdots$$

If we apply $Q_{R-p}$, to the exact sequence

$$0 \to Q_{R-p}(\mathcal{M}) \to E_0(Q_{R-p}(\mathcal{M})) \to \frac{E_0(Q_{R-p}(\mathcal{M}))}{Q_{R-p}(\mathcal{M})} \to 0,$$

we have again the same exact sequence, so $\frac{E_0(Q_{R-p}(\mathcal{M}))}{Q_{R-p}(\mathcal{M})}$ is $\sigma$-injective an $\sigma$-torsionfree. Then repeating this process, it is possible to prove that all the $E_i(Q_{R-p}(\mathcal{M}))$ are $\sigma_{R-p}$-torsionfree, then they are $\sigma$-torsionfree, and for any $i \geq 0$ we have $H^i_\sigma(Q_{R-p}(\mathcal{M})) = 0$. If we consider now the long exact sequence

$$\cdots \to H^i_\sigma(Q_{R-p}(\mathcal{M})/\mathcal{M}) \to H^{i+1}_\sigma(\mathcal{M}) \to H^{i+1}_\sigma(Q_{R-p}(\mathcal{M})) \to \cdots$$

it is clear the conclusion. ■
Corollary 3.3. Let $R$ be a left classical ring, in which $\sigma_{R-p}$ is perfect for every prime ideal $p$, $M$ a left $R$-module and $\sigma$ a symmetric torsion theory. Then

$$\sigma - \text{domdim}(M) \leq \text{cl} - \text{dim}(M).$$

References