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SOME REMARKS ON ALMOST FINITELY GENERATED NILPOTENT GROUPS

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Dedicated to the memory of Pere Menal

Abstract

We identify two generalizations of the notion of a finitely generated nilpotent. Thus a nilpotent group G is fgp if G_p is fg as p-local group for each p; and G is fg-like if there exists a fg nilpotent group H such that $G_p \cong H_p$ for all p. Then we have proper set-inclusions

$$\{fg\} \subset \{fg\text{-like}\} \subset \{fgp\}.$$

We examine the extent to which fg-like nilpotent groups satisfy the axioms for a Serre class. We obtain a complete answer only in the case that $[G, G]$ is finite. (The collection of fgp nilpotent groups is known to form a Serre class in the extended sense).

1. Introduction

Recently (see [**CH1**], [**CH2**]) a certain natural generalization of the concept of a *finitely generated* (fg) nilpotent group has been studied. Thus, given a prime p, we say that the p-local group M is fg *as p-local group* if there exists a fg group N such that $N_p = M$. We then say that the nilpotent group G is fg *at every prime*, or fgp, if G_p is fg as p-local group for every prime p. Plainly fg groups are fgp, but the converse fails. It was shown in [**H2**] that fgp groups form a Serre class in the extended sense of [**HR**].

Among the fgp (nilpotent) groups we may pick out those groups G such that there exists a fg group H with $G_p \cong H_p$, all p. We then call G *fg-like* or, more specifically, *H-like*. It is plain that we have *strict* inclusions

$$\{fg \text{ groups}\} \subset \{fg\text{-like groups}\} \subset \{fgp \text{ groups}\}.$$

For if B is the subgroup of $\mathbb{Q}$ generated by the rationals $(\frac{1}{p}$, all $p)$, then B is not fg but is plainly $\mathbb{Z}$-like; and if $A = \bigoplus_p \mathbb{Z}/p$ then A is plainly fgp, but it cannot be fg-like since the torsion subgroup of any fg-like group must be finite.

Moreover, this example shows that the fg-like groups do not form a Serre class, since we have a short exact sequence.

$$\mathbb{Z} \rightarrowtail B \twoheadrightarrow A.$$

However, for abelian groups, the only axiom which fails is that which asserts that a quotient group of a member of the class is again a member of the class. This we show in the next section, where we regard the Serre axioms relating to a short exact sequence as the *principal* axioms and the remaining axioms as *subsidiary*. However, the fact that a subgroup of an abelian fg-like group is fg-like, and the fact that an abelian extension of an fg-like group by an fg-like group is fg-like, both follow from an easy characterization of abelian fg-like groups as those abelian fgp groups whose torsion subgroups are finite. For nilpotent groups we do not know if this characterization holds; certainly fg-like nilpotent groups have finite torsion subgroups, but we do not know whether fgp nilpotent groups with finite torsion subgroups are necessarily fg-like. If the characterization held then the corresponding Serre axioms for fg-like nilpotent groups could be proved just as easily as in the abelian case.

In Section 3 we go as far as we can in the nilpotent case. Of course, the homology axiom holds; for the homology groups of a fg nilpotent group are fg, and, if G is K-like, with K fg, then $H_k(G)$ is $H_k(K)$-like. As to the principal axioms, one of couse is false and, as to the others, we are only able to prove them in the case of those groups G such that $[G, G]$ is finite. This we do by means of our main theorem, Theorem 3.1, which asserts that if $Q = G/N$ with N finite, then G is fg-like if and only if Q is fg-like. From this it readily follows (Theorem 3.5) that if G is fgp with TG finite and if $[G, G]$ is finite, then G is fg-like.

In an Appendix we show that Schur's Theorem, namely, that if G is a group with G/ZG finite, where ZG is the center of G, then $[G, G]$ is finite, admits a converse if G is nilpotent fgp. Moreover, precisely the same primes enter into the orders of G/ZG and $[G, G]$.

2. The Abelian case

We have already observed that a quotient of an fg-like abelian group need not be fg-like. We will show in this section that this is the only axiom for a Serre class which fails. As to the *subsidiary* axioms, this follows from the following composite theorem.

