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L^1 AND L^∞-ESTIMATES WITH A LOCAL WEIGHT FOR THE $\bar\partial$-EQUATION ON CONVEX DOMAINS IN $\mathbf{C}^n$

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Abstract

We construct a defining function for a convex domain in $\mathbf{C}^n$ that we use to prove that the solution-operator of Henkin-Romanov for the $\bar\partial$-equation is bounded in L^1 and L^∞-norms with a weight which reflects not only how much near is the point to the boundary of the domain but also how much convex is the domain near the point. We refine and localize the weights that Polking uses in **[Po]** for the same type of domains because they depend only on the euclidean distance to the boundary and don't take into account of the geometry of the domain.

1. Introduction and statement of results

This paper deals with the L^p-estimates for the solutions of the equation $\bar\partial u = f$, where f is a $\bar\partial$-closed $(0,1)$-form, for a certain class of pseudoconvex domains.

Let D be a bounded convex domain in $\mathbf{C}^n$ of class C^3 defined by a function ρ and denote by ∂D its boundary.

For $x \in \partial D$, the Hessian of ρ in x, $H\rho(x)$, restricted to the tangent space to the boundary in x, $T_x(\partial D)$, is the second fundamental quadratic form of the boundary. We write m for its convexity lower bound (c.l.b.), that is,

$$m(x) = \inf_{\lambda \in T_x(\partial D)} \frac{H\rho(x)(\lambda,\lambda)}{|\lambda|^2}.$$

We know that m is an intrinsic quantity, that is, it does'nt depend on the defining function, but only depends on the boundary of the domain.

This c.l.b. "measures" for every point of the boundary how much convex is the domain in a neighborhood of the point; for example, where the domain is flat, m is zero and, on the other hand, m is strictly positive in a neighborhood of a strictly convexity point.

We obtain L^1 and uniform estimates with a weight that is expressed in two different ways: near the flat pieces it depends on the distance of the point to the boundary and far of them, it is given in terms of this c.l.b. evaluated in the projection of the point to the boundary. In this way the local behavior of the boundary of the domain appears in weighted-estimates.

We denote by $\pi(.)$ and $d(.)$ the projection and the distance, respectively, of the point to the boundary. From now on, all not distinguished positive constants will be denoted by c. We shall put diam(D) for the diameter of the domain.

In this paper we prove the following three theorems:

Theorem 1. *Let f be a $\bar\partial$-closed $(0,1)$-form in $L^\infty(D)$. Then, there exists a solution of the equation $\bar\partial u = f$ such that*

$$\text{if } n=2, \quad \| u|\log[m(\pi(.))d(.)]|^{-1} \|_{L^\infty(D)} \leq c \| f \|_{L^\infty(D)}$$
$$\text{and for } n>2, \quad \| u[m(\pi(.))^3 + d(.)]^{n-2} \|_{L^\infty(D)} \leq c \| f \|_{L^\infty(D)} .$$

Theorem 2. *For $n = 2$, let f be a $\bar\partial$-closed $(0,1)$-form such that $f|\log[m(\pi(.))d(.)]|$ belongs to $L^1(D)$. Then, there exists a solution of the equation $\bar\partial u = f$ such that*

$$\| u \|_{L^1(D)} \leq c \| f|\log[m(\pi(.))d(.)]| \|_{L^1(D)} .$$

If $n>2$ and $f[m(\pi(.))^3 + d(.)]^{-n+2}$ belongs to $L^1(D)$, then

$$\| u \|_{L^1(D)} \leq c \| f[m(\pi(.))^3 + d(.)]^{-n+2} \|_{L^1(D)} .$$

Observe that in the strictly convex case, $m(\pi(.))$ is bounded below in D and our theorems give the well-known L^1 and L^∞-estimates without weights.

The following result is proved by Polking in [**Po**]:

Theorem 3. *For $n = 2$, let f be a $\bar\partial$-closed $(0,1)$-form in $L^p(D)$, $1 < p < \infty$. Then, there exists a solution of the equation $\bar\partial u = f$ such that*

$$\| u \|_{L^p(D)} \leq c \| f \|_{L^p(D)} .$$

If $\Omega \subset \mathbf{C}^n$ is a weakly pseudoconvex domain with smooth boundary, Bonneau and Diederich ([**Bo Di**]) have proved, recently, the existence of solutions operators H_n for $\bar\partial$ on Ω, such that

$$\| H_2 f \|_{L^1(\Omega)} \leq c \| f|\log d(.)| \|_{L^1(\Omega)}$$
$$\text{and for } n>2, \quad \| H_n f \|_{L^1(\Omega)} \leq c \| f d(.)^\beta \|_{L^1(\Omega)} \quad \text{with } \beta < -n/2.$$

