POWER-CANCELLATION OF
CW-COMPLEXES WITH FEW CELLS

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Dedicated to Pere Menal, in memoriam

Abstract

In this paper, we use the fact that the rings of integer matrices have the power-substitution property in order to obtain a power-cancellation property for homotopy types of CW-complexes with one cell in dimensions 0 and 4n and a finite number of cells in dimension 2n.

Introduction

Cancellation in homotopy is closely related to the genus. Recall that the genus of a homotopy type X is the set of all homotopy types Y such that the p-localizations of X and Y are homotopy equivalent for all primes $p : X_p \simeq Y_p$ [9]. If X and Y are in the same genus, we shall write $X \sim Y$. All the known examples of non-cancellation occur for homotopy types in the same genus. A relevant result in this line is the following.

Theorem 1 (Wilkerson [12]). Let $X, Y, W$ be nilpotent co-$H$-complexes with finitely generated homology. Then

i) $X \vee Y \simeq X \vee W$ implies $Y \sim W$, and

ii) $\bigvee_k X \simeq \bigvee_k Y$ implies $X \sim Y$.

In [10] Mislin characterized the genus of nilpotent co-$H$-spaces with finitely generated homology in the following way

$X \sim Y$ if and only if $X \vee \bigvee S^{m_i} \simeq Y \vee \bigvee S^{m_j}$,

where the dimensions of the spheres are determined by the homotopy groups of X and Y. A similar result had been obtained by Molnar [11]
for $CW$-complexes with two cells, $S^n \cup_f e^m$, with attaching map $f$ of finite order in $\pi_{m-1}(S^n)$. See also [7].

In [1] Bokor develops matricial methods in order to deal with spaces $S^{2n} \cup_f e^{4n}$ such that the attaching map is of infinite order. This lead him to prove the following.

**Theorem 2 (Bokor [1]).** Let

$$X_i = S^{2n} \cup_{f_i} e^{4n}, \quad Y_i = S^{2n} \cup_{g_i} e^{4n}, \quad i = 1, \ldots, H.$$ 

Suppose that $f_i, g_j$ represent elements of infinite order in $\pi_{4n-1}(\bigvee S^{2n})$ if and only if $i \leq h$ and $j \leq h'$. Then

$$\bigvee_{i=1}^H X_i \simeq \bigvee_{j=1}^H Y_j$$

if and only if

i) $h = h'$

ii) $\bigvee_{i=1}^{h+1} X_i \simeq \bigvee_{j=1}^{h+1} Y_j$

iii) There exists a permutation $\tau$ of $\{1, \ldots, h\}$ such that $X_i \simeq Y_{\tau(i)}$ for all $i \leq h$.

In [8] we tried to extend Bokor's result to complexes with one cell in dimensions 0 and $4n$ and $k$ cells in dimension $2n$. For $n \geq 2$, the result turned out not to be true. However, if we restrict ourselves to the $p$-local case, the theorem holds. The proof uses a technical lemma that is also used in the proof of a cancellation property for the ring $\mathcal{M}_k(\mathbb{Z}_p)$ of matrices over the $p$-localization $\mathbb{Z}_p$ of the ring $\mathbb{Z}$ ([4, Th.2]). For our purpose, we state the following version of the main theorem in [8].

**Theorem 3.** Let

$$X_i = \bigvee_{i=1}^k S^{2n} \cup_{f_i} e^{4n}, \quad Y_i = \bigvee_{i=1}^k S^{2n} \cup_{g_i} e^{4n}, \quad i = 1, \ldots, H.$$ 

Suppose that $f_i, g_j$ represent elements of infinite order in $\pi_{4n-1}(\bigvee^k S^{2n})$ if and only if $i \leq h$ and $j \leq h'$. Then

$$\bigvee_{i=1}^H X_i \simeq \bigvee_{i=1}^H Y_i$$
if and only if

i) $h = h'$

ii) $\bigvee_{h} X_i \sim \bigvee_{h} Y_i$

iii) $\bigvee_{h+1} X_i \sim \bigvee_{h+1} Y_i$.

In this paper we use a property of the rings $M_k(\mathbb{Z})$, the (left) power-substitution property, to prove the following theorems.

Theorem I. Let

$$X = \bigvee_{k} S^m \cup f e^m, \quad Y = \bigvee_{k} S^m \cup g e^m.$$  

Assume that $f, g$ are suspension elements of finite order in $\pi_{m-1}(\bigvee S^n)$. Then if $X \sim Y$, there is a positive integer $t$ such that

$$\bigvee_{i} X \sim \bigvee_{i} Y.$$  

Observe that the converse holds by Theorem 1.

