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LOCAL COHOMOLOGY IN CLASSICAL RINGS*

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In memoriam of Prof. Pere Menal i Brufal

Abstract

The aim of this paper is to stablish the close connection between prime ideals and torsion theories in a non necessarily commutative noetherian ring. We introduce a new definition of support of a module and characterize some kinds of torsion theories in terms of prime ideals. Using the machinery introduced before, we prove a version of the Mayer-Vietoris Theorem for local cohomology and stablish a relationship between the classical dimension and the vanishing of the groups of local cohomology on a classical ring.

In this paper we show the relationship between prime ideals and torsion theories on a left noetherian, non necessarily commutative, ring R. The techniques we use are based on the prime ideals associated to a left R-module M and on its support, which we will define here in Section 2. All the rings in this paper are left noetherian.

In Section 1, we provide the interaction between the associated prime ideals of a left R-module M and if it is torsion or torsionfree, we give a characterization of symmetric torsion theories which is useful to characterize stable and symmetric torsion theories. Recall that a torsion theory σ is symmetric if for every $\mathbf{a} \in \mathcal{L}(\sigma)$, there is a two-sided ideal $\mathbf{b} \in \mathcal{L}(\sigma)$ such that $\mathbf{b} \subseteq \mathbf{a}$, and σ is stable is the class of all σ-torsion left R-modules is closed under taking essential extensions. In Section 2, we introduce the support of a left R-module and give its basic properties. We apply, in Section 3, the technique introduced before to the local cohomology. First we prove a version of the Mayer-Vietoris Theorem to

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local cohomology, and secondly, we establish the relationship between the classical dimension of a left R-module and the dominant dimension relative to a torsion theory σ.

Let us recall some results on associated prime ideals and tertiary decomposition of modules on a left noetherian ring. These results are avalaible in the literature, but we include them here for completeness.

Let R be a left noetherian ring, a left R-module M is *prime* if $Ann_R(N) = Ann_R(L)$ for every submodule L of N. Let M be a left R-module, a prime ideal $\mathbf{p}$ of R is *associated to* M if there is a prime submodule N of M such that $\mathbf{p} = Ann_R(N)$. The set of all prime ideals associated to M is called $Ass_R(M)$. Every maximal element in the set of all the annihilators of non zero submodules of M is a prime ideal associated to M, so if M is a non zero left R-module, then $Ass_R(M) \neq \emptyset$.

Let M be a non zero left R-module, and $\mathbf{p}$ a prime ideal, we say M is $\mathbf{p}$-*cotertiary* if $Ass_R(M) = \{\mathbf{p}\}$. A proper submodule N of M is $\mathbf{p}$-*tertiary in* M if M/N is $\mathbf{p}$-coterciario. The left R-module M (resp. a submodule N of M) is *cotertiary* (resp. *tertiary*) if it is $\mathbf{p}$-cotertiary (resp. $\mathbf{p}$-tertiary) for some prime ideal $\mathbf{p}$.

Lemma 0.1. *Let M be a left R-module. The following statements are equivalent:*

1. *M is $\mathbf{p}$-cotertiary.*
2. *$Ann_M(\mathbf{p})$ is essential in M and $\mathbf{p}$ contains all the two-sided ideals that annihilate some non zero submodule of M.*

Proof: $1 \Rightarrow 2$). If M is $\mathbf{p}$-cotertiary, then $Ass_R(M) = \{\mathbf{p}\}$, let $H \subseteq M$ be a non-zero submodule such that $H \cap Ann_M(\mathbf{p}) = 0$, so $Ass_R(H) \subseteq Ass_R(M) = \{\mathbf{p}\}$, and $Ass_R(H) = \{\mathbf{p}\}$, which is a contradiction.

