Publicacions Matemàtiques, Vol 36 (1992), 541–567.

SELF-INJECTIVE VON NEUMANN REGULAR SUBRINGS AND A THEOREM OF PERE MENAL

CARL FAITH[1]

In memoria nobilissimi omnium-Pere

Abstract

This paper owes its origins to Pere Menal and his work on Von Neumann Regular (= VNR) rings, especially his work listed in the bibliography on when the tensor product $K = A \otimes_K B$ of two algebras over a field k are right self-injective (= SI) or VNR. Pere showed that then A and B both enjoy the same property, SI or VNR, and furthermore that either A and B are algebraic algebras over k (see [M]). This is connected with a lemma in the proof of the *Hilbert Nullstellensatz*, namely, a finite ring extension $K = k[a_1, \dots, a_n]$ is a field only if $a_1, \dots, a_n$ are algebraic over k.

In this paper, we follow Pere's lead in just the one property, namely SI, applied to a VNR subring A of a right SI ring K. Pere proved in essence that all that is required is for K to be split-flat over A (= K is a flat left module and A is a direct summand of K as a right A-module). By a theorem of Bourbaki-Lambek [L], the inclusion functor mod-$K \hookrightarrow$ mod-A preserves injectives iff ${}_AK$ is flat. Then (when K_K is injective), A_A is injective iff A_A splits in K_A, equivalently, A_A has no proper essential extensions in K_A (cf. Proposition 1.1).

Pere's theorem on tensor products over an arbitrary commutative ring k similarly can be generalized:

[1] A part of this paper was written in Spring 1986 during my visit at Centre de Recerca Matemàtica (CRM) of the Institut d'Estudis Catalans in Universitat Autònoma de Barcelona (UAB). I owe much to Pere Menal for inviting me to visit UAB and CRM, and for our subsequent collaborations. (See my memento of Pere Menal i Brufal at the close of this article).

Menal's general theorem. *(1.7). If a right SI ring $K = A \otimes_k B$ is a tensor of faithful k-algebras A and B and is split-flat over a commutative ring k, then A and B (also k) are right SI.*

The proof requires a further bit of homological algebra (1.3, 1.4, 1.5 and 1.6). *Moreover, Theorem 1.3 shows that only B need be split flat over k for the conclusion that A is right SI.*

The corollary that follows shows in one part of Pere's theorem **[M]** that *the field k can be replaced by any commutative self-injective VNR k.*

Corollary. *(1.8). If $K = A \otimes_k B$ is right SI faithful algebra over a SI VNR ring k, then A and B are right SI.*

The effect of the hypothesis that K is faithful over k is that $k \hookrightarrow K$, hence the SI requirement for k is equivalent to: k has no proper essential extensions in K; and in this case K is split over k. (And K is flat because k is VNR).

We next turn our attention to a **maximal** VNR **subring** A, abbreviated **max** VNR in K. Then if K is a right nonsingular module over A, by a theorem of R. E. Johnson, $Q = Q^r_{\max}(R)$ embeds uniquely in K (see, e.g., Lemma 3.1), hence either $K = Q$, or else $A = Q$ is right SI. Expressed otherwise, when K is nonsingular either A is right SI, or else K is VNR and $= Q$. Thus, A is then right SI assuming that A is inessential or that K is not VNR (Corollaries 3.3 A-C).

Furthermore if K is a VNR, *even without assuming that K is right nonsingular over A*, we have:

Theorem. *If A is a max VNR in a VNR right SI ring $K \neq Q^r_{\max}(R)$, then A is right SI provided that K is left strongly bounded ($= SB$) in that sense that every left ideal $\neq 0$ contains an ideal $\neq 0$.*

This theorem follows from the preceding result for K singular over A, without assuming SB; and otherwise from Theorem 5.5. We also have:

Theorem 5.6. *If an Abelian VNR ring K is right SI, and right singular over a max VNR subring A, then A is SI. Moreover,*

$$K \approx K_1 \times A$$

where K_1 is a skew field.

The next theorem follows from Utumi's theorem **[U3]**, **[U4]** that characterizes a continuous VNR ring A by the property that A contains all idempotents of $Q = Q^r_{\max}(A)$. (K Abelian in the theorem means that all idempotents are central).

Theorem 3.3E. *A max VNR subring A of a right self-injective Abelian (e.g., commutative) ring is continuous.*

What happens when a max VNR subring A does not contain all central idempotents of K? This is decided by Corollary 2.8: K must be VNR and either $K = Q$ or A is right self-injective.

0. Introduction

I. Wedderburn splitting and semisimple subrings.

A number of conditions on a ring K imply that a VNR subring A is right SI, without assuming that K is right SI, or that A is maximal, and we cite several below.

(I.1a) K contains no infinite sets of orthogonal indempotents.

(I.1b) K has the ascending condition on right (or left) annihilators, denoted $acc\perp$ (or $\perp acc$),

(I.1c) K is subring of a right (or left) Noetherian ring.

Note (c) $\Rightarrow$ (b) $\Rightarrow$ (a) for any subring A.

Furthermore, a VNR ring A with no infinite sets of orthogonal idempotents is semisimple Artin, and then every right or left A-module is injective.

(I.2) The center, cen K, of a VNR is VNR, and, moreover, right SI when K is VNR right SI by a theorem of Armendariz-Steinberg [**A-S**].

(I.3) If K is right SI, then by a theorem of Utumi [U], $\bar{K} = K/\operatorname{rad} K$ is right SI and VNR. Thus any VNR subring A such that $K = A + (\operatorname{rad} K)$ is necessarily right self-injective inasmuch as $A \cap (\operatorname{rad} K = 0$, hence $A \approx \bar{K}$. When this occurs, we say that K has WEDDERBURN SPLITTING. For any (not necessarily SI) finite dimensional algebra K over a field, this always occurs when K is a separable algebra ([J]), and then $\bar{A}$ is semisimple by (1.1) above.

II. Two cases of max VNR's.

The study of a max VNR subring A in a ring K devolves into two cases:

(II.1) A contains all central idempotents. *This must occur if K is not VNR,*

(II.2) $K = A[e] = A + Ae$ for a central idempotent e.

In this case K must be a VNR, and, moreover, a ring epic of A^2. (Theorem 4.5 and Corollary 4.3).

III. Case 1: A contains all central idempotents.

(III.0) Theorem. *Let A be a subring of a ring K that contains all central idempotents (e.g., suppose $A \supseteq \operatorname{Cen} K$).*

(III.1) If K is a ring product

$$K = \prod_{i \in I} K_i$$

then (III.0) implies:

$$A = \prod_{i \in I} A_i$$

where $A_i = A \cap K_i \ \forall i \in I$.

(III.2) Furthermore, A is then max VNR in K iff A_i is max VNR in $K_i \ \forall i \in I$.

When K is VNR, then A is said to be a **genuine max** VNR in K if A is maximal in the set of all VNR subrings $\neq K$.

(III.3) Corollary. *If A is a VNR subring containing all central idempotents of a VNR ring K, then (III.1) holds, and then, A is a genuine max VNR of K iff there exists $i_0 \in I$ such that A_{i_0} is a genuine max VNR of K_{i_0} and $A_i = K_i \ \forall i \in I - \{i_0\}$. Moreover:*

$$K = Q^r_{\max}(A) \text{ iff } K_{i_0} = Q^r_{\max}(A_{i_0}).$$

The proofs are trivial and are omitted.