The paper is organized as follows. Section two is devoted to construct from ρ a defining function $q \in \mathcal{C}^2(\bar{D})$, strictly convex in the interior of D and such that, for every point of D and any vector in $\mathbf{C}^n$, the c.l.b. of its Hessian depends on m and d. We remark that for a weakly pseudoconvex domain our method does not apply to obtain similar results because it is not true that there exists a defining function strictly plurisubharmonic in the interior of the domain. In section three we decompose the solution-operator of Henkin-Romanov as a finite sum of volume integral operators, written in terms of q, and we do the necessary computations of every addend-operator in order to obtain all the types of estimates.

2. Construction of the defining function q

In this section we situate ourselves in the real space $\mathbf{R}^{2n}$.

First of all, for $y \in D$ fixed, let p_y be defined by

$$p_y(x) := \inf \{k \geq 0 / x - y \in kD\},$$

where kD is calculated respect to y; p_y is the norm that has D as the unit ball when y is the origin.

For x different from y, we put $w_k := y + (x - y)/k$. Now $\rho(w_k) = 0$ defines implicitly $p_y(x)$.

We put $z := y + (x - y)/p_y(x)$. Observe that z is the point of the boundary obtained by continuing from x the straight line that joins x and y.

If we differentiate implicitly in $\rho(w_k) = 0$, it is a computation to obtain the gradient and the Hessian of p_y in terms of the gradient and Hessian of ρ, respectively:

$$\nabla p_y(x) = \frac{p_y(x)}{\nabla\rho(z)(x - y)} \nabla\rho(z). \tag{1}$$

$$Hp_y(x)(\lambda, \lambda) = \frac{1}{(\nabla\rho(z)(x - y))^3} H\rho(z)(v, v), \tag{2}$$

$\forall \lambda \in \mathbf{R}^{2n}$, where $v = (\lambda\nabla\rho(z))(x - y) - ((x - y)\nabla\rho(z))\lambda \in T_z(\partial D)$.

We have $p_y \in \mathcal{C}^3(\mathbf{R}^{2n} \setminus \{y\})$, $p_y\,|_{\partial D} = 1$ and $D = \{x/p_y(x) < 1\}$.

Since $\nabla\rho(z)(x - y) \geq 0$, it follows from (1) that $\nabla p_y(x) \neq 0$, for $x \in \partial D$, and from (2) that $Hp_y(x)(\lambda, \lambda) \geq 0$, for $x \in \partial D$ and $\lambda \in \mathbf{R}^{2n}$.

But p_y is not a candidate for q because it is not strictly convex in the interior; note that given a direction λ, $Hp_y(x)(\lambda,\lambda)$ is zero on the line $x = y + t\lambda$, $t \in \mathbf{R}$, because p_y is linear on the straight lines through y.

Fixed a point x and a direction λ, there are points y such that $Hp_y(x)(\lambda,\lambda)$ is equal to zero (when z belongs to a flat piece) and others that aren't. So we think in obtaining q as a convex combination of all p_y (as an average in y).

We define q in the following way

$$q(x) := \int_D \omega(y) p_y(x) dm(y),$$

where ω is a positive weight that we will determinate below.

Observe that $\nabla q(x) \neq 0$, for $x \in \partial D$, because $\nabla p_y(x) \neq 0$.

We need to choice ω such that $\int_D \omega(y)dm(y) = 1$ for $q\,|_{\partial D} = 1$ and $D = \{x/q(x) < 1\}$; on the other hand, ω must be zero in the boundary in a way that compensates the explosion of the second gradient of p_y when we approximate to the boundary, because we want $q \in C^2(\bar{D})$.

Proposition 1.

$$|\nabla^2 p_y(x)| = O\left(\frac{1}{\| x - y \| \, d(y)^3}\right).$$

Proof: It is a computation to obtain:

$$\frac{\partial^2 p_y(x)}{\partial x_k \partial x_i} = -\frac{\partial \rho(z)}{\partial x_i}\frac{1}{(\nabla\rho(z)(x-y))^2}\sum_l (x_l - y_l)\frac{\partial^2 \rho(z)}{\partial x_k \partial x_l} +$$

$$+\frac{\partial \rho(z)}{\partial x_i}\frac{1}{(\nabla\rho(z)(x-y))^3}\frac{\partial \rho(z)}{\partial x_k}\sum_{l,r}(x_l - y_l)(x_r - y_r)\frac{\partial^2 \rho(z)}{\partial x_r \partial x_l} +$$

$$+\frac{1}{\nabla\rho(z)(x-y)}\frac{\partial^2 \rho(z)}{\partial x_k \partial x_i} - \frac{1}{(\nabla\rho(z)(x-y))^2}\frac{\partial \rho(z)}{\partial x_k}\sum_l (x_l - y_l)\frac{\partial^2 \rho(z)}{\partial x_l \partial x_i}.$$