Theorem 2.1. *Let A be M-like and B be N-like (where all groups are abelian). Then*

(i) *$A \otimes B$ is $M \otimes N$-like.*
(ii) $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(M, N)$.
(iii) *$H_k(A)$ is $H_k(M)$-like.*

Proof:

(i) We have, for any prime p, $(A \otimes B)_p = A_p \otimes B_p \cong M_p \otimes N_p = (M \otimes N)_p$.
(ii) Similarly, for any prime p, $\mathrm{Tor}(A, B)_p \cong \mathrm{Tor}(M, N)_p$; but $\mathrm{Tor}(A, B)$ is a torsion group, so $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(M, N)$.
(iii) We have, for any prime p, and $k \geq 1$, $H_k(A)_p = H_k(A_p) \cong H_k(M_p) = H_k(M)_p$.

If $k = 0$, the assertion is trivial. ∎

As to the *principal* axioms, we have only to prove (a) that a subgroup of an fg-like abelian group is fg-like; and (b) that an abelian extension of an fg-like group by an fg-like group is fg-like. We base these results on the following obvious characterization of abelian fg-like groups. (Compare Theorem 1.3 of [**CH2**]).

Proposition 2.2. *Let A be an abelian group. Then A is fg-like if and only if A is fgp and TA is finite.*

Proof: If A is M-like with M fg, then $TA \cong TM$ which is evidently finite. Thus if A is fg-like, then A is fgp and TA is finite. Conversely, suppose that TA is finite and A is fgp. Then FA is $\mathbb{Z}^k$-like, for some k, so that (see Theorem 1.1 of [**CH2**]) $\mathrm{Ext}(FA, TA) = 0$ and $A = TA \oplus FA$. It follows that A is M-like, where $M = TA \oplus \mathbb{Z}^k$. ∎

Corollary 2.3.

(a) *A subgroup of an fg-like abelian group is fg-like.*
(b) *An abelian extension of an fg-like group by an fg-like group is fg-like.*

Proof: Since fgp abelian groups form a Serre class [**H2**], it remains only, in the light of Proposition 2.2, to show that the property of having a finite torsion subgroup is preserved by subgroups and extensions. This is absolutely clear for subgroups. For extensions, we observe that, if

$$G' \rightarrowtail G \overset{\pi}{\twoheadrightarrow} G''$$

is exact, so is

$$TG' \rightarrowtail TG \twoheadrightarrow \pi(TG),$$

and $\pi(TG) \subseteq TG''$. Thus, if TG' and TG'' are finite, so is TG. ∎

Remark. We notice that the proof of Corollary 2.3 works in the nilpotent case, provided the characterization of Proposition 2.2 remains valid for nilpotent groups. Certainly fg-like nilpotent groups are fgp with finite torsion subgroups, but we have not been able to establish the converse. Thus we have also not succeeded in generalizing Corollary 2.3 to nilpotent groups.

3. The Nilpotent case

Our basic result in the nilpotent case is the following.

Theorem 3.1. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be a short exact sequence of nilpotent groups with N finite. Then G is fg-like if and only if Q is fg-like.*

Proof: Suppose G is K-like, with K fg. We may then assume that $TG = TK$ and that, $\forall p$, $G_p = K_p$. Now $N \subseteq TG = TK \subseteq K$ and N_p is normal in $K_p (= G_p)$ for all p. It thus follows from [**HM**] that N is normal in K. Set $L = K/N$, so that L is fg. For all p, we have

$$\begin{array}{ccccc} N_p & \rightarrowtail & G_p & \twoheadrightarrow & Q_p \\ \| & & \| & & \\ N_p & \rightarrowtail & K_p & \twoheadrightarrow & L_p \end{array},$$

yielding isomorphisms $Q_p \cong L_p$, so that Q is L-like.