To see how a VNR right SI ring K decomposes into a ring product, see [**G**, p. 110]. If the right SI ring K is a PI-ring, then by a theorem of Armendariz and Steinberg [**A-S**], $K = \Pi_{i \in I} K_i$, where each K_i is a full matrix ring over an Abelian VNR ring. (Cf. Theorem 5.3 in Section 5 below). Furthermore, in this case, K is an Azumaya algebra over its center by theorems of [**A-S**] and [**S**], see Theorem 2.4c.

Since $A + B$ is VNR for any VNR subring and VNR ideal B of K, then any max VNR A in a non VNR ring K must contain the maximal regular ($= VNR$) ideal $M(K)$. Moreover, the implication (III.1) above then establishes:

(III.4) Theorem. *If $K = M(K) \times K_2$, where $M(K)$ is the maximal regular ideal, and $K_2 \neq 0$, then any max VNR A splits:*

$$A = M(R) \times A_2$$

where A_2 is a max VNR in K_2 and $M(K_2) \neq 0$.

Thus, by **[F4]**, in the next corollary, we reduce the question of when A is SI to when A_2 is SI:

(III.5) Corollary. *If K is a right and left SI ring that is not a VNR, then K and any max VNR A decompose as stated in (3.4). In this case, $M(K)$ and K_2 are 2-sided SI, and, moreover, A is SI iff A_2 is SI.*

Actually, $M(K)$ splits off for any 2-sided continuous ring K (see [**F4**]). For a VNR right SI ring K, we apply a fundamental theorem of Utumi [**U3**] to decompose $K = K_1 \times K_2$, such that K_1 is Abelian (= strongly regular), K_2 has no nonzero Abelian ideals, and K_2 is generated by idempotents.

This is a corollary to Theorem 2 of [**U3**], and is cited in [**U4**, Theorem 3.2].

(III.6) Theorem. *If K is a VNR right SI ring, not a centralizing extension of a genuine max VNR A, then the stated Utumi decomposition induces a decomposition $A = A_1 \times A_2$, where $A_i = A \cap K_i$ is a genuine max VNR for exactly one $i \in \{1, 2\}$, and $A_j = K_j$ for $j \neq i$, $j = 1$, or 2.*

IV. Case 2: Does not contain all central idempotents.

In this case, $K = A[e] = A = eA$ for a central idempotent e, hence K must be VNR, and, as it happens, a ring epic of A^2 (Theorem 4.5 and Corollary 4.3). Furthermore, once again, we find that A is right SI when $K \neq Q$:

(IV.1) Theorem. *(2.7) If K is a right self-injective VNR, and if K is a centralizing extension of a genuine max VNR A, then either*

(1) $\quad$ *A is right SI,*

or

(2) $\quad$ $K = Q^r_{\max}(A)$.

(IV.2) Corollary. *Under the same assumptions, then A is right SI iff A is inessential in K as a right A-module.*

We next show that A need not be right SI in some cases.

(IV.3) Example. (Cf. (4.10)) If R is any subring of a right self-injective prime ring K, and if R contains a nonzero left ideal H of K, then $K = Q = Q^r_{\max}(R)$.

Proof: If $0 \neq a, b \in K$, then $aL \neq 0$ by primeness of K. Then, there exists $x \in L \subseteq R$ with $0 \neq ax \in L \subseteq R$ and $bx \subseteq R$. So $K = Q$. (See, e.g., [**F**, Chapter 8]). If follows that if A is any VNR subring of K containing R, then A is right SI iff $A = K$.

To give a specific example let $K = \operatorname{End} V_D$ for a right vector space V over a field D, and let H be the right socle of K (= the sum of all minimal right ideals). Since H is an essential right ideal, and VNR, then $R = D + H$ is VNR and essential in K as a right R-module, then $Q^r_{\max}(R) = K$ for any VNR subring $A \supseteq R$. Thus, no genuine max VNR containing R can be right SI.

In Theorem 5.7 we record the following curiosity: if a genuine max VNR subring A does not contain a maximal ideal of a right self-injective VNR ring K, then K is a subdirect product of simple homomorphic images of A (cf. Corollary 4.3). ■

Acknowledgements.

Thanks to Ken Goodearl for pointing out that Corollary 7 of [**F3**] requires the nonsingular hypothesis of the preceding Theorem 6. (This was left out inadvertently).

Grateful thanks to Manuel Castellet, the Director of CRM, for inviting me to CRM and for authorizing the Algebra semesters of Spring 1982 and Fall 1989.

To the amazing and beautiful Elena Carles, I am immensely indebted for her many efforts on behalf of my wife, Molly Sullivan and myself, and for helping to make our tenures in Catalunya so pleasant, and (for me) mathematically productive.

Finally, one can not say enough in praise of Pere Menal's students who contributed so much to the exciting mathematical atmosphere at UAB and CRM, not to say the jolly fun they generate at any occasion! (See Section 7).

Added in Proof. A theorem of Pere's, given by Pere's student, Claudi Busqué in [**B**], shows that any SI VNR ring R is a product of algebras over fields. This may be useful in studying the sequel.

1. Self-injective split-flat extensions, algebras and tensor products

If A is a subring of K then is a **split (resp. flat)** extension provided that A is isomorphic to a direct summand of K as a right A-module (resp., K is a flat left A-module). Thus, by a theorem of Azumaya, K is a **split extension** iff K generates mod-A. (See [**F5**, p. 145, Cor. 3.27(a)]). Furthermore, K is said to be **split-flat** over A, if K is both a split and flat extension of A. If K is an algebra over a commutative ring k, then K is a **split (flat) algebra** if K is a split (flat) extension of k. This implies, in either case, that K is faithful over k, equivalently that the canonical map $k \to K$ is an embedding.

1.1. Proposition. *If K is right self-injective, and if K is a flat left A-module for a subring A, then K is an injective right A-module. Moreover, in this case A is right self-injective iff K is right split over A, equivalently, A is a direct summand of K in mod-A.*

Proof: See [**F3**]. ∎

1.2A. Corollary. *Let K be right self-injective, and G be a finite group of automorphisms of unit order. Then, if $A = K^G$ is VNR, it is right self injective.*

Proof: K is flat over A since A is VNR, and a splits in K (both sides) by Maschke's theorem (e.g., see [**F5**, p. 475, Theorem 13.21]). ∎

The next result generalizes a theorem of [**P**]. (See Open Questions, Section 6).

1.2B. Corollary. *If a VNR K is right SI, and G is a group of automorphism of unit order, then the fix ring $A = K^G$ is right SI.*

1.3. Theorem. *If A and B are algebras over k, and if $K = A \otimes_K B$ is right self-injective, and if B is a split-flat algebra over k, then A is right self-injective.*

Proof: Since k splits in A, then $A \approx A \otimes_k k$ splits in K. Specifically: if $B \approx k \oplus X$ in mod-k, then for $Y = A \otimes_k X$

$$K \approx A \otimes_k B \approx (A \otimes_k k) \oplus (A \otimes_k X) \approx A \oplus Y$$

in mod-A. Moreover, K is left flat over A, since if $0 \to U \to V$ is exact in mod-A, then

$$U \otimes_A K \approx U \otimes_A (A \otimes_k B) \approx (U \otimes_A A) \otimes_k B \approx U \otimes_k B.$$

Since B is flat over k, then

$$0 \to U \otimes_k B \to V \otimes_k B$$

is exact, hence

$$0 \to U \otimes_A K \to V \otimes_A K$$

is exact.