Theorem II. Let

$$X_i = \bigvee_{k} S^{2n} \cup f_i e^{4n}, \quad Y_i = \bigvee_{k} S^{2n} \cup g_i e^{4n}, \quad i = 1, \ldots, H.$$  

Assume that $f_i, g_j$ represent elements of infinite order in $\pi_{4n-1}(\bigvee S^{2n})$ if and only if $i \leq h$, $j \leq h'$. Then if

$$\bigvee_{H} X_i \sim \bigvee_{H} Y_i$$

we have $h = h'$ and there exist positive integers $s, t_1, \ldots, t_h$ and a permutation $\tau$ of $\{1, \ldots, h\}$ such that

$$\bigvee_{i} X_i \sim \bigvee_{\tau(i)} Y_i$$

for all $i \leq h$, and

$$\bigvee_{H} \left(\bigvee_{h+1} X_i\right) \sim \bigvee_{H} \left(\bigvee_{h+1} Y_i\right).$$
Power-substitution

A ring $E$ satisfies the "(left) power-substitution property" if given $xa + b = 1$ in $E$ there exist a positive integer $t$ and a matrix $T \in M_t(E)$ such that $I_t a + T b$ is an unit in $M_t(E)$. Here $I_t$ is the identity $t \times t$ matrix. All subrings of the rationals $\mathbb{Q}$ and in particular $\mathbb{Z}$, as well as all the matrix rings $M_n(\mathbb{Z})$, $n \geq 1$, satisfy power-substitution ([5, (2.9) and (3.4)]). So, given

$$XA + B = I_n$$

in $M_n(\mathbb{Z})$, there exist a positive integer $t$ and a matrix $T \in M_{nt}(\mathbb{Z})$ such that

$$I_t \otimes A + T(I_t \otimes B)$$

is invertible. Here $\otimes$ denotes the Kronecker product

$$I_t \otimes A = \begin{pmatrix}
A & 0 & \cdots & 0 \\
0 & A & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A
\end{pmatrix}$$

Power-substitution implies power-cancellation in the following sense. If $A, B, C$ are right $R$-modules and $\text{End}_R(A)$ has the power-substitution property, then $A \otimes B \cong A \otimes C$ implies $B^t \cong C^t$ for some positive integer $t$.

**Proof of Theorem I**

Since $f$ and $g$ are suspension elements, they represent elements in

$$\bigoplus_{m-1}(S^n) \subset \pi_{m-1}(S^n).$$

So, $f$ and $g$ are determined by column matrices $x(f)$ and $x(g)$ with entries in $\pi_{m-1}(S^n)$. It was shown in [7] that the mapping cones of $f$ and $g$, $X$ and $Y$, are in the same genus if and only if there exists an integer matrix $A$ such that

$$x(g) = Ax(f)$$

and $(\det A, l) = 1$, where $l$ denotes the order of $f$.

Let $B$ be an integer matrix such that $AB = \det A I_k$, and take $r, s$ such that $r \det A + sl = 1$. Since $\mathbb{Z}$ has the power-substitution property, there exist an integer $c$ and a matrix $C$ such that

$$I_c \det A + Cl$$

is invertible.
Then
\[ I_k \otimes (I_c \det A) + (I_k \otimes C l) = (I_c \otimes B)(I_c \otimes A) + (I_k \otimes C l) = V \]
is invertible and, by the power-substitution property of $M_{k}(\mathbb{Z})$, we obtain an integer $d$ and a matrix $D$ such that
\[ I_d \otimes (I_c \otimes A) + D(I_d \otimes V^{-1}(I_k \otimes C l)) = U \]
has an inverse. So applying power-substitution once more, we obtain an invertible matrix of the form
\[ Q = I_c \otimes (I_{cd} \otimes A) + E(I_c \otimes U^{-1}D(I_d \otimes V^{-1}(I_k \otimes C l))) = I_t \otimes A + Tl \]
where $t = cde$ and $T = E(I_c \otimes U^{-1}D(I_d \otimes V^{-1}(I_k \otimes C l)))$.

Now
\[ Qx(f \vee \ldots \vee f) = Q \begin{pmatrix} x(f) & 0 & \ldots & 0 \\ 0 & x(f) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x(f) \end{pmatrix} = \begin{pmatrix} Ax(f) & 0 & \ldots & 0 \\ 0 & Ax(f) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A\pi(f) \end{pmatrix} = x(g \vee \ldots \vee g). \]

Hence
\[ \bigvee X \simeq \bigvee Y. \]

