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ON THE UNIT-1-STABLE RANK OF RINGS OF ANALYTIC FUNCTIONS

Joan Josep Carmona, Julià Cufí and Pere Menal

Abstract

In this paper we prove a general result for the ring $H(U)$ of the analytic functions on an open set U in the complex plane which implies that $H(U)$ has not unit-1-stable rank and that has some other interesting consequences. We prove also that in $H(U)$ there is no totally reducible elements different from the zero function.

Introduction

Let A be a commutative ring with unity. A pair of elements $(a_1, a_2) \in A^2$ is said to be unimodular if there exists $(b_1, b_2) \in A^2$ such that $a_1b_1 + a_2b_2 = 1$. We will denote by $U_2(A)$ the set of all unimodular pairs and by $U_1(A) = A^{-1}$ the set of inversible elements of A. One says that the unimodular pair (a_1, a_2) is reducible if it is possible to find $x \in A$ such that $a_1 + xa_2 \in A^{-1}$. The ring A is said to have stable rank 1 if each unimodular pair in A is reducible in A. This is a special case of the concept of stable rank n introduced by Bass **[1]**. This notion has been useful in treating some problems in K-theory. Moreover Vasershtein **[12]** has calculated the stable rank of rings of continuous functions and rings of differentiable functions in $\mathbb{R}^n$ and related it to the topological dimension of the domain space.

Concerning to rings of holomorphic functions P. Jones, D. Marshall and T. Wolff **[3]** proved that the disc algebra has stable rank 1. Previously L. A. Rubel **[7]** had observed that the same is true for the ring $H(U)$ of holomorphic functions on the open set $U \subset \mathbb{C}$. Different proves of these results can be found in the paper of G. Corach and F. Suárez

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[2] , where the case of rings of holomorphic functions of several complex variables is also considered. The more dificult problem to decide if the algebra $H^\infty(\mathrm{D})$ of bounded analytic functions in the unit disc D, has stable rank 1 has recently been answered positively by S. Treil **[10]** .

In **[4]** P. Menal and J. Moncasi introduced the concept of unit-1-stable rank . A unimodular pair $(a_1, a_2) \in U_2(A)$ is said to be totally reducible if there exists an element $u \in A^{-1}$ such that $a_1 + ua_2 \in A^{-1}$. The ring A is said to have unit-1-stable rank if each unimodular pair in A is totally reducible. In **[7]** L. A. Rubel proved that $H(\mathrm{C})$ has not the unit-1-stable rank property.

The question to decide if the disc algebra has unit-1-stable rank arose and was studied by R. Mortini and R. Rupp and by ourselves. Mortini and Rupp communicated to us the negative answer to this question and using some ideas of their proof we obtain a more general result that has some other interesting consequences. This is the content of the first part of the present paper.

The result of Mortini and Rupp appears in **[5]** where the question to characterize the totally reducible elements of the disc algebra is also considered. They find a sufficient condition for an element to be totally reducible in this algebra. Given a ring A one says that an element $a \in A$ is totally reducible if for each $b \in A$ such that the pair $(a, b) \in U_2(A)$ then (a, b) is a totally reducible pair. In the second part of this paper we consider the totally reducible elements of the ring $H(U)$. In this case the situation is completely different from the disc algebra because we show that the zero function is the only totally reducible element of $H(U)$.

We are able to obtain these kind of algebraic properties of rings of analytic functions by using deep theorems of the function theory of one complex variable.

Unit-1-stable rank

From now on we deal with the ring $H(U)$. If $a \in H(U) \setminus \{0\}$, then we denote by Z_a, the discret closed set in U of the zeros a. Each zero is considered with the corresponding multiplicity. So when we write $Z_a = Z_b$ we mean that a and b have the same zeros with the same multiplicity. We recall that a pair (a, b) with $a, b \in H(U)$ is unimodular if only if $Z_a \cap Z_b = \emptyset$.

We can prove the following general result.

Theorem 1. *Assume that* $(a_n), (b_n), (c_n), (d_n)$ *are sequences of elements of* $H(U)$ *satisfying*

i) $a_nb_n + c_nd_n = 1$ *for all* $n \geq 1$ *and* b_n, d_n *are invertible elements of* $H(U)$.

ii) *The sequences* $(a_n), (c_n)$ *are uniformly convergent on compact subsets of* U *to* $a, c \in H(U) \setminus \{0\}$ *respectively.*

Then either $Z_a \cap Z_c = \emptyset$ *or* $Z_a = Z_c$.