Since K is therefore left flat and a right generator over A, then by Proposition 1.1, A is right self-injective. ■

1.4. Lemma. *If $K = A \otimes_k B$ is a flat k-algebra, and if $\otimes_k B$ is a faithful functor, then A is a flat k-module. A sufficient condition for faithfulness of $\otimes_k B$ is for B to be a k-split algebra.*

Proof: Let $0 \to X \to Y$ be exact in mod-R, and let

$$0 \longrightarrow U \longrightarrow X \otimes_k A \xrightarrow{\text{can}} Y \otimes_k A$$

be exact. Then

$$0 \longrightarrow U \otimes_k B \longrightarrow X \otimes_k K \longrightarrow Y \otimes_k K$$

is exact, but flatness of K then implies that $U \otimes_k B = 0$, whence $U = 0$ by faithfulness of ${}_kB$, so $0 \to X \otimes_k A \to Y \otimes_k A$ is exact. This proves that A is flat over k.

Next suppose that B is split over R, $k \approx R \oplus X$ in mod-R. Then obviously $\otimes_k B$ is faithful, since it has a faithful subfunctor: $\otimes_k \approx 1_{\text{mod-}R}$. ■

1.5. Theorem. *A tensor product $A \otimes_k B$ of modules over a commutative ring k generates mod-k iff both A and B generate mod-k.*

Proof: A module A generates mod-k iff the trace ideal $T_k(A) = k$, where for a k-module A, $T_k(A)$ is the image of the canonical map $A^* \otimes_k A \to k$, where A^* is the k-dual module of A. Now, tensoring over k, we have

$$\begin{aligned} K^* \otimes K &= A^* \otimes B^* \otimes A \otimes B \\ &= (A^* \otimes A) \otimes (B^* \otimes B) \end{aligned}$$

hence the trace ideal

$$T(K) = T(A)T(B).$$

For, if $\varphi_1 : A^* \otimes A \to k$ and $\varphi_2 : B^* \otimes B \to k$ are canonical, then the canonical map $K^* \otimes K \to k$ is $\varphi = \varphi_1 \otimes \varphi_2$ and so

$$T(K) = \operatorname{im} \varphi = \varphi_1 \otimes \operatorname{im} \varphi_2 = T(A)T(B)$$

it follows that $T(K) = k$ iff $T(A) = T(B) = k$, proving the theorem. ■

1.6. Corollary. *If $K = A \otimes_k B$ is a split algebra over k, then so are A and B. Moreover, if K is a split-flat algebra, then so are A and B.*

Proof: A and B are split algebras by Theorem 1.5 and Azumaya's Theorem cited at the beginning of this section and then A and B are flat algebras by Lemma 1.4. ■

1.7. Menal's General Tensor Theorem. *If K is a right self-injective split-flat algebra over a commutative ring k, and if $K = A \otimes_k B$, for subalgebras A and B, then A and B are right self-injective split-flat algebras.*

Proof: Corollary 1.6 and Theorem 1.3. ■

1.8. Corollary. *If $K = A \otimes_k B$ is a right SI faithful algebra over a SI commutative VNR k, then A and B are right SI.*

Proof: K is split-flat over k, so Menal's General Theorem applies. ■

2. Azumaya subalgebras and centralizing extensions

In this section we demonstrate that an Azumaya algebra A over k is (right and left) self-injective if A embeds in a right self-injective split-flat k-algebra. This follows from the more general theorem (Theorem 1.7) stating that if $K = A \otimes_k B$ is right self-injective "split-flat" algebra over k then A and B are right self-injective, and hence, then so is k.

We also note that any self-injective ring VNR ring K with polynomial identity ($= PI$) is Azumayan over its center (Theorem 2.4C). This follows easily from **[A-S]** and **[S]**.

For use below, if A is an algebra over a commutative ring k, the *enveloping algebra* A^e is defined to be the tensor product $A \otimes_k A^0$ of A and the opposite algebra A^0. Then the operation defined by the rule $X(a \otimes b^0) = bxa$ for any $a, b, x \in A$ defines A as a canonical (right) A^e-module. There is a canonical defined by $h : A^e \to \mathrm{End}_k A$

$$xh(a \otimes b^0) = bxa$$

and the image of h is the subring $A_d[A_s]$ of $\mathrm{End}_k A$ generated by the subring A_d consisting of all right, and the subring A_s consisting of all left, multiplications of the k-module A by elements of A.

2.0. Azumaya algebra definition and Theorem. An algebra A over a commutative ring k is called an *Azumaya algebra* if A satisfies the equivalent conditions.

(Az 1) A is a projective module over the enveloping algebra $A^e = A \otimes_k A^0$.

(Az 2) A is a finitely generated projective module over k, and $A^e = \mathrm{End}_k A$ canonically.

(Az 3) A^e is Morita equivalent to k.

(Az 4) A is finitely generated module over k, and for all maximal ideals m of k, the factor algebra A/mA is a central simple k/m-algebra.

(Az 5) A is finitely generated projective and central over k and every ideal I of A is of the form $I = I_0A$, where $I_0 = I \cap k$.

(Az 6) A generates mod-A^e and $k = \mathrm{End}_{A^e} A$.

When this is so, then Cen $A = k$.

Proof: Most of this is due to Azumaya (Az 1) over local k, and the carry-over to general k is in [**A-O**]. Also see [**B**].

The equivalence of (Az 5) with (Az 1) is a theorem of Rao [**R**]. Moreover, the Morita equivalence (Az 3) implies (Az 6). Conversely (Az 6) $\Rightarrow$ (Az 2), (Az 6) $\Rightarrow$ (Az 2) a theorem of Morita ([**F5**, p. 190, Theorem 4.1]) and Azumaya (in [**Az**]). ∎

2.1. Proposition. *If A is an Azumaya algebra over a commutative ring k, then the f.a.e.*

(2.1.A) *A is right self-injective.*
(2.1.B) *A is left self-injective.*
(2.1.C) *k is self-injective.*
(2.1.D) *A^e is right self-injective.*
(2.1.E) *A^e is self-injective.*

Proof: k is Morita equivalent to A^e, hence A^e is right self-injective iff k is self-injective, and this implies A^e is then left self-injective. Moreover, since Azumaya algebras are flat generators over k (in fact, finitely generated faithful projective modules over k (in fact, finitely generated faithful projective modules over k), then $A^e \approx A \otimes_k A^0$ implies via Theorem 1.3 (or Theorem 1.7) that A and A^0 are self-injective (both sides). ∎

2.2. Azumaya Theorem. *If A is an Azumaya algebra over a commutative ring k, and if A is a subalgebra of an algebra K over k, then $K \approx A \otimes_k A'$, where A' is the centralizer of A in K.*

Proof: [**B**, p. III-28, Corollary 4.3]. ∎

2.3. Corollary. *If K is right self-injective, and if a subring A is an Azumaya algebra over a subring k of $C = \operatorname{Cen} K$, then (a) A is right self-injective, (b) k is self-injective, and (c) A is left self-injective.*

Proof: $K = A \otimes_k A'$ by the theorem, and then A and A' are right self-injective by Theorem 1.7. Then k, also A on the left, is self-injective by Proposition 2.1. ■

We need a now classical result.

2.4A. Lemma (Azumaya). *If K is an Azumaya algebra over k, then for any proper ideal I of K, the subalgebra K/I is central over $R/(I \cap R)$.*

Proof: Suppose that $x \in K$ and that $xy - yx \in I$ for all $y \in K$. Let m be a maximal ideal of k so that $mK \supseteq I$. Then $\bar{K} = K/mK$ is central simple over $\bar{k} = k/mk$, consequently $\bar{x} \in \bar{k}$ proving that $x \in k$. This shows that $k/k \cap I$ is the center of K/I. ■

2.4B. Proposition. *If K is an Azumaya algebra over k, and if A is a (max) VNR subring, then $A \cap k$ is VNR.*

Proof: Write $A_0 = A \cap R$, and let $a \in A_0$. Then, by regularity of A, there exists $x \in A$ such that $axa = a$. Since $a \in k$, then $xa^2 = a$, if $y \in K$, then

$$yxa^2 = ya = ay = xa^2y = xya^2$$

so $yx - xy$ annihilates $a^2K = aK$. This implies that in K/aK that x maps onto an element of the center of K/aK, so the lemma implies that $x \in k$. Thus, $x \in A_0$ proving that A_0 is VNR.