Conversely, suppose that Q is L-like, with L fg; and suppose further that N is commutative. Then the sequence $N \rightarrowtail G \twoheadrightarrow Q$ determines a nilpotent action of Q on N, and then the extension represents an element $\eta = [G] \in H^2(Q; N)$, where N is regarded as a (nilpotent) Q-module. Now there are associated nilpotent actions of Q_p on N_p, for all p, such that

$$(3.1) \quad H^2(Q;N) = H^2(Q; \prod_p N_p) = \prod_p H^2(Q;N_p) \cong \prod_p H^2(Q_p;N_p);$$

and, under this isomorphism, η corresponds to $\{\eta_p\}$, where $\eta_p \in H^2(Q_p; N_p)$ is represented by

$$N_p \rightarrowtail G_p \twoheadrightarrow Q_p.$$

We may, as before, assume that $L_p = Q_p$ and then we let L act on N_p via $e_p : L \to L_p$. This action is, of course, nilpotent, and so is the induced action of L on $N = \prod_p N_p$. Thus we obtain, paralleling (3.1), an isomorphism

$$H^2(L;N) \cong \prod_p H^2(L_p;N_p)(= \prod_p H^2(Q_p;N_p)) \tag{3.2}$$

The isomorphisms (3.1), (3.2) compose to an isomorphism $H^2(Q;N) \cong H^2(L;N)$, under which η corresponds to, say, ζ, where ζ is represented by the sequence $N \rightarrowtail K \twoheadrightarrow L$. Then K is nilpotent and certainly fg. Moreover, for each p, we have an isomorphism

$$\begin{array}{ccccc} N_p & \rightarrowtail & G_p & \twoheadrightarrow & Q_p \\ \| & & \wr\| & & \| \\ N_p & \rightarrowtail & K_p & \twoheadrightarrow & L_p \end{array},$$

showing that G is K-like.

We now complete the converse argument by induction on nil N. For the sequence $N \rightarrowtail G \twoheadrightarrow Q$ gives rise to a sequence

$$N/[N,N] \rightarrowtail G/[N,N] \twoheadrightarrow Q,$$

so that, by what we have just proved, $G/[N,N]$ is fg-like. Finally we consider the sequence

$$[N,N] \rightarrowtail G \twoheadrightarrow G/[N,N].$$

Since $[N,N]$ is finite with nil $[N,N] <$ nil N, our inductive hypothesis allows us to infer that G is fg-like. ∎

Remark. The abelian version of Theorem 3.1 is trivial. For then Corollary 2.3(b) assures us that G is fg-like if Q is fg-like; and, in the other direction, the finiteness of N tells us that TG maps *onto* TQ, so Q is fg-like by Proposition 2.2.

Corollary 3.2. *Let G be a nilpotent group with finite commutator-subgroup. Then G is fg-like if and only if G_{ab} is fg-like.*

Proof: Of course we need no assumption on $[G,G]$ to infer that G_{ab} is fg-like if G is fg-like; for if G is K-like, then G_{ab} is K_{ab}-like. However, in the other direction, we must apply Theorem 3.1. ∎

Theorem 3.3. *Let G be fg-like and let $H \subseteq G$ be a subgroup such that $H \cap [G,G]$ is finite. Then H is fg-like. In particular, all subgroups of G are fg-like if $[G,G]$ is finite.*

Proof: Since G is fg-like so is G_{ab}. Then $H/H \cap [G,G] \subseteq G_{ab}$, so that, by Corollary 2.3(a), $H/H \cap [G,G]$ is fg-like. We now use Theorem 3.1 to infer that H is itself fg-like. ■

Theorem 3.4. *Let $G' \rightarrowtail G \twoheadrightarrow G''$ be a short exact sequence of nilpotent groups in which G', G'' are fg-like. Then, if $[G,G]$ is finite, G is fg-like.*

Proof: We have the short sequence of abelian groups

$$G'/G' \cap [G,G] \rightarrowtail G_{ab} \twoheadrightarrow G''_{ab}$$

Moreover, G''_{ab} and $G'/G' \cap [G,G]$ are fg-like, the latter by Theorem 3.1. Thus, by Corollary 2.3(b), G_{ab} is fg-like. Hence, by Corollary 3.2, G is fg-like. ■

Remark. Note that the class of nilpotent groups with finite commutatorsubgroup is not closed under (nilpotent) extensions. Thus if Γ is the free nilpotent group of class 2 on 2 generators, then Γ has a central subgroup $[\Gamma, \Gamma]$ which is free cyclic with quotient free abelian of rank 2.