Proof: First we prove that (a_nb_n) is a normal sequence in $D \setminus Z_{ac}$. Fix a point $z_0 \in D \setminus Z_{ac}$. Then there exists a closed disc $\bar{\Delta} \subset U \setminus Z_{ac}$ with center z_0 and a number $\delta > 0$ such that $|a_n(z)c_n(z)| \geq \delta$, for $z \in \bar{\Delta}, n \geq \nu$, ν large enough.

Consider the sequence given by (a_nb_n) if $n \geq \nu$. We will prove that this sequence is normal in the classical sense in Δ . Since b_n is invertible and a_n has no zeros in Δ we see that a_nb_n never takes the value $0, n \geq \nu$. If $a_n(z)b_n(z) = 1$ for some $z \in \Delta$, it follows from i) that $c_nd_n(z) = 0$. But d_n is invertible and $c_n(z) \neq 0$ if $n \geq \nu$ and this is a contradiction. By Montel's Theorem [**8**, p. 350] , $(a_nb_n)_{n \geq \nu}$ is a normal sequence in Δ and so is $(a_nb_n)_{n\geq 1}$. Since z_0 was an arbitrary point of the open set $U \setminus Z_{ac}$, it follows from [**8**, p. 51] that $(a_nb_n)_{n \geq 1}$ is normal in $U \setminus Z_{ac}$.

Assume $Z_a \cap Z_c \neq \emptyset$ and let us fix a point $\alpha \in Z_a \cap Z_c$. Take a closed disc $\Delta_1 \subset U$ with center α and such that $(\Delta_1 \setminus \{\alpha\}) \cap Z_{ac} = \emptyset$.

The result will follow if we prove that any $\beta \in Z_{ac}$ is a common zero of a and c with the same multiplicity in a than in c. For such a β let $\bar{\Delta}_2$ be a closed disc with center β and such that $(\bar{\Delta}_2 \setminus \{\beta\}) \cap Z_{ac} = \emptyset$. Let $K = \bar{\Delta}_1 \cup \bar{\Delta}_2$. By ii) (c_n) converges uniformly to c in K and $(1/c_n)$ converges uniformly on ∂K. Since $d_n = (1 - a_nb_n)/c_n$ we know that a partial sequence of (d_n) is either uniformly convergent or uniformly divergent on ∂K. In the first case this partial sequence (d_n) is uniformly bounded in $\partial\bar{\Delta}_1$ and, by the maximum modulus principle, also in $\bar{\Delta}_1$. Then a partial sequence of $(d_n(\alpha))$ is convergent and the corresponding partial of $((c_nd_n)(\alpha))$ tends to 0. Since $b_n = (1 - c_nd_n)/a_n$ and $(\frac{1}{a_n}) \to \frac{1}{a}$ on ∂K we get that same partial sequence of (b_n) is uniformly bounded on K. Therefore $(a_n(\alpha)b_n(\alpha))$ tends to 0 and this together with the fact that $(c_n(\alpha)d_n(\alpha))$ tends to 0 contradicts i). Therefore (d_n) is uniformly divergent on ∂K. Since d_n is invertible, by the minimum modulus principle we get that (d_n) is uniformly divergent on K. Since $d_n^{-1} = a_nb_nd_n^{-1} + c_n$ we get that $(-a_nb_nd_n^{-1})$ converges uniformly to c in K. Also (a_n) tends to a uniformly on K and since $Z_{ac} \cap \partial\bar{\Delta}_2 = \emptyset$, by Hurwitz's Theorem [**8**, p. 158] the function a has the same zeros than c in Δ_2. ∎

The following simple result shows that the hypothesis about the invertibility of b_n and d_n cannot be weakned.