A ring K is *biregular* provided that for all $a \in K$ every principal ideal KaK is generated by a central idempotent. Evidently, every simple ring is biregular, i.e., a biregular ring need not be regular. ■

We note the following characterization of Azumaya algebras over VNR's.

Szeto's Theorem. ([S]) *A ring K is an Azumaya algebra over a commutative VNR subring C iff K is a biregular ring finitely generated as a module over its center C.*

We also note:

Renault's Theorem. ([**Re2**]). *A VNR right SI ring K is biregular iff every prime ideal is maximal.*

Cf. [**Re2**, Proposition 3.7], which states that $Q^r_{\max}(R)$ is biregular when R is reduced.

2.4C. Theorem. *A VNR right SI ring K with PI is an Azumaya algebra over its center k.*

Proof: The center k of K is a VNR ring, and by [**A-S**, Theorem 3.7 and Corollary 3.2]) K is a biregular ring that is a finitely generated projective module over its center k. It follows from Szeto's Theorem that K is an Azumaya algebra over k. ■

2.4D. Notes. 1. Theorem 2.4C is implicit in [**A-S**] where the *ideal correspondence*

$$\phi\begin{cases} \text{Ideals } k & \longrightarrow \text{Ideals} K \\ I & \longrightarrow IK \end{cases}$$

is proved a bijection with $\phi^{-1}(J) = J \cap k$ for an ideal J of K. This is a consequence of the biregularity of K see [**A-S**], footnote added in proof. (Also see [**A-S**, Corollary 3.6], where the ideal correspondence is proved using the fact that K is a direct sum of finitely many Azumaya algebras!).

2. Using 1, Theorem 2.4C also follows from (Az 5) of Theorem 2.0.

2.4E. Corollary. *Let A be a VNR central subalgebra of a VNR right SI algebra K with PI over its center k. Then A is self-injective iff A is an Azumaya algebra over k.*

Proof: If A is Azumaya over k, then A is SI by Corollary 2.3. Conversely, if A is SI, then A is Azumaya over k by Theorem 2.4C. ■

2.4F. Remark. In [**A-S**], the PI's have coefficients in the centroid of K, but since K is SI, the centroid is k. (At least one coefficient must be a unit).

Centralizing extensions.

An ring K is a *central extension* of a subring A provided that K is the subring $A[C]$ generated by A and $C = \operatorname{Cen} K$. If $K = A[A']$ is generated by A and its centralizer $A' = \operatorname{Cen}_K A$, then we say that K is a *centralizing extension* of A. By Theorem 2.1, K is a centralizing extension of any subalgebra that is Azumaya over a subring k of $\operatorname{Cen} K$.

An idempotent $e \in K$ *centralizes* a subring A if, a belongs to the centralizer of A in K.

2.5. Lemma. *If A is a VNR subring of a ring R, then so is $A[e] = A + Ae$ for any idempotent e of K that centralizes A.*

Proof: A ring B is VNR if there exists a VNR ideal I so that B/I is VNR. (In this case every factor ring and every ideal is VNR). In our case, let $B = A + Ae$, and then $I = Ae$, is an ideal of B and is a VNR ring since it is a homomorphic image of A. Moreover, $B/I \approx A/(I \cap A)$ is also a homomorphic image of A, so B is VNR. ■

If $N \subseteq M$ are A-modules, then $N \underset{\text{ess}}{\subset} M$ denotes that N is an *essential submodule* of M.

If M is a right A-module, then the *singular submodule* $\operatorname{sing} M_A$ is defined by

$$\operatorname{sing} M_A = \{m \in M \mid \operatorname{ann}_A m \underset{\text{ess}}{\subset} A\}$$

where, as stated,

$$\operatorname{ann}_A m = \{a \in A | ma = 0\}.$$

Now $S = \operatorname{sing} M_A$ is a fully invariant submodule of M, that is, $bs \in S$ for all $b \in \operatorname{End} M_A$ and all $s \in S$. This implies that $\operatorname{sing} A_A$ is an ideal. If $\operatorname{sing} M_A = 0$, then M is said to be *nonsingular*, otherwise *singular*.

2.6A. Lemma. *If K is a ring and A is a subring, then the right singular A-submodule $S = \operatorname{sing} K_A$ is a (K, A)-submodule of K, hence a left ideal of K.*

Proof: S is a fully invariant A-submodule of K. ■

2.6. Proposition. *If K is a right self-injective ring and right nonsingular over a subring A, then $S = \operatorname{sing} K_A$ is an ideal of K under any of the conditions:*

(1) *K is a central extension of A.*
(2) *$K = A[A']$ is the subring generated by A and the centralizer A' of A in K.*
(3) *$A \subseteq C = \operatorname{Cen} K$.*

Proof: Since S is a (K, A)-submodule, then S is an ideal iff $SK \subsetneq S$. It suffices to prove (2), since $A' \supseteq C$ so $K = A[C] \Rightarrow K = A[A']$. Also, $A \subseteq C \Rightarrow A' = K$ so $K = A[A']$ in this case too.

(2) If $b \in A'$ and $s \in S$, then

$$sb(s^{\perp} \cap A) = s(s^{\perp} \cap A)b = 0$$

so $sb \in S$, that is, S is a right $A[A']$-submodule of K. Then $K = A[A']$ implies that S is an ideal. ■

2.7. Theorem. *If K is a right self-injective VNR, and a centralizing extension of a genuine max VNR subring A, then either*

(1) *A is right self-injective; or else*
(2) *$K = Q^r_{\max}(A)$*

Proof: If $K = A[A']$ is a centralizing extension of A, then S is an ideal of K, hence S is a VNR ring-1. If $S \neq 0$, then $S + A$ is a VNR subring properly A, so $K = S + A$ by maximality of A. Thus, A splits in K in mod-A (also in A-mod), so A is injective by Proposition 2.1. If $S = 0$, then Corollary 2.3 A applies. ■

2.8. Corollary. *If K is right self-injective, and if A is a genuine maximal VNR not containing all central (or centralizing) idempotents in K, then K is a VNR ring, and either A is right self-injective, or else $K = Q^r_{\max}(A) = A + Ae$, for a central idempotent e.*

Proof: Suppose that $e = e^2$ belong to the centralizer A' of A in K but non in A. Then, since $A + Ae$ is a regular ring by Lemma 2.5, $K = A + eA = A[e]$ is a central extension of A, so the theorem applies. ■

3. Self-injective rings nonsingular over regular subrings

This section is mainly devoted to rings described by the section title, but in some instances other conditions are considered, namely K flat over a nonsingular subring A, or K Galois over regular subrings, without assuming K nonsingular over A.

We also consider dense flat maximal subrings of self-injective rings.

If $M \supset N$ are modules over A, and if $x \in M$, then

$$(x : N) = \{a \in A | xa \in N\}$$

is the **conductor** of x in N. If N is an essential submodule of M, then $(x : N)$ is an essential right ideal of A. The annihilator of x in A is $(x : 0)$, also denoted by $\operatorname{ann}_A x$.

A subring A of K is *left (right) essential* if $A \underset{ess}{\subset} K$ as a left (right) A-module. Otherwise A is *left (right) inessential* in K.