$2 \Rightarrow 1$). Let $\mathbf{p}, \mathbf{q} \in Ass_R(M)$, and $N \subseteq M$ a prime submodule such that $\mathbf{q} = Ann_R(N)$, if $Ann_R(\mathbf{p})$ is essential in M, then $0 \neq N \cap Ann_M(\mathbf{p})$, and $\mathbf{q} \subseteq \mathbf{p}$, so we have $\mathbf{q} = \mathbf{p}$. ■

Let M be a non zero finitely generated left R-module. A *tertiary decomposition* of a submodule N of M is a finite family of tertiary submodules $\{N_1, \dots, N_r\}$ such that

1. $N = N_1 \cap \cdots \cap N_r$.
2. The decomposition is irreducible.
3. If $Ass_R(M/N_i) = \{\mathbf{p}_i\}$, then $\mathbf{p}_i \neq \mathbf{p}_j$ for $i \neq j$.

Lesieur and Croisot proved in [5] that if M is a non zero finitely generated R-module, then every submodule N of M has a tertiary decomposition, and if $N = N_1 \cap \cdots \cap N_r = L_1 \cap \cdots \cap L_s$ are two tertiary decomposition of N in M, then $r = s$ and $\{\mathbf{p}_1, \dots, \mathbf{p}_r\} = \{\mathbf{q}_1, \dots, \mathbf{q}_s\}$.

More results related to tertiary submodules can be found in Stenström's book [**8**].

1. Torsion theories and associated prime ideals

In this section we will study the relationship between torsion theories and prime ideals on *left noetherian* rings, we will consider mainly symmetric torsion theories.

Let R be a ring and σ a torsion theory in R-mod, we define

$$\mathcal{Z}(\sigma) = \{\mathbf{p} \in Spec(R);\ R/\mathbf{p} \in \mathcal{T}_\sigma\}.$$

and

$$\mathcal{K}(\sigma) = \{\mathbf{p} \in Spec(R);\ R/\mathbf{p} \in \mathcal{F}_\sigma\}.$$

In some cases $\{\mathcal{Z}(\sigma), \mathcal{K}(\sigma)\}$ is a partition of $Spec(R)$.

Lemma 1.1. [4] *Let R be a ring and σ a torsion theory in R-mod, then either $\mathbf{p} \in \mathcal{Z}(\sigma)$ or $\mathbf{p} \in \mathcal{K}(\sigma)$.*

Lemma 1.2. *Let R be a ring and σ a torsion theory in R-mod, then for every σ-torsionfree left R-module M we have $Ass_R(M) \subseteq \mathcal{K}(\sigma)$.*

Proof: If $\mathbf{q} \in Ass_R(M)$ and $\mathbf{q} \notin \mathcal{K}(\sigma)$, then $\mathbf{q} \in \mathcal{Z}(\sigma)$; therefore there is $m \in M$ such that $\mathbf{q} = Ann_R(Rm)$ and an epimorphism $R/\mathbf{q} \to Rm$; since $R/\mathbf{q}$ is σ-torsion, so Rm is also σ-torsion, which is a contradiction, therefore it must be $Ass_R(M) \subseteq \mathcal{K}(\sigma)$. ■

To prove the converse it is necessary to put conditions on the torsion theory, as we will see later.

Proposition 1.3. *Let R be a ring and σ a torsion theory in R-mod, then the following statements are equivalent for any left R-module M:*

1. *σ is symmetric.*
2. *If M is σ-torsion, then $Ass_R(M) \subseteq \mathcal{Z}(\sigma)$.*
3. *If $Ass_R \subseteq \mathcal{K}(\sigma)$, then M is σ-torsionfree.*

Proof: $1 \Rightarrow 2$). Let $\mathbf{p} \in Ass_R(M)$, then there is $N \subseteq M$ such that $\mathbf{p} = Ann_R(N)$, we can assume N is cyclic and generated by an element n, therefore $\mathbf{p}$ is the biggest two-sided ideal contained in $Ann_R(n) \in \mathcal{L}(\sigma)$, so $\mathbf{p} \in \mathcal{L}(\sigma)$.