Proposition 1. *Let $a, c \in H(U)$. Then there exist sequences (a_n), (b_n), (c_n), (d_n) such that (a_n) and (c_n) are almost uniformly convergent to a and c respectively such that $a_n b_n + c_n d_n = 1$ and b_n is invertible for all $n \geq 1$.*

Proof: For each $\lambda > 0$ let us consider the set $A_\lambda = \{z \in U | a(z) + \lambda = 0\} \cap Z_c$. Since $\lambda \neq \lambda'$ implies $A_\lambda \cap A_{\lambda'} = \emptyset$, the set of λ such that $A_\lambda \neq \emptyset$ is at most countable. Therefore we can choose a sequence (λ_n) of positive numbers that converges to 0 such that $A_{\lambda_n} = \emptyset$. Put $a_n(z) = a(z) + \lambda_n$ and $c_n = c, n \geq 1$. Cleary the sequences (a_n) and (c_n) converge uniformly to a and c and $Z_{a_n} \cap Z_c$ is empty for all $n \geq 1$. Since $H(U)$ has stable rank 1 this implies that there are b_n invertible and d_n such that $a_n b_n + c_n d_n = 1$. ■

Proposition 1 says in particular that the topological stable rank of $H(U)$, in the sense of Reiffel **[6]**, is 2.

From the Theorem 1 we can deduce the result of Mortini and Rupp **[5]**.

Corollary 1. *Let f be a nonzero element of $H(U)$. Then f has some zero in U if and only if there is a positive integer n such that the unimodular pair $(f, 1 - nf^2)$ is not totally reducible in $H(U)$.*

Proof: Since $(nf)f + (1 - nf^2) = 1$ it is clear that $(f, 1 - nf^2)$ is totally reducible when f has no zeros.

Conversely, assume there exist $u_n \in H(U)^{-1}$ such that $v_n = fu_n + 1/n - f^2 \in H(U)^{-1}$ for all $n \geq 1$. Then $1 = fu_n v_n^{-1} + (1/n - f^2)v_n^{-1}$. It follows from Theorem 1 that f has no zeros. ■

Corollary 2. *Let A be a subring of $H(U)$. If $f \in A$ is totally reducible in A, then f has no zeros in U or f is identically* 0.

Let $\varphi : A \to B$ be a ring homomorphism. We say that φ has *stable rank* 1 provided that for any $x, y \in A$ with $xA + yA = A$ there exists $c \in S$ with $\varphi(x) + \varphi(y)c \in B^{-1}$. If c can be chosen to be invertible in B, then we say that φ has unit-1-stable rank.

Corollary 3. *Let U be an open set of* C *and let A be a ring. If $\varphi : A \to H(U)$ is a ring homomorphism with unit-1-stable rank, then* $\varphi(A) \subset$ C.

Proof: Corollary 1 implies that $\varphi(a) \in H(U)^{-1}$ when $a \in A, a \neq 0$. Let $a \neq 0$ and assume that $\varphi(a)$ is not constant. Then $\varphi(a)(U)$ contains an algebraic number and so there is a nonzero polynomial $P \in \mathbb{Z}[t]$ such that $P(\varphi(a))$ has some zero in U. Since $P(\varphi(a)) = \varphi(P(a))$ we conclude that $P(a) = 0$ and so $P(\varphi(a)) = 0$. This shows that $\varphi(a)$ takes only finitely many values and so it must be constant which is a contradiction. ■

Corollary 4. *Let $\Sigma \subset \mathbb{C}[z]$ be the set of all polynomials without zeros in the closed unit disc. Then the ring $R = \mathbb{C}[z]_\Sigma$ has stable rank 1 but not unit-1-stable rank.*

Proof: Clearly we can view R as a subring of the disc algebra $A(\mathbb{D})$. If (a, b) is a unimodular pair in R , there exists an element $f \in A(\mathbb{D})$ such that $a + fb$ is invertible **[2]** **[3]**. Since f can be approximated uniformly by polinomials we can assume that f itself is a polynomial. Then $a + bf \in R$ and R has stable rank 1. It follows from Corollary 3 that R has not unit-1-stable rank. ■

Another applications of Theorem 1 are some results that guarantee the existence of a fixed disc contained in the image of the unit disc for each element of some classes of functions. In this line we recall the classical results of Bloch [**11**, p. 262], Koebe [**9**, p. 197] and also the interesting one referred in [**9**, p. 502]. Here we consider the class of functions fg where f is a fixed function and g is a holomorphic function without fixed points in an open set and also the class of all the funcions $f.g$ where $f \in S$ and g is as before. We write S for the set of all $f \in H(\mathbb{D})$ such that f is one to one and $f(0) = 0, f'(0) = 1$.