**Proof of Theorem II**

First of all, recall that the homotopy type of each $X_i = \bigvee S^{2n} \cup_{f_i} e^{2n}$ is determined by the $k \times k$ integer matrix $H(f_i)$ of the associated Hilton-Hopf quadratic form and one one-column matrix $x(\sum f_i)$, the entries of which are suspension elements in $\pi_{4n}(S^{2n+1})$ [1]. Moreover, $\bigvee^H X_i \simeq \bigvee^H Y_i$ if and only if there exists a homotopy commutative diagram
\[
\begin{array}{ccc}
\bigvee^H S^{4n-1} & \xrightarrow{f_1 \vee \ldots \vee f_k} & \bigvee^H S^{2n} \\
\phi \downarrow & & \downarrow \phi \\
\bigvee^H S^{4n-1} & \xrightarrow{g_1 \vee \ldots \vee g_H} & \bigvee^H S^{2n} 
\end{array}
\]
with \( \vartheta \) and \( \varphi \) homotopy equivalences. Let \( \theta_{i,j} \) be the degree of

\[
S^{4n-1} \xrightarrow{\text{in}_j} S^{4n-1} \xrightarrow{\vartheta} S^{4n-1} \xrightarrow{\varphi} S^{4n-1}. 
\]

The matrix \((\theta_{ij}) = \theta\) characterizes the homotopy class of \( \vartheta \).

Let now \( \phi_{i,j} \) be the matrix of

\[
\sqrt{S^{2n}} \xrightarrow{\text{in}_j} \sqrt{\sqrt{S^{2n}}} \xrightarrow{\varphi} \sqrt{\sqrt{S^{2n}}} \xrightarrow{\varphi} \sqrt{S^{2n}}.
\]

The homotopy commutativity of the previous diagram is equivalent to the following condition \([1],[8]\).

\[
\phi_{j,i} H(f_i) \phi_{j,i}^t = \theta_{j,i} H(g_j) \quad \text{for all } i,j
\]

(*)

\[
\phi_{j,i} H(f_i) \phi_{j,i} = 0 \quad \text{if } l \neq j
\]

\[
\phi_{j,i} x(\sum f_i) = \theta_{j,i} x(\sum g_j) \quad \text{for all } i,j
\]

where \( \phi_{j,i}^t \) is the transpose of \( \phi_{j,i} \).

We know that \( f_i \) is of finite order if and only if \( H(f_i) = 0 \). Hence, \( \theta_{j,i} = 0 \) for \( j \leq h', \ i > h \). Thus, \( \det \vartheta = \pm 1 \) implies \( h \geq h' \) and by symmetry \( h = h' \).

Now write the matrices of \( \vartheta \) and \( \varphi \) in the form

\[
\vartheta = \begin{pmatrix} \Theta_1 & 0 \\ \Theta_3 & \Theta_4 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}.
\]

They are invertibles. Thus

\[
C_1 \Phi_1 + C_2 \Phi_3 = I \quad C_3 \Phi_2 + C_4 \Phi_4 = I
\]

where \( \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \phi^{-1} \). By the power-substitution property of \( M_{kh}(\mathbb{Z}) \) and \( M_{kh-h}(\mathbb{Z}) \), we can find positive integers \( r, s \) and matrices \( R, S \) such that

\[
A = I_r \otimes \Phi_1 + R(I_r \otimes C_2 \Phi_3)
\]

\[
B = I_s \otimes \Phi_4 + S(I_s \otimes C_3 \Phi_2)
\]

are invertibles. Consider the diagrams
where \( \zeta, \xi, \alpha \) and \( \beta \) are homotopy equivalences with matrices \( I_r \otimes \Theta_1, I_s \otimes \Theta_4, A \) and \( B \) respectively. By checking the matricial conditions mentioned above (*), we can see that these diagrams are homotopy commutative. (See [8] for the details in a quite similar case). So, these diagrams induce homotopy equivalences

\[
\begin{align*}
\begin{array}{c}
\bigvee (\bigvee X_i) \xrightarrow{r \cdot h} \bigvee (\bigvee S^{2n}) \\
\zeta \\
\bigvee (\bigvee S^{4n-1}) \xrightarrow{r \cdot h} \bigvee (\bigvee S^{2n})
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\bigvee (\bigvee S^{4n-1}) \xrightarrow{r \cdot h} \bigvee (\bigvee S^{2n}) \\
\xi \\
\bigvee (\bigvee S^{4n-1}) \xrightarrow{r \cdot h} \bigvee (\bigvee S^{2n})
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\bigvee (\bigvee g_i) \xrightarrow{s \cdot h} \bigvee (\bigvee S^{2n}) \\
\bigvee (\bigvee S^{4n-1}) \xrightarrow{s \cdot h} \bigvee (\bigvee S^{2n})
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\bigvee (\bigvee f_i) \xrightarrow{s \cdot h} \bigvee (\bigvee S^{2n}) \\
\bigvee (\bigvee S^{4n-1}) \xrightarrow{s \cdot h} \bigvee (\bigvee S^{2n})
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\bigvee (\bigvee g_i) \xrightarrow{s \cdot h} \bigvee (\bigvee S^{2n}) \\
\bigvee (\bigvee S^{4n-1}) \xrightarrow{s \cdot h} \bigvee (\bigvee S^{2n})
\end{array}
\end{align*}
\]