$2 \Rightarrow 1$). Let $\mathbf{a} \in \mathcal{L}(\sigma)$, we consider a chain

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = R/\mathbf{a},$$

such that M_{i+1}/M_i is $\mathbf{p}_i$-cotertiary and $\mathbf{p}_i(M_{i+1}/M_i) = 0$. We have that every M_{i+1}/M_i is σ-torsion, therefore $\mathbf{p}_i \in Ass_R(M_{i+1}/M_i) \subseteq \mathcal{Z}(\sigma) \subseteq \mathcal{L}(\sigma)$, and so $\mathbf{p}_1 \cdots \mathbf{p}_n(R/\mathbf{a}) = 0$, therefore $\mathbf{p}_1 \cdots \mathbf{p}_n \subseteq \mathbf{a}$; finally, since the product of elements of $\mathcal{L}(\sigma)$ is also in $\mathcal{L}(\sigma)$, we have that σ is symmetric.

$2 \Rightarrow 3$). If M is not σ-torsionfree, so $\sigma(M) \neq 0$, and $\emptyset \neq Ass_R(\sigma(M)) \subseteq Ass_R(M) \subseteq \mathcal{K}(\sigma)$; on the other hand $\sigma(M)$ is σ-torsion, so $Ass_R(\sigma(M)) \subseteq \mathcal{Z}(\sigma)$, which is a contradiction.

$3 \Rightarrow 2$). Let M be a σ-torsion R-module, if $Ass_R(M) \not\subseteq \mathcal{Z}(\sigma)$, then there is some $\mathbf{q} \in Ass_R(M)$, and $\mathbf{q} \in \mathcal{K}(\sigma)$. Therefore there is $0 \neq N \subseteq M$ such that $Ass_R(N) = \{\mathbf{q}\}$, by the hypothesis we have $N \in \mathcal{F}_\sigma$, which is a contradiction. ■

As a consequence, for any left R-module M and any symmetric torsion theory σ, we have M is σ-torsionfree if, and only if, $Ass_R(M) \subseteq \mathcal{K}(\sigma)$. The analogous result for σ-torsion modules will characterize stable and symmetric torsion theories. To reverse the condition (2) in Proposition 1.3 we need consider a new condition on σ.

Lemma 1.4. *Let R be a ring and σ a stable torsion theory in R-mod; for any left R-module M, if $Ass_R(M) \subseteq \mathcal{Z}(\sigma)$, then M is σ-torsion.*

Proof: We can assume M is finitely generated, let us consider a tertiary decomposition of 0 in M,

$$0 = N_1 \cap \ldots \cap N_n,$$

with N_i $\mathbf{p}_i$-tertiary in M, $1 \leq i \leq n$, $Ass_R(M) = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ and $\mathbf{p}_i \neq \mathbf{p}_j$ if $i \neq j$; so there is a monomorphism $M \to \oplus_{i=1}^n M/N_i$. Let X be a $\mathbf{p}$-cotertiary left R-module with $\mathbf{p} \in \mathcal{L}(\sigma)$, so $Ann_X(\mathbf{p})$ is essential in X, for every $x \in Ann_X(\mathbf{p})$ we have $x \in \sigma(X)$, then $Ann_X(\mathbf{p}) \subseteq \sigma(X)$, and since σ is stable, X is σ-torsion. As a consequence every M/N_i is σ-torsion and so M is σ-torsion. ■

Lemma 1.5. *Let R be a ring and σ a symmetric stable torsion theory in R-mod, then for every left R-module M we have:*

1. $Ass_R(\sigma(M)) = Ass_R(M) \cap \mathcal{Z}(\sigma)$.
2. $Ass_R(M/\sigma(M)) = Ass_R(M) \cap \mathcal{K}(\sigma)$.
3. $Ass_R(M) = Ass_R(\sigma(M)) \cup Ass_R(M/\sigma(M))$.