The worst estimate corresponds to the second term, its numerator is, clearly, $O(\| x - y \|^2)$ and $\nabla\rho(z)(x-y) = \nabla\rho(z)p_y(x)(z-y) = p_y(x)\mathrm{dist}(y, T_z(\partial D)) \geq p_y(x)d(y)$. Now $p_y(x) \geq c \| x - y \|$ because $\| z - y \| \leq \mathrm{diam}(D)$. ■

Lemma 1. *For $\epsilon > 0$, if we put $\omega(y) := Cd(y)^{2+\epsilon}$, where $C := 1/\int_D d(y)^{2+\epsilon}dm(y)$, then q is a defining function of D and $q \in C^2(\bar{D})$.*

More generally, it is possible to choose $\omega(y) := C'd(y)^2 h(d(y))$, where the function h is such that the integral $\int_0^1 (h(t)/t)dt$ is finite.

Proof: We situate ourselves in $\mathbf{R}^2$ because it is the most unfavourable case.

For $E \subseteq D$, let $I(E)$ be defined by

$$I(E) := \int_E \frac{dm(y)}{\| x - y \| \, d(y)^{1-\epsilon}}$$

By proposition 1, $|\nabla^2 q(x)| = O(I(D))$.

For all $x \in \partial D$, $1/ \| x - y \| \, d(y)^{1-\epsilon}$ are uniformly integrative functions on D because if we take local coordinates (u, v) with center x, where u represents the distance d and v is taken on ∂D, we reduce $I(D)$ to the area's integral

$$\int_{-1}^{1} \int_0^1 \frac{dudv}{(u + |v|)u^{1-\epsilon}},$$

which is finite.

For all $x \notin \partial D$, we prove now that $\forall \eta > 0$, $\exists \delta > 0$ such that if the measure of E, $|E|$, is smaller than δ, then $I(E) \leq \eta$.

Given a region E, we put $E_1 := \{y \in E / d(y) \geq \| x - y \|\}$. Now,

$$I(E_1) \leq \int_{E_1} \| x - y \|^{\epsilon - 2} \, dm(y) \leq |E_1|^{\frac{\epsilon}{2}}.$$

$$\text{So, if } |E_1| < \delta_1 = (\eta/2)^{\frac{2}{\epsilon}}, \text{ then } I(E_1) \leq \eta/2.$$

If $y \in E \setminus E_1$, we put $\bar{x} \in \partial D$ for the point where $d(x)$ is attained. Now, $\| \bar{x} - y \| \leq \| x - y \| + \| x - \bar{x} \| = \| x - y \| + d(x) \leq \| x - y \| + d(y) + \| x - y \| \leq 3 \| x - y \|$. Then we may replace x by $\bar{x}$ and, so, $\exists \delta_2 > 0$ such that if $|E \setminus E_1| < \delta_2$, then $I(E \setminus E_1) \leq \eta/2$.

Put $\delta = \min(\delta_1, \delta_2)$. ∎

Proposition 2. *For $x \in D$ fixed, we choose for every point $y \in D$ the coordinates $r := \| x - y \|$ and z. Then, if $\sigma_{\partial D}(z)$ denotes the area's measure on ∂D, we have*

$$dm(y) \geq c r^{2n-1} \frac{|\nabla \rho(z)(x - z)|}{\| x - z \|^{2n}} dr d\sigma_{\partial D}(z).$$

Proof: We denote by S the unit sphere of $\mathbf{R}^{2n}$. If $w \in S$, let $\lambda(w)$ be such that $\rho(x - \lambda(w)w) = 0$. Now, for $j = 1, \cdots, 2n$, one has

$$\sum_{k=1}^{2n} \frac{\partial \rho}{\partial z_k} \left(\lambda(w)\delta_{kj} + \frac{\partial \lambda}{\partial w_j} w_k \right) = 0, \text{ that is, } \frac{\partial \rho}{\partial z_j} \lambda(w) + \frac{\partial \lambda}{\partial w_j} \sum_{k=1}^{2n} \frac{\partial \rho}{\partial z_k} w_k = 0.$$