We may now assume that all the given spaces \( X_i \) and \( Y_i, \ i = 1, \ldots, H \), have attaching maps of infinite order. Let \( r \) be the maximum rank of the matrices \( H(f_i) \) and \( H(g_i) \) and assume that the rank of \( H(g_1) \) is \( r \). Since \( \det \theta = \pm 1, \ \theta_{ii} \neq 0 \), for some \( i \) and, hence, \( \phi_{ii} H(f_i) \phi_{ii}^T = \theta_{ii} H(g_1) \) has rank \( r \). Thus \( \text{rk} H(f_i) = r \). Now, from the rest of the matricial conditions (*), one gets \( \theta_{ji} = 0 \) if \( j \neq 1 \), and \( \theta_{ii} = \pm 1 \). For simplicity we suppose \( i = 1 \).

Let us write the matrices of \( \theta \) and \( \phi \) in the form

\[
\theta = \begin{pmatrix} \theta_{11} & \Theta_3 \\ 0 & \Theta_4 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_{11} & \Phi_2 \\ \Phi_3 & \Phi_4 \end{pmatrix}
\]

\[
\begin{align*}
\begin{array}{c}
\bigvee (\bigvee X_i) \simeq \bigvee (\bigvee S^{2n}) \\
\bigvee (\bigvee Y_i) \simeq \bigvee (\bigvee S^{2n}) \\
\bigvee (\bigvee X_i) \simeq \bigvee (\bigvee Y_i)
\end{array}
\end{align*}
\]
and let $\phi^{-1} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$. Thus

$$C_1\phi_1 + C_2\Phi_3 = I \quad \text{and} \quad C_3\Phi_2 + C_4\Phi_4 = I.$$ 

As before, we can find invertible matrices of the form

$$A = I_r \otimes \phi_1 + R(I_r \otimes C_2\Phi_3) \quad \text{and} \quad B = I_s \otimes \Phi_4 + S(I_s \otimes C_3\Phi_2).$$

Consider homotopy commutative diagrams

$$
\begin{array}{ccc}
S^{4n-1} & \overset{r}{\longrightarrow} & \bigvee S^{2n} \\
\zeta \downarrow & & \downarrow \alpha \\
\bigvee S^{4n-1} & \overset{r}{\longrightarrow} & \bigvee S^{2n}
\end{array}
$$

$$
\begin{array}{ccc}
\bigvee S^{4n-1} & \overset{r}{\longrightarrow} & \bigvee S^{2n} \\
\zeta \downarrow & & \downarrow \alpha \\
\bigvee S^{4n-1} & \overset{r}{\longrightarrow} & \bigvee S^{2n}
\end{array}
$$

where $\zeta$, $\xi$, $\alpha$, and $\beta$ are homotopy equivalences with matrices $\theta_{11}I_r$, $I_3 \otimes \Theta_4$, $A$ and $B$ respectively. These diagrams induce homotopy equivalences

$$
\begin{array}{ccc}
\bigvee X_1 \simeq \bigvee Y_1 & \quad \text{and} \quad & \bigvee \left( \bigvee X_i \right) \simeq \bigvee \left( \bigvee Y_i \right).
\end{array}
$$

This, by induction, completes the proof of Theorem II. ■

Remark. The following question naturally arises in the study of cancellation: is $X \sim Y$ equivalent to $\bigvee X \sim \bigvee Y$? (See [6] for an algebraic result of this kind). Theorem I and Theorem 1 give a positive answer for some co-$H$-spaces with few cells. For mapping cones $X$ and $Y$ of
elements of infinite order in $\pi_{4n−1}(\vee^k S^n)$, it follows from [8] Theorem 1 that $\bigvee X \simeq \bigvee Y$ implies $X \sim Y$. In this case, however, the converse is not always true. If, for instance, $f$ and $g$ are such that

$$H(f) = \begin{pmatrix} 2p & 0 \\ 0 & 2q \end{pmatrix}, \quad H(g) = \begin{pmatrix} 2 & 0 \\ 0 & 2pq \end{pmatrix}, \quad \sum f = 0 \text{ and } \sum g = 0,$$

where $p \neq q$ are prime integers, then $X \sim Y$ and $X \not\simeq Y$ (see [2]). But, since $H(f)$ and $H(g)$ are of maximum rank, it follows from [8] Theorem 2 that $\bigvee X \not\simeq \bigvee Y$ for all integers $t$.

References

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