Proof: Let $\mathcal{P} = Ass_R(M) \cap \mathcal{Z}(\sigma)$, then there is a submodule N of M such that $Ass_R(N) = Ass_R(M) \cap \mathcal{Z}(\sigma)$ and $Ass_R(M/N) = Ass_R(M) \cap \mathcal{K}(\sigma)$. Then we have $Ass_R(N) \subseteq \mathcal{Z}(\sigma)$, so N is σ-torsion, and $Ass_R(M/N) \subseteq \mathcal{K}(\sigma)$, so M/N is σ-torsionfree, it follows that $N = \sigma(M)$. ■

This result can be used to provide a characterization of stable symmetric torsion theories in the following way.

Proposition 1.6. *Let R be a ring and σ a symmetric torsion theory in R-mod, then σ is stable if, and only if, $\mathcal{T}_\sigma = \{M;\ Ass_R(M) \subseteq \mathcal{Z}(\sigma)\}$.*

It is possible to characterize non necessarily symmetric torsion theories σ such that $Ass_R(M) \subseteq \mathcal{Z}(\sigma)$ implies M is σ-torsion, like those torsion theories satisfying a property of the Artin-Rees type [**6**].

Lemma 1.7. *Let R be a ring and σ a symmetric torsion theory in R-mod, then*

$$\sigma = \bigwedge\{\sigma_{R-\mathbf{p}};\ \mathbf{p} \in \mathcal{K}(\sigma)\}.$$

More generally, it is possible to associate to a set of prime ideals $\mathcal{K} = \{\mathbf{p}_i;\ i \in I\}$ a symmetric torsion theory $\sigma_{\mathcal{K}}$ defined by $\sigma_{\mathcal{K}} = \wedge\{\sigma_{R-\mathbf{p}_i};\ i \in I\}$. It is arise the following question: When is $\mathcal{K}(\sigma_{\mathcal{K}}) = \mathcal{K}$?. we call a set $\mathcal{K}$ of prime ideals is *generically closed* if for any pair of prime ideals $\mathbf{p} \subseteq \mathbf{q}$ such that $\mathbf{q} \in \mathcal{K}$ we have $\mathbf{p} \in \mathcal{K}$. It is clear that for *any* symmetric torsion theory σ we have $\mathcal{K}(\sigma)$ is generically closed. The next Proposition answer the above question.

Proposition 1.8. *Let R be a ring, then there is a bijection between generically closed subset of $Spec(R)$ and symmetric torsion theories σ in R-mod.*

Proof: Let $\mathcal{K} \subseteq Spec(R)$, we define $\sigma_{\mathcal{K}} = \wedge\{\sigma_{R-\mathbf{p}};\ \mathbf{p} \in \mathcal{K}\}$, then it is straightforward to show that $\mathcal{K} = \mathcal{K}(\sigma_{\mathcal{K}})$. Now the result follows from Lemma 1.7. ■

2. Torsion theories and the support of a module

Let M be a left R-module, we define the *support* of M as

$$Supp_R(M) = \{\mathbf{p} \in Spec(R);\ M \text{ is not } \sigma_{R-\mathbf{p}}\text{-torsion}\}.$$

Lemma 2.1. *Let $0 \to M' \to M \to M'' \to 0$ a exact sequence of left R-modules, then*

$$Supp_R(M) = Supp_R(M') \cup Supp_R(M'').$$

Proof: If $\mathbf{p} \notin Supp_R(M)$, then M is $\sigma_{R-\mathbf{p}}$-torsion, so M' and M'' are $\sigma_{R-\mathbf{p}}$-torsion and $\mathbf{p} \notin Supp_R(M') \cup Supp_R(M'')$. Conversely, if $\mathbf{p} \notin Supp_R(M') \cup Supp_R(M'')$, then M' and M'' are $\sigma_{R-\mathbf{p}}$-torsion, so M is $\sigma_{R-\mathbf{p}}$-torsion and $\mathbf{p} \notin Supp_R(M)$. ■