Proposition 2. *There exists a universal constant $r > 0$ such that the disc $D(0, r)$ is contained in the image of every function fg, where $f \in S$ and $g \in H(\mathbb{D})$ has no fixed points in $\mathbb{D}$.*

Proof: Assume our conclusion is false. Then for each positive integer n, there exist z_n with $|z_n| < \frac{1}{n}, f_n \in S$ and g_n without fixed points such that $f_n g_n - z_n \in H(\mathbb{D})^{-1}$. Write $g_n(z) = z - h_n(z)$, where $h_n \in H(\mathbb{D})^{-1}$. We obtain

$$1 = f_n u_n + (z f_n - z_n) v_n^{-1}, \qquad \text{with} \quad u_n, v_n \in H(\mathbb{D})^{-1}.$$

But S is a normal class [**9**, p. 200] , so there exits a partial sequence of (f_n) uniformly convergent to some $f, f \in S$. Applying Theorem 1 we conclude that $Z_f = Z_{zf}$, a contradiction. ■

Proposition 3. *Let U be an open set with $0 \in U$ and let f be a function that is neither invertible nor zero. Then there exits a constant $r = r(f) > 0$ such that the disc $D(0, r)$ is contained in the image of every function fg, where $g \in H(U)$ has no fixed points in U.*

Proof: If the statement is not true, then for each n there would exist $|z_n|$, with $|z_n| < \frac{1}{n}$ and g_n without fixed points such that $f.g_n - z_n \in H(U)^{-1}$. Proceeding as before we obtain

$$1 = fu_n + (zf - z_n)v_n^{-1}, \qquad \text{with} \quad u_n, v_n \in H(U)^{-1}.$$

Now we aply Theorem 1 and the conclusion follows. ∎

The constant that appears in Proposition 2 is less or iqual than $\frac{1}{4}$, by Koebe's Theorem. Considering the functions $f(z) = z, g(z) = e^{z-1}$ one can see that $r \leq \frac{1}{e^2}$. It would be interesting to find a constructive proof of Proposition 2 and the best value of r.

Totally reducible elements

We are going now to consider the totally reducible elements of the ring of analytic functions in an open set. Let A be a Banach algebra. For this special case every element $u \in A^{-1}$ is totally reducible. In fact , for each $f \in A$, the pair (u, f) is unimodular and $u + \epsilon f \in A^{-1}$ if $\epsilon < \frac{1}{\|fu^{-1}\|}$. For the disc algebra $A(\mathrm{D})$ formed by the functions which are continuous on $\bar{\mathrm{D}}$ and holomorphic in D, Mortini and Rupp proved that each outer function in $A(\mathrm{D})$ is totally reducible [**5**].

The situation is completely different for the ring $H(U)$ as the following theorem shows.

Theorem 2. *Let U be an open set in C. A function $f \in H(U)$ is totally reducible in $H(U)$ if and only if f is the zero function in U.*

For the proof we consider separately the two different cases $U = \mathrm{C}$ and $U \neq \mathrm{C}$.

Proof for the first case: Let f be totally reducible in $H(\mathrm{C})$. We know, using Corollary 2, that $f = e^h$ with h an entire function. If the unimodular pair (e^h, z) would be totally reducible then there would exist $k, l \in H(\mathrm{C})$ such that

$$e^h e^k + ze^l = 1.$$

The function ze^l never takes the value 1 and takes the value 0 only once at the origen. By the great Picard Theorem [**8**, p. 353] ze^l must be a polynomial and this contradicts the fact that it never takes the value 1. ■

To prove the Theorem 2 for $U \neq \mathbb{C}$ first of all we remark that the existence of some $f \in H(U), f \neq 0$ totally reducible implies that each function in $H(U)^{-1}$ is also totally reducible and this existence is equivalent to an interpolation problem as the following lemma shows.

Lemma. *Let U be an open set of $\mathbb{C}$. Then the following are equivalent.*

i) *There exists a function $f \in H(U)^{-1}$ which is totally reducible in $H(U)$.*

ii) *For each closed and discrete set $\{z_n\}$, counting every z_n with some multiplicity, there exists a function $h \in H(U)$ whose zeros are $\{z_n\}$, with the corresponding multiplicity, and such that h never takes the value 1 in U.*

iii) *Each function $g \in H(U)^{-1}$ is totally reducible in $H(U)$.*

Proof of the lemma: i) ⇒ ii) Let $f \in H(U)^{-1}$ be totally reducible. Given the set $\{z_n\}$ take $k \in H(U)$ with $\{z_n\}$ as its zero set. By i) there are $a, b \in H(U)^{-1}$ such that $af + bk = 1$. So $a = \frac{1-bk}{f}$ and $h = bk$ satisfies the requirements of ii).