3.1. Lemma. *If a right self-injective ring K is right nonsingular over a VNR subring A, then $Q = Q^r_{\max}(A)$ embeds uniquely as a subring of K*

containing A. A sufficient condition is that A is a right or left essential VNR subring of K.

Proof: This is essentially a theorem of R. E. Johnson (see, e.g. [**F**, Chapter 8]). We give a proof here for completeness and comprehensibility. By a theorem of Auslander-Harada, every (right and left) A-module is flat (see, e.g. [**F5**, p. 434, Theorem 11.4]). By Lemma 2.6A, $S = \operatorname{sing} K_A$ is a (K, A)-submodule of K. Since $S \cap A = 0$ by nonsingularity of A, then $S = 0$, when A is left or right essential in K. In this case, K is right non-singular over A.

Since K is (right and left) flat over A, by Theorem 1.1 then K is injective in mod-A, hence contains an injective hull $E(A) = E$ of A. We first prove that E is the unique injective hull of A contained in K: If F were another one, then there is an isomorphism $f : E \to F$ of A-modules. But if $x \in E$, the conductor of x in A is an essential right ideal I (notation $I = (x : A)$) and

$$xa = f(xa) = f(x)a \; \forall a \in I$$

so $x - f(x)$ annihilates I, hence belongs to the singular submodule $\operatorname{sing}_A K$. Since K is nonsingular, then $\operatorname{sing}_A K = 0$, so $f(x) = x \; \forall x \in E$, hence $F = E$.

Now as in [**F**, Chapter 8, Theorems 1 and 2], E has the structure of the maximal quotient ring $Q = Q_{\max}(A)$, so it remains to show that the ring product $x \cdot y$ in Q coincides with that in K for all $x, y \in E$. To do this if I the conductor of y in A, then

$$(x \cdot y)a = x(ya) = x(ya) = (xy)a$$

so $x \cdot y - xy$ annihilates an essential right ideal I, hence $x \cdot y - xy \in \operatorname{sing} K_A = 0$. This proves that $x \cdot y = xy \; \forall x, y \in E$ and therefore Q embeds in K. ∎

3.2. Proposition. *If A is a semisimple subring of a ring K, then K is right and left nonsingular over A.*

Proof: The only essential one-sided ideal of a semisimple ring A is A. Since $1 \in A$, then both right and left A-singular A submodules of K are zero. ∎

3.3A. Corollary. *If K is a right self-injective ring right nonsingular over a maximal VNR A, then either $K = Q^r_{\max}(A)$, in which case K is VNR, or A is right self-injective.*

Proof: K is right nonsingular over A, so $Q = Q^r_{\max}(A)$ embeds in K by the theorem. However, Q is a regular right self-injective ring, so either $Q = K$, or else $Q = A$ is right self-injective. ■

3.3B. Corollary. *If A is a right essential VNR subring of a right self-injective ring K, then $K = Q^r_{\max}(A)$, hence K is VNR.*

Proof: A is right essential, hence K is a nonsingular right A-module, so Lemma 3.1 applies, that is, $K \supseteq Q^r_{\max}(A)$, hence $K = Q^r_{\max}(A)$. ■

3.3C. Corollary. *If K is a right self-injective ring right nonsingular over a max VNR A, and if either K is not VNR or A is right inessential, then A is right self-injective.*

Proof: In either case, $K \neq Q$, hence $A = Q$ by Corollary 3.3A. ■

3.3D. Corollary. *Let K be right SI. If every nonzero left ideal of K contains a nonzero central idempotent then K is right nonsingular over any right nonsingular subring A containing all central idempotents, hence the conclusions of Corollary 3.3C hold.*

Proof: The right singular submodule $S = \operatorname{sing} K_A$ is a left ideal of K satisfies $S \cap A = 0$ since A is right nonsingular. But $S \neq 0$ implies the existence of a central idempotent $e \neq 0$ in S which violates $S \cap A = 0$.

If every idempotent of top $K = K/J$ lifts to an idempotent of K, we say that *top idempotents lift*. A ring K is *Abelian* if all idempotents are central. ■

3.3E. Theorem. *If K is an Abelian self-injective ring, then a max VNR A is continuous.*

Proof: We may assume K is not VNR. By Lemma 2.5, A contains all idempotents. By a theorem of Utumi [**U1**], all top idempotents of a self-injective ring lift, so $\bar{A}$ contains all idempotents of $\bar{K} = K/J$. By Lemma 3.3D, $\bar{K}$ is nonsingular over $\bar{A} \approx A$, hence contains $Q_{\max}(\bar{A}) \approx Q_{\max}(A)$, and therefore A is continuous by Utumi's Theorem ([**U3**], [**U4**]). ■

3.3F. Corollary. *If K is a right self-injective ring nonsingular over a maximal commutative VNR A, then either $K = Q^r_{\max}(A)$, or else A is self-injective.*

Proof: By Theorem 3.1, $Q = Q_{\max}(A)$ embeds in K. Since Q is commutative VNR and self-injective, then by maximality of A, either $K = Q$ or $A = Q$. Since Q is SI, the latter implies that A is. ■

3.3G. Corollary. *If K is a right SI ring that is right nonsingular over a max commutative VNR A, and if either (1) K is noncommutative or (2) K is not VNR, or (3) A is inessential, then A is SI.*

Proof: Same as Corollary 3.3C. ■

3.3H. If a max commutative VNR A is right or left essential in a right SI ring K then $K = Q_{\max}(A)$ is commutative.

Proof: K is right nonsingular over A by Theorem 3.3. Since A is essential but $\neq K$, then A cannot be injective, hence $A \neq Q_{\max}(A) = K$, by Corollary 3.3D. ■

4. Central idempotents and direct products

In this chapter we prove theorems on a maximal VNR subring A of a direct product $K = \Pi_{\alpha\in\Lambda} K_\alpha$.

The first result restates Theorem (III.0) of the Introduction.

4.1. Trivial Lemma. *1. If a subring A contains all central idempotents of a ring K, then any direct product representation $K = \Pi_{\alpha\in\Lambda} K_\alpha$ induces a direct product representation $A = \Pi_{\alpha\in\Lambda} A_\alpha$ where $A_\alpha = K_\alpha \cap A$. (If e_α is the unit element of $K_{\alpha'}$ then $A_\alpha = e_\alpha A$).*

2. If K is VNR, then A is genuine max VNR in K iff there exists $\beta \in \Lambda$ such that $A_\alpha = K_\alpha$ for all α except $\alpha = \beta$ and A_β is a genuine max VNR in K_β.

4.2. Theorem. *Let A be a genuine max VNR of K, and let e be any central idempotent.*

(1) *If $e \notin A$, then $eA = eK$ and $(1-e)A = (1-e)K$, and*

$$K \approx eA \times (1-e)A \approx A/(A \cap (1-e)A) \times A/A \cap eA.$$

(2) *If $e \in A$, then either (2a) or (2b) holds:*

(2a) $eK = eA$, whence

$$K = eA \times (1-e)K$$

(2b) $(1-e)K = (1-e)A$ whence

$$K = eK \times (1-e)A.$$

Moreover, any idempotent $e \in \mathrm{Cen}_K A$ is central.

Proof: By Lemma 2.5, $B = A + eA$ is a regular ring, hence in case $e \notin A$ necessarily $B = K$ by maximality of A. The rest of (1) is evident.

Since $K \neq A$, then either $eK \neq eA$ or $(1-e)K \neq (1-e)A$, when $e \in C \cap A$. Hence in case $(1-e)K \neq (1-e)A$, then by Lemma 2.5, $B = A + (1-e)K$ is a VNR subring properly containing A, hence $B = K$, so (2a) holds, and (2b) holds in the contrary case.