Proposition 2.2. *Let* $\mathbf{a}$ *be a left ideal of* R, *and* $\tilde{\mathbf{a}}$ *the bigger two-sided ideal contained in* $\mathbf{a}$, *then*

$$Supp_R(R/\mathbf{a}) = V(\tilde{\mathbf{a}}).$$

Proof: Let $\mathbf{p} \notin Supp_R(R/\mathbf{a})$, then $R/\mathbf{a}$ is $\sigma_{R-\mathbf{p}}$-torsion and $\mathbf{a} \in \mathcal{L}(\sigma_{R\setminus\mathbf{p}})$, so $\tilde{\mathbf{a}} \not\subseteq \mathbf{p}$, therefore $\mathbf{p} \notin V(\tilde{\mathbf{a}})$. The converse is obvious because all the implications are reversible. ■

Corollary 2.3. *Let* M *be a finitely generated left* R*-module, then*

$$Supp_R(M) = V(Ann_R(M)).$$

Proof: Let $M = Rm_1 + \ldots + Rm_n$, then we have the identities:

$$\begin{aligned} Supp(M) &= \bigcup_{i=1}^{n} Supp(Rm_i) = \bigcup_{i=1}^{n} Supp(R/Ann_R(m_i)) = \\ &= \bigcup_{i=1}^{n} V(\widetilde{Ann_R(m_i)}) = \bigcup_{i=1}^{n} V(Ann_R(Rm_i)) = \\ &= V(\bigcap_{i=1}^{n} Ann_R(Rm_i)) = V(Ann_R(M)). \ ■ \end{aligned}$$

A left noetherian ring R is called *left classical* if all symmetric torsion theories are stable. For this kind of rings we can show a strong connection between associated prime ideals and prime ideals in the support of a left R-module.

Lemma 2.4. *Let* R *be a left classical ring and* M *a left* R*-module, then for every prime ideal* $\mathbf{p}$ *we have* $\mathbf{p} \in Supp_R(M)$ *if, and only if, there is* $\mathbf{q} \in Ass_R(M)$ *such that* $\mathbf{q} \subseteq \mathbf{p}$.

Proof: Let $\mathbf{p} \in Supp_R(M)$, then M is not $\sigma_{R-\mathbf{p}}$-torsion, so $Ass_R(M) \cap \mathcal{K}(\sigma_{R-\mathbf{p}}) \neq \emptyset$ and there is $\mathbf{q} \in Ass_R(M)$ such that $\mathbf{q} \subseteq \mathbf{p}$. The converse is easy. ■

Corollary 2.5. *Let R be a left classical ring and M a left R-module, then $Ass_R(M) \subseteq Supp_R(M)$, and the two families have the same minimal elements.*

Proposition 2.6. *Let R be a left classical ring and σ a symmetric torsion theory in R-mod, then we have:*

$$\mathcal{T}_\sigma = \{M \in R-mod;\ Supp_R(M) \subseteq \mathcal{Z}(\sigma)\}.$$

Proof: Since R is a left classical ring, then M is σ-torsion if, and only if, $Ass_R(M) \subseteq \mathcal{Z}(\sigma)$. If $Supp_R(M) \subseteq \mathcal{Z}(\sigma)$, therefore $Ass_R(M) \subseteq Supp_R(M) \subseteq \mathcal{Z}(\sigma)$ and M is σ-torsion. On the other hand, if M is σ-torsion and $\mathbf{p} \in Supp_R(M)$, then there is $\mathbf{q} \in Ass_R(M) \subseteq \mathcal{Z}(\sigma)$ such that $\mathbf{q} \subseteq \mathbf{p}$, so $\mathbf{p} \in \mathcal{Z}(\sigma)$ and $Supp_R(M) \subseteq \mathcal{Z}(\sigma)$. ■

Proposition 2.7. *Let R be a left classical ring and M a left R-module, then $Supp_R(M) = Supp_R(E(M))$.*