From $z_i = x_i - \lambda(w)w_i$, it follows that

$$\frac{\partial\lambda}{\partial w_j} = \frac{-\lambda(w)\frac{\partial\rho}{\partial z_j}}{\sum_k \frac{\partial\rho}{\partial z_k} w_k} = \frac{-\lambda^2(w)\frac{\partial\rho}{\partial z_j}}{\sum_k \frac{\partial\rho}{\partial z_k}(x_k - z_k)} \text{ and } \left|\frac{\partial\lambda}{\partial w_j}\right| = O\left(\frac{\| x - z \|^2}{|\nabla\rho(z)(x-z)|}\right).$$

We put $d\sigma_{\partial D}(z)$ in terms of $dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_{2n}$ and we use that $-dz_i = \lambda(w)dw_i + d\lambda(w)w_i$ to write these products in terms of $dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_{2n}$. Hence, we obtain two types of coefficients, $\lambda(w)^{2n-1}$ and $\lambda(w)^{2n-2}\frac{\partial\lambda}{\partial w_k}w_l$. Then, if $d\sigma_S(w)$ represents the area's measure on S, one has

$$d\sigma_{\partial D}(z) \le c \| x - z \|^{2n-1} \left(1 + \frac{\| x - z \|}{|\nabla\rho(z)(x-z)|}\right) d\sigma_S(w).$$

Now the second term absorbs the first and we must take now into account that $dm(y) = r^{2n-1} dr d\sigma_S(w)$. ■

Lemma 2. *q is strictly convex in the interior of D. Moreover,*

$$Hq(x)(\lambda,\lambda) \ge c[d(x)^{2n+1+\epsilon} + m(\pi(x))]|\lambda|^2, \ \forall\lambda \in \mathbf{R}^{2n}.$$

Proof: Put $T(\lambda, x, z) := \sin^2(\lambda, x - z)\cos^2(\nabla\rho(z), <\lambda, x - z>)$, where $<\lambda, x - z>$ denotes the plane generated by the vectors λ and $x - z$.

We have, by the relation (2),

$$Hq(x)(\lambda,\lambda) \ge C \int_D \frac{m(z)|v|^2 d(y)^{2+\epsilon} dm(y)}{(\nabla\rho(z)(x-y))^3}.$$

Using the explicit formula for v,

$$\begin{aligned} Hq(x)(\lambda,\lambda) &\ge C|\lambda|^2 \int_D \frac{m(z)|\nabla\rho(z)|^2 \| x - y \|^2 \, T(\lambda,x,z) d(y)^{2+\epsilon} dm(y)}{(\nabla\rho(z)(x-y))^3} = \\ &= C|\lambda|^2 \int_D \frac{m(z)|\nabla\rho(z)|^2 \| x - z \|^3 \, T(\lambda,x,z) d(y)^{2+\epsilon} dm(y)}{(\nabla\rho(z)(z-x))^3 \| x - y \|}. \end{aligned}$$

Now we use the proposition 2 and we integrate on the ball of center x and radius $d(x)/2$ (then $d(y) \simeq d(x)$),

$$\begin{aligned} &Hq(x)(\lambda,\lambda) \ge \\ &\ge c d(x)^{2+\epsilon}|\lambda|^2 \int_{\partial D}\int_0^{\frac{d(x)}{2}} \frac{m(z)|\nabla\rho(z)|^2 T(\lambda,x,z) r^{2n-2} dr d\sigma_{\partial D}(z)}{(\nabla\rho(z)(z-x))^2 \| x - z \|^{2n-3}} \ge \\ &\qquad \ge c d(x)^{2+\epsilon}|\lambda|^2 \int_{\partial D}\int_0^{\frac{d(x)}{2}} \frac{m(z) T(\lambda,x,z) r^{2n-2} dr d\sigma_{\partial D}(z)}{\| x - z \|^{2n-1}}. \end{aligned}$$

Now $\| x - z \|$ is bounded above by diam(D) and we integrate with respect to r,

$$Hq(x)(\lambda,\lambda) \geq cd(x)^{2n+1+\epsilon}|\lambda|^2 \int_{\partial D} m(z)T(\lambda,x,z)d\sigma_{\partial D}(z).$$

This last integral is a positive function of x because $m(z)$ is equal to zero on the flat pieces only (it is positive on a set of positive measure) and $T(\lambda,x,z)$ is equal to zero for z belonging to a set of zero measure only; on the other hand, such function is continuous on $\bar{D}$ ($\leq \int_{\partial D} m(z) = c$) and so it is bounded below. This proves $Hq(x)(\lambda,\lambda) \geq cd(x)^{2n+1+\epsilon}|\lambda|^2$.