Now let $e \in \mathrm{Cen}_K A$. One easily shows that $B_1 = eKe + A$ is a VNR subring containing A, and similarly for $B_2 = (1-e)K(1-e) + A$. If $B_1 \neq A$, then $K = eKe + A$, hence $(1-e)K = (1-e)A$ so $(1-e)Ke = 0$. By symmetry $eK(1-e) = 0$, so e is central. Similarly when $B_2 \neq A$, then $1-e$, whence e, is central. Finally $B_1 = B_2 = A$ is impossible since $A \neq K$. ■

4.3. Corollary. *Let A be a genuine max VNR of a VNR ring K, and let e be a central idempotent. If $e \notin A$, then K is an epic image of A^2, hence is VNR; and if $e \in A$, then either eA or $(1-e)A$ is a direct factor of K.*

4.4. Theorem. *If A is a genuine maximal subring of a VNR ring K. Let $K = K_1 \times K_2$, and let e_i denote the identity of K_i, $i = 1, 2$. If one $e_i \notin A$, say $e_1 \notin A$, then $e_i A = K_i$, $i = 1, 2$, so*

(1) $$K = e_1 A \times e_2 A.$$

Moreover, if one hence both $e_i \in A$, $i = 1, 2$, then

(2a) $$K = e_1 A \times K_2$$

or

(2b) $$K = K_1 \times e_2 A.$$

In (2a), $e_1 A$ is a genuine max VNR of $K_{1'}$, and in (2b), $e_2 A$ is a genuine max VNR of K_2.

Proof: Follows from Theorem 4.2 and the Trivial Lemma 4.1. ■

4.5. Theorem. *If A is a maximal (commutative) VNR subring of a non-VNR ring K, then A contains all central idempotents, and moreover all idempotents of K that centralize A.*

Proof: By Lemma 2.5, $B = A + Ae$ is a VNR subring of K, hence $B = A$ by maximality (also in case A is a maximal commutative VNR subring), so $e \in A$ for all idempotents $e \in A'$. Since $A' \supset C = \mathrm{cen}\, K$, then A contains all central idempotents. ■

4.6. Corollary. *If A is a maximal (commutative) VNR subring of a non-VNR ring K, and if $C = \operatorname{Cen} K$ is generated by idempotents then $A \supseteq C$. Moreover, if A' is generated by idempotents, then $A \supseteq A'$. In case A is right SI, then so is A' in the latter case.*

Proof: If A' is generated by idempotents, then $A \supseteq A'$ by the theorem. If A is right SI, then so is $A' = \operatorname{Cen} A$ by the theorem of [**A-S**]. ∎

4.7. Corollary. *If A is a maximal commutative VNR subring of a non VNR ring K, and if A' is generated by idempotents then A is a maximal commutative subring of K.*

Proof: Since $A \supseteq A' \supseteq A$, then $A = A'$ is a maximally commutative subring in K. ∎

4.8. Corollary. *A commutative ring K generated by idempotents is a VNR ring.*

Proof: A union of VNR rings is VNR, so if we deny the theorem, then K contains a genuine maximal VNR subring A. By the corollary, however, A contains $\operatorname{Cen} K = K$. ∎

4.9. Corollary. *If A is a maximal commutative VNR subring of a noncommutative VNR ring, and if $\operatorname{Cen} K$ (resp. A') is generated by idempotents, then $A \supseteq \operatorname{Cen} K$ (resp. $A = A'$ is a maximal commutative subring).*

Proof: Same. ∎

We use the concept of a *dense submodule* M of a right A-module K, equivalently, K is a *rational extension* of M. The condition is that

$$\operatorname{Hom}_A(T/M, K) = 0$$

for all submodules T of K containing M. See [**L2**] or [**F**], [**F2**] for other characterizations, and the background for the following.

K is *right torsion free* over A provided that no nonzero element of K annihilates a dense right ideal of A. When A is right nonsingular, then a right ideal I is dense iff I is an essential right ideal, so torsionfreeness is equivalent to nonsingularity over non-singular rings.

A subring A of K is *right dense* if A is a dense right A-submodule of K, and A is *left flat* if K is a flat left A-module.

4.1′ Theorem. *If K is a right self-injective ring and right torsionfree over a left flat subring A then $Q = Q^r_{\max}(A)$ embeds uniquely as a subring of K containing A.*

Proof: The proof is the same *mutatis mutandi* as the proof of Theorem 3.1.

Similarly the following corollaries are proved as the corresponding corollaries above. ∎

4.3′A. Corollary. *If K is a right self-injective ring torsionfree over a left flat maximal subring A, then either $K = Q$, or else $A = Q$.*

4.3′B. Corollary. *If A is a right dense left flat subring of a right self-injective ring K, then $K = Q$.*

Corollary 4.3′C does not have enough content to restate.

4.3′F. Corollary. *If K is a right self-injective ring is right or left torsion free over a flat maximal commutative subring A, then either $K = Q$, or $A = Q$.*

4.3′G. Corollary. *If a right self-injective K has a dense flat maximal commutative subring A, then $K = Q$, hence K is commutative and Q is self-injective.*

4.10. Proposition. *If K is a right self-injective ring, and if a max VNR subring A contains either a faithful left ideal L of K or a dense right ideal I of K, then $Q^r_{\max}(A) = K$.*

Proof: If $0 \neq x \in K$, then $xL \neq 0$, hence $xA \cap A = 0$, so A is a right essential subring of K, and $Q^r_{\max}(A) = K$ by Corollary 3.3B.

Inasmuch as K is a rational extension of I, then $K = Q^r_{\max}(I)$ and it follows that $K = Q^r_{\max}(A)$. ∎

Note. The max VNR hypothesis is not needed in the second instance.

5. Right self-injective rings singular over regular subring

A ring K is **left bounded** if every essential left ideal contains an ideal; K is left **strongly bounded** (= left SB) if every nonzero left ideal a nonzero ideal. If K is both left and right SB, we say that K is SB.

A simple Artinian ring K is SB, since it has no essential one-sided ideals $\neq K$, but not left SB since K has no non zero ideals $\neq K$.

A ring K is **left (right) duo** if every left (right) ideal is an ideal. A left duo ring is thus trivially left SB. A **duo** ring is both left and right duo, hence SB. Any Abelian VNR K is duo, hence SB, since, e.g. if L is a left ideal, and $x \in L$, then $Kx = Ke$ for an idempotent $e \in \operatorname{Cen} K$. Thus, $x \in Ke = eK$ so

$$xK \subseteq Kx = Ke \subseteq L$$

proving L is an ideal.

A ring K has **bounded index** ($= BI$) n provided that every nilpotent element has index $\leq n$, but at least one has index n. Then K is a BI **ring**, on **has** BI.

A VNR is **unit regular** ($= UR$) provided that for every $a \in K$ there is a unit $x \in K$ such that $axa = a$. Then we say R is a $UVNR$ **ring**.

A ring R is **Dedekind Finite** (= directly finite in [**G**, p. 49]) if $xy = 1$ for $x, y \in R$ iff $yx = 1$. Any Abelian VNR ring K is UR; any $UVNR$ ring K is DF, and any $n \times n$ full matrix ring over an Abelian, or UR, or DF, VNR is again DF. (See [**G**, p. 50, prop. 5.2 and 5.3]).

If a ring K has no infinite set of orthogonal idempotents, then K is Dedekind finite by a theorem of Jacobson (see e.g., [**F6**, p. 85] ff.).

5.1. Remark. If I is an ideal of a ring K, then K is VNR iff both I and K/I is VNR. Moreover, any (left, right) ideal I_0 of I is a (left, right) ideal of K.