Proof: We apply Lemma 2.4. ■

Corollary 2.8. *Let R be a left classical ring, M a left R-module and*

$$0 \to E_0(M) \to E_1(M) \to \cdots$$

a minimal injective resolution of M. Then for every $i \geq 0$ we have

$$Supp_R(E_i(M)) \subseteq Supp_R(M).$$

3. Local cohomology and Krull dimension

Let R be a ring and σ a torsion theory in R-mod, it is well known that σ determines a left exact functor

$$\sigma : R-mod \to R-mod,$$

if we derive on the right the functor σ, we have a sequence of functors $\{H^n_\sigma(-)\}_{n\geq 0}$, and for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of left R-modules, there is a long exact sequence

$$\begin{aligned} 0 \to \sigma(M') \to \sigma(M) \to \sigma(M'') \to H^1_\sigma(M') \to \dots \\ \dots \to H^{n-1}_\sigma(M'') \to H^n_\sigma(M') \to H^n_\sigma(M) \to H^n_\sigma(M'') \to \dots \end{aligned}$$

Of course $H^n_\sigma(M) \in \mathcal{T}_\sigma$ for all left R-modules M and $n \geq 0$.

If σ is stable, then $H^n_\sigma(M) = 0$ for any σ-torsion left R-module M and $n \geq 1$, so $H^n_\sigma(M) \cong H^n(M/\sigma(M))$. One can also prove the main result: for any left R-module M, we have an exact sequence

$$0 \to \sigma(M) \to M \to Q_\sigma(M) \to H^1_\sigma(M) \to 0,$$

where $Q_\sigma(M)$ is the localization of M in the torsion theory σ. In finishing this short summary on local cohomology, one can prove that H^{n+1}_σ is naturarly isomorphic to R^nQ_σ, the n-th right derived functor of Q_σ.

Proposition 3.1. *Let R be a ring and σ, τ two symmetric and stable torsion theories in R-mod, then for every left R-module M there is an exact sequence*

$$\cdots \to H^i_{\sigma\wedge\tau}(M) \to H^i_\sigma(M) \oplus H^i_\tau(M) \to H^i_{\sigma\vee\tau}(M) \to H^{i+1}_{\sigma\wedge\tau}(M) \to \cdots$$

Proof: Let $\mathbf{a} \in \mathcal{L}(\sigma)$ and $\mathbf{b} \in \mathcal{L}(\tau)$, then there is an exact sequence

$$0 \to R/(\mathbf{a} \cap \mathbf{b}) \to R/\mathbf{a} \oplus R/\mathbf{b} \to R/(\mathbf{a} + \mathbf{b}) \to 0$$

If we apply $Hom_R(-, M)$, we have a long exact sequence

$$\begin{aligned} 0 \to Hom_R(R/(\mathbf{a}+\mathbf{b}), M) &\to Hom_R(R/\mathbf{a} \oplus R/\mathbf{b}, M) \to \\ \to Hom_R(R/(\mathbf{a}\cap\mathbf{b}), M) &\to Ext^1_R(R/(\mathbf{a}+\mathbf{b}), M) \to \cdots \end{aligned}$$

Since direct limits are exact, there is an exact sequence

$$\begin{aligned} 0 \to \varinjlim_{\mathbf{a},\mathbf{b}} Hom_R(R/(\mathbf{a}+\mathbf{b}), M) &\to \varinjlim_{\mathbf{a},\mathbf{b}} Hom_R(R/\mathbf{a} \oplus R/\mathbf{b}, M) \to \\ \to \varinjlim_{\mathbf{a},\mathbf{b}} Hom_R(R/(\mathbf{a}\cap\mathbf{b}), M) &\to \varinjlim_{\mathbf{a},\mathbf{b}} Ext^1_R(R/(\mathbf{a}+\mathbf{b}), M) \to \cdots \end{aligned}$$

We know there are isomorphisms

$$\varinjlim_{\mathbf{a},\mathbf{b}} Ext^i_R(R/\mathbf{a}, M) \cong \varinjlim_{\mathbf{a}\in\mathcal{L}(\sigma)} Ext^i_R(R/\mathbf{a}, M) \cong H^i_\sigma(M),$$

and we will prove the isomorphisms

$$\varinjlim_{\mathbf{a},\mathbf{b}} Ext^i_R(R/(\mathbf{a}\cap\mathbf{b}), M) \cong H^i_{\sigma\vee\tau}(M).$$