Theorems 3.3 and 3.4 suggest that, in the study of fg-like groups, nilpotent groups G such that $[G,G]$ is finite behave very much like abelian groups (which may be regarded as special cases). In the light of Proposition 2.2, this view is reinforced by the following theorem.

Theorem 3.5. *Suppose G is nilpotent with $[G,G]$ finite. Then G is fg-like if and only if G is fgp with TG finite.*

Proof: As remarked in the Introduction, if G is fg-like then, even without the hypothesis that $[G,G]$ is finite, G is fgp with TG finite. Suppose conversely that G is fgp with TG finite and that $[G,G]$ is finite. Consider the short exact sequence.

$$TG \rightarrowtail G \twoheadrightarrow FG.$$

Since $[G,G]$ is finite, $[G,G] \subseteq TG$, so that FG is abelian. Thus FG is torsionfree, abelian and fgp, which implies that FG is $\mathbb{Z}^k$-like for some integer $k \geq 0$. The conclusion now follows from Theorem 3.1 (or Theorem 3.4).

Of course, as hinted in the Introduction, it is also not difficult to deduce Theorems 3.3 and 3.4 from Theorem 3.5. ■

4. Appendix: Groups with finite commutatorsubgroup

A famous theorem due to I. Schur asserts that if G/ZG is finite, then $[G,G]$ is finite. A study of its proof (we give a homological proof below) shows that more is true. Given any finite group N, let $\tau(N)$ be the set of primes dividing the order of N. Then we may conclude that $\tau[G,G] \subseteq \tau(G/ZG)$. In the context of nilpotent groups, we have a stronger result.

Theorem 4.1. *Let G be an fgp nilpotent group. Then G/ZG is finite if and only if $[G,G]$ is finite and then $\tau(G/ZG) = \tau[G,G]$.*

Proof: We first prove Schur's Theorem. Thus if G/ZG is finite then $G'/G' \cap ZG$ is finite, where $G' = [G,G]$, and $\tau(G'/G' \cap ZG) \subseteq \tau(G/ZG)$. Thus it remains to prove that $G' \cap ZG$ is finite, with $\tau(G' \cap ZG) \subseteq \tau(G/ZG)$. Now $H_2(G/ZG)$ is finite with $\tau(H_2(G/ZG)) \subseteq \tau(G/ZG)$. Moreover there is an exact sequence

$$H_2(G/ZG) \longrightarrow ZG \longrightarrow G_{ab},$$

which shows that $G' \cap ZG$ is a homomorphic image of $H_2(G/ZG)$. We conclude that $G' \cap ZG$ is finite with $\tau(G' \cap ZG) \subseteq \tau(G/ZG)$, as required. Indeed, we have shown that $\tau[G,G] \subseteq \tau(G/ZG)$, as claimed.

Conversely, suppose that G is an fgp nilpotent group with G' finite. Set $\tau = \tau(G')$. Then, for all $g, h \in G$, $[g,h]^n = 1$ for some τ-number n. If $b = [g,h]$, we have $gb = h^{-1}gh$ so that, by Corollary 6.2 of [**H1**], $g^{n^c} = (h^{-1}gh)^{n^c} = h^{-1}g^{n^c}h$, where $\operatorname{nil} G \leq c$. Thus $g^{n^c} \in ZG$, so that G/ZG is a torsion group with bounded exponent dividing n^c. But G is fgp, so G/ZG is fgp. It follows that each $(G/ZG)_p$ is finite and $(G/ZG)_p$ is trivial unless $p|n^c$. We conclude that G/ZG is a finite group with $\tau(G/ZG) \subseteq \tau = \tau[G,G]$. This completes the proof of the theorem. ■

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