Proof: See [**G**, p.2, Lemma 1.3] for the first statement. The proof shows that if $a \in I$, and if $axa = a$ for some $x \in K$, then $y = xax$ is an element of such that $aya = a$. Now let I_0 be, e.g., a left ideal of I, and $a \in I_0$. Then $e = ya$ satisfies $e^2 = e \in Ka$, and then

$$ae = aya = a \in Ka$$

so that

$$Ka = Ke = (Ky)a \subseteq Ia \subseteq I_0,$$

that is, I_0 is a left ideal of K. ■

5.2. Proposition. *Let $S = \operatorname{Sing} K_A$ denote the right singular submodule of a ring K over a right nonsingular subring A.*

Then:

1. *S is a left ideal of K such that $SA \subseteq S$ and $S \cap A = 0$.*
2. *If $e = e^2 \in S$, then $eK \cap A = Ke \cap A = 0$, $(1-e)A \approx A$, and $(1-e)A \cap A$ is an essential right ideal of A.*
3. *If K is injective in mod-A (e.g., if K is right self-injective, and a flat left A-module), then $(1-e)K$ contains an injective hull of $(1-e)A \approx A$ in mod-A.*

Proof: The singular submodule of a module is fully invariant, so S is a (K, A)-submodule of K since $K \hookrightarrow \operatorname{End} K_A$ canonically. Since A is right singular, then $S \cap A = 0$, and hence $Ke \cap A = 0$ and also $eA \cap A = 0$. Since $eK \cap A \subsetneq eA \cap A$, then $eK \cap A = 0$. Thus $(1-e)A \approx A$. Furthermore $(1-e)A \cap A$ is an essential right ideal of A since it is the annihilator in A of $e \in S$. This proves (1) and (2), and (3) follows from injective module-theory, since, under the hypotheses, K whence $(1-e)K$ is injective over A. (The parenthetic assertion comes from Proposition 1.1). ■

5.3. Theorem. *Let K be a right self-injective VNR ring. The following conditions are equivalent:*

(1) *K is BI.*
(2) *K is PA.*
(3) *K/M is artinian for all maximal ideals.*
(4) *K is isomorphic to a finite product of full matrix rings over abelian regular rings.*

In this case K is self-injective.

Proof: See [**G**, p. 79]. (1) $\Rightarrow$ (2) is a theorem of Utumi, (1) $\Rightarrow$ (4) is a theorem of Kaplansky [**K**], assuming all K/M have the same index, and Armendariz-Steinberg [**A-S**] in the general case. (1) $\Rightarrow$ (3) is a remark of Goodearl's (loc. cit.), and the last assertion is a theorem of Utumi [**U2**]. ■

5.4. Theorem. *Let L be a right self-injective VNR ring.*

(1) *(Renault) If K is Dedekind finite then: $(*)$ every nonzero ideal contains a central idempotent.*
(2) *(Utumi [**U1**]) If K is also left $(\aleph_0-)$ self-injective then K is Dedekind finite, hence $(*)$ holds.*
(3) *(Utumi) If K is a PA, equivalently, BI ring, then K is left self-injective, hence $(*)$ holds.*

Proof: See [**G**, p. 105, Theorem 9.25 and 9.29] for (1) and (2), and [**G**, p. 70, Corollary 6.22] for (3). Also see *op. cit.*, pp. 70, 79 and 109 for specific references. ■

5.5. Theorem. *If a right self-injective VNR ring K is right singular over a max VNR A, and if K is left strongly bounded, then A is a right self-injective, and the right singular A-Submodule S of K is a minimal ideal such that $K/S \approx A$, and*

$$K = S + A \text{ and } S \cap A = 0.$$

Proof: By Proposition 5.2, S is a left ideal of K such that $S \cap A = 0$. Let L be a nonzero ideal contained in S. Then, L is a regular ideal and by Remark 5.1, $L + A$ is a regular ring $\supset A$ so $L + A = K$. Since $S \supset L$, and $S \cap A = 0$, then $S = L + (S \cap A) = L$ is a minimal ideal and $K = S + A$. Since A is a direct summand of K in mod-A, then A is right self-injective by Proposition 1.1 and clearly $K/S \approx A$. ■

5.6. Theorem. *If a right SI Abelian VNR ring K is right singular over a max VNR subring A, then A is right SI, and isomorphic to a ring direct factor K_2 of K:*

$$K = K_1 \times K_2 \approx K_1 \times A$$

where K_1, the singular right A-submodule of K is a skew field.

Proof: K is Abelian, hence SB, hence Theorem 5.5 applies A is a right SI, and $K_1 \cap A = 0$, where $K_1 = \operatorname{sing} K_A$ is a (K, A)-submodule of K. Since K is Abelian, hence duo, then every one-sided ideal of K is an ideal. By Theorem 5.5, K_1 is a minimal ideal of K, and by Remark 5.1, every (left, right) ideal I of K_1 is a (left, right) ideal of K, hence $I = K_1$. This shows that K_1 is a skew field.

Moreover, if $e = e^2 \in K_1$ is a nonzero idempotent, then e is central and

$$K_1 = eK = Ke = eKe$$

splits off of K as a ring factor:

$$K = K_1 \times K_2$$

where $K_2 = (1 - e)K$. Now, since $K/K_1 \approx A$ by Theorem 5.5, then $K_2 \approx A$ as rings, in fact since $K_1 \cap A = 0$, then $eA \cap A = 0$, hence

$$K_2 = (1 - e)K = (1 - e)A \approx A.$$

Finally, by Theorem 5.3, both K and A are 2-sided SI. ■

Max K denotes the set of maximal ideals of K.

5.7. Theorem. *Let A be a genuine maximal VNR subring of directly finite right self-injective VNR ring K.*

(1) *If A contains a maximal ideal M of K, then A/M is a genuine maximal VNR subring of a simple right self-injective VNR ring K/M.*
(2) *Otherwise*

$$K \underset{\text{subdirect}}{\hookrightarrow} \Pi_{M_a \in \max K} K/M_a \approx \Pi_a A/(A \cap M_a)$$

i.e., K is isomorphic to a subdirect product of simple homomorphic images of A.

Proof: Theorem 9.32 of [**G**] implies K/M is right SI, so (1) is evident. If (1) fails, then M_a is a regular ideal $\not\subseteq A$, hence by Remark 5.1, $K = A + M_a \in \operatorname{Max} K$. By Corollary 9.26 of [**G**], necessarily $\cap_A M_A = 0$, hence K is a subdirect product as stated. ∎

Note. Corollary 9.26 is a consequence of a theorem of F. Maeda. See notes, p. 109 of [**G**], where also the origin of Theorem 9.32 is sketched.

6. Open Questions

There are many open problems associated with max VNR subrings of rings, and this paper barely scratches the surface. Here are sample questions.

Does Menal's Theorem hold for the general context of Menal's General Theorem 1.7? Namely, is A or B algebraic over k, if $A \otimes_k B$ is SI (or VNR)?

Can Theorem 5.6 be extended to a full $n \times n$ matrix $K = R_n$ over a right SI Abelian VNR ring R, i.e., if K is right-singular over a max VNR subring A, is A necessarily SI? If so, is A isomorphic to a ring direct factor of K?

What is the Galois Theory for SI VNR subrings of a VNR right SI ring K?

By Proposition 1.1, the invariant or Galois subring $A = K^G$ is also right SI whenever K is split-flat over A. If $|G| = n$ is a unit, then A is SI, when ${}_AK$ is flat, e.g. when A is VNR (Corollary 1.2A). This and Corollary 1.2B generalizes the theorem of A. Page [**P**].