$\mathbf{k} \in \mathcal{L}(\sigma \vee \tau)$ if, and only if, there are $\mathbf{a} \in \mathcal{L}(\sigma)$ and $\mathbf{b} \in \mathcal{L}(\tau)$ such that $\mathbf{ab} \subseteq \mathbf{k}$. Since σ is stable it satisfies the Artin-Rees property, and there is $\mathbf{a}_0 \in \mathcal{L}(\sigma)$ such that $\mathbf{a}_0 \cap \mathbf{b} \subseteq \mathbf{ab} \subseteq \mathbf{k}$, thus $\{\mathbf{a} \cap \mathbf{b};\ \mathbf{a} \in \mathcal{L}(\sigma), \mathbf{b} \in \mathcal{L}(\tau)\}$ is a cofinal subset of $\mathcal{L}(\sigma \vee \tau)$, therefore

$$\varinjlim_{\mathbf{a},\mathbf{b}} Ext^i_R(R/\mathbf{a} \cap \mathbf{b}, M) \cong \varinjlim_{\mathbf{k}} Ext^i_R(R/\mathbf{k}, M) \cong H^i_{\sigma \vee \tau}(M).$$

Finaly we can prove

$$\varinjlim_{\mathbf{a},\mathbf{b}} Ext^i_R(R/(\mathbf{a}+\mathbf{b}), M) \cong H^i_{\sigma \wedge \tau}(M).$$

If $\mathbf{a} \in \mathcal{L}(\sigma)$ and $\mathbf{b} \in \mathcal{L}(\tau)$, then $\mathbf{a}+\mathbf{b} \in \mathcal{L}(\sigma \wedge \tau)$ and $\{\mathbf{a}+\mathbf{b};\ \mathbf{a} \in \mathcal{L}(\sigma), \mathbf{b} \in \mathcal{L}(\tau)\}$ is a cofinal subset of $\mathcal{L}(\sigma \wedge \tau)$, so

$$\varinjlim_{\mathbf{a},\mathbf{b}} Ext^i_R(R/(\mathbf{a}+\mathbf{b}), M) \cong \varinjlim_{\mathbf{k}} Ext^i_R(R/\mathbf{k}, M) \cong H^i_{\sigma \wedge \tau}(M) \quad \blacksquare$$

Using the local cohomology, it is possible to establish a kind of dimension, the so called *σ-dominant dimension*. We say M has σ-dominant dimension greater or equal to n if $H^i_\sigma(M) = 0$ for all $0 \leq i < n$, or equivalently, the first n terms in a injective minimal resolution of M are σ-torsionfree.

Another dimension can be defined in (classical) left noetherian rings is the so called *classical dimension*. Let M be a left R-module, we define the classical dimension of M to be greater or equal to n, $cl-dim(M) \geq n$ if in $Supp(M)$ there is a strictly ascending chain of prime ideals

$$\mathbf{p}_0 \subset \mathbf{p}_1 \subset \ldots \subset \mathbf{p}_n.$$

And M has exactly classical dimension n if $cl - dim(M) \geq n$ and $cl - dim(M) \not\geq n-1$. In this way, we return to the definition of Krull dimension on commutative rings. It would be interesting to study how the classical dimension gives information on the structure of left noetherian rings. We will prove a theorem relating the classical dimension with the vanishing of some groups of local cohomology, and therefore with the σ-dominant dimension.