ii) ⇒ iii) Let $g \in H(U)^{-1}$ and let $l \in H(U)$ be arbitrary. The pair (g, l) is unimodular. Let $\{z_n\}$ be the zero set of l. By ii) there is some $h \in H(U)$ that takes the value 1 on $\{z_n\}$ and $h(z) \neq 0$. Now the functions $b = \frac{1-h}{l} \in H(U)^{-1}$ and $a = \frac{h}{g} \in H(U)^{-1}$ verify $ag + bl = 1$. ■

We need the following results [**11**, p. 215-204].

Theorem (Ahlfors). *Let $f \in H(\mathbb{D})$ and assume that*

$$\limsup_{r\to 1} \frac{T(r)}{\log \frac{1}{1-r}} = +\infty.$$

Then f takes any value infinitely often in $\mathbb{D}$ with one possible exception.

Here $T(r)$ is the characteristic function of Nevalinna [**11**, p. 196].

Theorem. *Let $f \in H(\mathbb{D})$ and let $\{z_n\}$ be its zero set. Assume that $\sum_{n=1}^{\infty}(1-|z_n|)^{1+\rho} = +\infty$, for all $\rho > 0$. Then $\limsup_{r\to 1} \frac{T(r)}{\log \frac{1}{1-r}} = +\infty$.*

Proof for the second case: Assume first that $U = \mathrm{D}$. Let (z'_n) be a sequence such that $\lim_{n\to\infty} z'_n = 1$ and $\sum_{n=1}^{\infty}(1-|z'_n|)^{1+\rho} = +\infty$ for all $\rho > 0$. Let $\{z_n\}$ be the sequence of the same points but doubling their multiplicity. We show that ii) of the lemma is not satisfied for $\{z_n\}$. Let $h \in H(\mathrm{D})$ be any function that vanishes on $\{z_n\}$. We have $h = h_0^2$ for some $h_0 \in H(\mathrm{D})$ vanishing at $\{z'_n\}$. By Ahlfors Theorem h_0 may omit only one value and so its square h takes any complex value infinitely often.

For a general open set U let Δ be any disc $\Delta \subset U$ such that there is a point $\alpha \in \partial\Delta \cap \partial U$. It suffices to take a sequence $\{z_n\}, z_n \in \Delta$ as before with $\lim_{n\to\infty} z'_n = \alpha$. For this sequence, condition ii) of the lemma is not satisfied in U, since it is not satisfied on Δ. ■

References

1. H. BASS, K-theory and stable algebra, *Publi. Math IHES* **22** (1964), 5–60.
2. G. CORACH AND F. DANIEL SUÁREZ, Stable rank in holomorphic functions algebras, *Illinois J. of Math.* **29** (1985), 627–639.
3. P. W. JONES, D. MARSHALL AND T. WOLFF, Stable rank of the disc algebra, *Proc. Amer. Math. Soc.* **96** (1986), 603–604.
4. P. MENAL AND J. MONCASI, K_1 of Von Neumann regular rings, *Journal of Pure and Applied Algebra* **33** (1984), 295–312.
5. R. MORTINI AND R. RUPP, Totally reducible elements in rings of analytic functions, *Comm. in Algebra* (to appear).
6. M. A. RIEFFEL, Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc.* **46** (1983), 303–333.
7. L. A. RUBEL, Linear compositions of two entire functions, *Amer. Math. Monthly* **85** (1978), 505–506.
8. S. SAKS AND A. ZYGMUND, "*Analytic functions*," Elsevier Pub. Company, 1971.
9. G. SANSONE AND J. GERRETSEN, "*Lectures on the theory of functions of a complex variable*," Wolters Noordhoff, 1969.
10. S. TREIL, The stable rank of the algebra H^∞ equals 1, Preprint Leningrad University (1991).
11. M. TSUJI, "*Potential theory in modern function theory*," Chelsea Publishing Company, New York, 1975.

12. L. N. VASERSTEIN, Stable rank of rings and dimensionality of topological spaces, *Functional Anal. Appl.* **5** (1971), 102–110.

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona)
SPAIN

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