7. Pere Menal i Brufal: A Memento

It would be impossible to fully express in this brief space the debts of friendship, both personal and mathematical, that I owe to the late Pere Menal.

It all began in 1981 when I wrote to Pere about his work [**M**] that I discussed in the Abstract on algebraic regular rings in connection with tensor products. He expressed amazement and delight that it was connected with the Hilbert Nullstellensatz.

After some years of correspondence, we began contemplating an algebra semester under the auspices of Manuel Castellet's CRM, and we went on to organize one in Spring 1986; another much larger conference followed in Fall 1989.

Much creative mathematics flowed out of these conferences, particularly the theorem of Pere Menal and Peter Vámos [**M-V**] that realized a three decades-old dream of ring theory: an embedding of any ring in an FP-injectve ring[1].

In the 1989 semester, Pere and I collaborated on a problem that eluded us individually for many years, and we finally found that what we were looking for: "A counter-example to a conjecture of Johns" which just appeared in the Proceedings of the American Mathematical Society (1992).

These are just two of the myriad collaborations that Pere Menal had with others: Jaume Moncasi, Pere Ara, Claudi Busqué, Ferran Cedó, Dolors Herbera, Rosa Camps (his talented students and colleagues) and Brian Hartley, Warren Dicks, Kenneth Goodearl, the aforementioned Peter Vámos, Boris Vaserstein (to mention but a few of his intense interactions with others).

In particular, Pere's paper with Dolors Herbera **[H-M]**, Ferran Cedó paper **[C]**, Dolors Herbera's paper with Poobhalan Pillay **[H-P]**, and her Doctoral Thesis at U.A.B., each greatly advanced our knowledge on subjects taken up in seminars during these conferences.

I am grateful that I happened to write to Pere back in 1981; otherwise, I might never have met this noble and gentle genius of Catalunya, who became an inspiration to so many.

Wherever Pere is now, I like to think that he is continuing his work that he left here on earth, and collaborating with the Great One.

References

[Az] G. AZUMAYA, On maximally central algebras, *Nagoya Math. J.* **2** (1951), 119–150.

[A-O] M. AUSLANDER AND O. GOLDMAN, The Brauer group of a commutative ring, *Trans. A.M.S.* **97** (1960), 367–409.

[A-S] E. ARMENDARIZ AND S. STEINBERG, Regular self-injective rings with PI, *Trans. Amer. Math. Soc.* **190** (1974), 417–425.

[B] H. BASS, "*Lectures on topics in algebraic K-theory*," Tata Inst. for Advanced Study, Colaba, Bombay, 1967.

[Bu] C. BUSQUÉ, Centers and two-sided ideals in right self-injective regular rings, *Journal of Algebra*, to appear.

[C-E] H. CARTAN AND S. EILENBERG, "*Homological Algebra*," Princeton, 1956.

[1]In her Rutgers University ph.D. thesis (1964), Barbara Osofsky had shown that in general the injective hull of a ring R could not be made into a ring containing R as a subring. It is also known that R can not always be embedded in a self-injective ring.

[C] F. CEDÓ, Zip rings and Mal'cev domains, *Comm. Alg.* **19** (1991), 1983–1991.

[F] C. FAITH, "*Injective Modules and Quotient rings*," Lecture Notes in Math. **49**, Springer-Verlag, 1967.

[F2] C. FAITH, "*Injective modules and injective quotient rings*," Lecture Notes in Pure and Applied Math. **72**, Marcel Dekker, 1982.

[F3] C. FAITH, Subrings of self-injective and FPF rings, in "*Advances in Non-Commutative ring theory*," Lecture Notes in Math. **951**, 1982, pp. 12–20.

[F4] C. FAITH, The maximal regular ideal of self-injective and continuous rings splits off, *Arch. der Math.* **44** (1985), 511–521.

[F5] C. FAITH, "*Algebra I: Rings, Modules and Categories*," Springer Verlag, Berlin, Heidelberg, New York, 1973. Corrected Reprint 1981.

[F6] C. FAITH, "*Algebra II: Ring Theory*," Springer Verlag, 1976.

[F-M] C. FAITH AND PERE MENAL, A counter-example to a conjecture of Johns, *Proc. Amer. Math. Soc.* **116** (1992), 21–26.

[F-P] C. FAITH AND S. PAGE, "*FPF Ring Theory: Faithful modules and generators of mod-R*," Lecture Notes of the London Math. Soc. **88**, Cambridge University Press, Cambridge, London, New York, Melbourne, Sydney, 1984.

[G] K. R. GOODEARL, "*Von Neumann regular rings*," Pitman, 1979.

[H] DOLORS HERBERA, On rings whose finitely generated faithful modules are generators, Ph.D. Thesis (in catalan), Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona (1992).

[H-M] DOLORS HERBERA AND PERE MENAL, On rings whose finitely generated faithful modules are generators, *J. of Algebra* **122** (1989), 425–438.

[H-P] DOLORS HERBERA AND P. PILLAY, Injective classical rings of quotients of the polynomial rings are quasi-Frobenius, *J. Pure and Applied Algebra*, to appear.

[J] NATHAN JACOBSON, "*The Structure of rings*," American Math. Society, Colloquium Publication **37**, Providence, R.I., 02940, 1956 (rev. ed. 1964).

[K] I. KAPLANSKY, Topological representation of algebras, *Trans. Amer. Math. Soc.* **68** (1950), 62–75.

[L] J. LAMBEK, A module is flat iff its character module is injective, *Canad. Math. Bull* **7** (1964), 237–243.

[L2] J. LAMBEK, "*Lectures on rings and modules*," Chelsea, New York.

[M] P. MENAL, On tensor products of algebras being Von Neumann regular or self-injective, *Comm. Alg.* **9(7)** (1981), 691–697.

[M-V] P. MENAL AND P. VÁMOS, Pure ring extensions and self-FP-injective rings, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), 447–458.

[P] A. PAGE, "*Actions de groupes*," Sem. P. Dubreil (1977-78), Lecture Notes in Math. **740**, Springer Verlag, 1979.

[R] R. L. RAO, Azumaya, Semisimple and ideal algebras, *Bull. Amer. Math. Soc.* **78** (1972).

[Re] G. RENAULT, Anneaux reguliers auto-injectifs a droite, *Bull. Soc. Math. France* **101** (1973), 237–254.

[Re2] G. RENAULT, Anneaux reguliers auto-injectifs a droite, *J. of Algebra* **36** (1975), 77–84.

[Re3] G. RENAULT, "*Actions des groupes et anneaux reguliers injectifs*," Proc. Conf. Waterloo (1978), Lecture Notes in Math. **734**, Springer-Verlag, 1979.

[S] G. SZETO, A characterization of Azumaya algebras, *J. Pure and Appl. Alg.* **9** (1976/77), 369–371.

[U1] Y. UTUMI, On continuous rings and self-injective rings, *Trans. Amer. Math. Soc.* **118** (1965), 158–173.

[U2] Y. UTUMI, On self-injective rings, *J. Algebra* **6** (1967), 56–64.

[U3] Y. UTUMI, On rings of which any one-sided quotent rings are two-sided, *Proc. Amer. Math. Soc.* **14** (1963), 141–147.

[U4] Y. UTUMI, On continuous regular rings and semisimple self-injective rings, *Canad. J. Math.* **12** (1960), 597–605.

[U5] Y. UTUMI, On quotient rings, *Osaka Math. J.* **8** (1956), 1–18.

Hill Center for Mathematical Sciences
Department of Mathematics
Rutgers, The State University
New Brunswick, NJ 08903
U.S.A.

Rebut el 22 de Gener de 1992