Theorem 3.2. *Let R be a left classical ring, in which $\sigma_{R-\mathbf{p}}$ is perfect for every prime ideal $\mathbf{p}$, M a left R-module and σ a symmetric torsion theory. If $cl - dim(M) = n$, then $H^i_\sigma(M) = 0$ for every $i > n$.*

Proof: Since every left R-module is a direct limit of finitely generated submodules, we can assume M is finitely generated, and since R is left noetherian, there is a finite chain of submodules of M

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that

1. Every M_i is tertiary in M_{i+1}.
2. M_{i+1}/M_i is annihilated by its associated prime ideal.

Then to prove the result we can reduce to the case in which M is $\mathbf{p}$-cotertiary and σ-torsionfree (annihilated by its associated prime ideal). If $n = 0$, then in any injective resolution of M,

$$0 \to M \to E_0(M) \to E_1(M) \to \dots,$$

we have $Ass_R(E_i(M)) \subseteq Supp_R(E_i(M)) \subseteq Supp_R(M) = \{\mathbf{p}\}$, and $E_i(M) \in \mathcal{F}(\sigma)$ for all $i \geq 0$, and so $H^i_\sigma(M) = 0$ for all $i > 0$. Let us consider $n > 0$, we assume the result is true for every left R-module N such that $cl - dim(N) = m < n$. Since $Ass_R(M) = \{\mathbf{p}\} \subseteq \mathcal{K}(\sigma)$, then M is $\sigma_{R-\mathbf{p}}$-torsionfree, we have then an exact sequence

$$0 \to M \to Q_{R-\mathbf{p}}(M) \to Q_{R-\mathbf{p}}(M)/M \to 0.$$

Because M is essential in $Q_{R-\mathbf{p}}(M)$, it follows

$$Ass_R(Q_{R-\mathbf{p}}(M)) = Ass_R(M) = \{\mathbf{p}\}.$$

If we apply now the Corollary 2.3 and Lemma 2.7. we have

$$Supp_R(M) = V(\mathbf{p}) = Supp_R(Q_{R\setminus\mathbf{p}}(M)),$$

since $\mathbf{p} \notin Supp_R(Q_{R-\mathbf{p}}(M)/M)$, then

$$Supp_R(Q_{R-\mathbf{p}}(M)/M) \subseteq Supp_R(Q_{R-\mathbf{p}}(M)) \setminus \{\mathbf{p}\} = Supp_R(M) \setminus \{\mathbf{p}\},$$

and $m = cl - dim(Q_{R-\mathbf{p}}(M)/M) < cl - dim(M) = n$. By induction we have

$$H^i_\sigma(Q_{R-\mathbf{p}}(M)/M) = 0 \text{ for } i > m.$$

We consider a minimal injective resolution of $Q_{R-\mathbf{p}}(M)$,

$$0 \to Q_{R-\mathbf{p}}(M) \to E_0(Q_{R-\mathbf{p}}(M)) \to E_1(Q_{R-\mathbf{p}}(M)) \to \dots$$

If we aply $Q_{R-\mathbf{p}}$, to the exact sequence

$$0 \to Q_{R-\mathbf{p}}(M) \to E_0(Q_{R-\mathbf{p}}(M)) \to \frac{E_0(Q_{R-\mathbf{p}}(M)}{Q_{R-\mathbf{p}}(M)} \to 0,$$

we have again the same exact sequence, so $\frac{E_0(Q_{R-\mathbf{p}}(M))}{Q_{R-\mathbf{p}}(M)}$ is σ-injective an σ-torsionfree. Then repeating this process, it is possible to prove that all the $E_i(Q_{R-\mathbf{p}}(M))$ are $\sigma_{R-\mathbf{p}}$-torsionfree, then they are σ-torsionfree, and for any $i \geq 0$ we have $H^i_\sigma(Q_{R-\mathbf{p}}(M)) = 0$. If we consider now the long exact sequence

$$\cdots \to H^i_\sigma(Q_{R-\mathbf{p}}(M)/M) \to H^{i+1}_\sigma(M) \to H^{i+1}_\sigma(Q_{R-\mathbf{p}}(M)) \to \cdots$$

it is clear the conclusion. ■

Corollary 3.3. *Let R be a left classical ring, in which $\sigma_{R-\mathfrak{p}}$ is perfect for every prime ideal* $\mathfrak{p}$, *M a left R-module and σ a symmetric torsion theory. Then*

$$\sigma - domdim(M) \leq cl - dim(M).$$

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