If $x \in \partial D$, then $m(z) = m(\pi(x))$ and

$$Hq(x)(\lambda,\lambda) \geq Cm(\pi(x))|\lambda|^2 \int_D \frac{T(\lambda,x,z)d(y)^{2+\epsilon}dm(y)}{\nabla\rho(z)(x-y)}.$$

This integral is a positive function of x and continuous on $\bar{D}$ ($\leq \int_D \frac{d(y)^{1+\epsilon}dm(y)}{\|x-y\|} = c$); so it is bounded below. This proves $Hq(x)(\lambda,\lambda) \geq cm(\pi(x))|\lambda|^2$ when $x \in \partial D$.

For all $x \notin \partial D$, we define $U := \{z \in \partial D / \| z - \pi(x) \| \leq m(\pi(x))\}$ and also the cone $L(x) := \{y \in D / z \in U\}$ (observe that $L(x)$ increases when x approaches to ∂D). If $z \in U$, then $m(z) \simeq m(\pi(x))$. If we integrate on $L(x)$ only, we have:

$$Hq(x)(\lambda,\lambda) \geq Cm(\pi(x))|\lambda|^2 \int_{L(x)} \frac{T(\lambda,x,z)d(y)^{2+\epsilon}dm(y)}{\nabla\rho(z)(x-y)} \geq cm(\pi(x))|\lambda|^2,$$

where the last inequality is proved as before. ∎

3. Estimates for the Henkin-Romanov solution-operator

We suppose D defined by the function q. Put $w'(\zeta) := \sum_{i=1}^n (-1)^{i-1}\zeta_i \bigwedge_{j\neq i} d\zeta_j$, $w(\zeta) := d\zeta_1 \wedge ... \wedge d\zeta_n$, $P_i(\zeta) := \partial q(\zeta)/\partial\zeta_i$, $P(\zeta) := (P_1(\zeta),...,P_n(\zeta))$ and $F(\zeta,z) := \langle P(\zeta), \zeta - z\rangle$. Put

$$\eta := (1-t)\frac{\bar{\zeta}-\bar{z}}{|\zeta-z|^2} + t\frac{P(\zeta)}{F(\zeta,z)},\ 0 \leq t \leq 1, \text{ and}$$

$$B(\zeta,z) := w'\left(\frac{\bar{\zeta}-\bar{z}}{|\zeta-z|^2}\right) \wedge w(\zeta).$$

The Henkin-Romanov solution-operator ([**He Ro**]), call it T, is given by $Tf(z) = T_1 f(z) + T_2 f(z)$, where:

$$T_1 f(z) = c_n \int_{\partial D \times [0,1]} f(\zeta) \wedge w'(\eta) \wedge w(\zeta) \text{ and}$$

$$T_2 f(z) = -c_n \int_D f(\zeta) \wedge B(\zeta, z).$$

Here c_n is a constant that depends only on the dimension.

In [**Ca**] and [**Br Ca**] the following lemma is proved:

Lemma 3.

$$T_1 f(z) = \sum_{k=0}^{n-2} c_{n,k} \int_D f(\zeta) \wedge G_k(\zeta, z), \ z \in D,$$

where $c_{n,k}$ are constants and for $a(\zeta, z) = -q(\zeta) + F(\zeta, z)$,

$$\begin{aligned} G_k(\zeta, z) = &\left(\frac{(k+1-n)\bar\partial_\zeta |\zeta - z|^2}{|\zeta - z|^{2n-2k} a(\zeta, z)^{k+1}} - \frac{(k+1)\bar\partial_\zeta a(\zeta, z)}{|\zeta - z|^{2n-2-2k} a(\zeta, z)^{k+2}} \right) \wedge \\ &\wedge \partial_\zeta |\zeta - z|^2 \wedge \partial q(\zeta) \wedge (\bar\partial \partial q(\zeta))^k \wedge (\bar\partial \partial |\zeta - z|^2)^{n-2-k} + \\ &+ \frac{1}{|\zeta - z|^{2(n-1-k)} a(\zeta, z)^{k+1}} [\partial q(\zeta) \wedge (\bar\partial \partial q(\zeta))^k \wedge (\bar\partial \partial |\zeta - z|^2)^{n-k-1} + \\ &+ \partial_\zeta |\zeta - z|^2 \wedge (\bar\partial \partial q(\zeta))^{k+1} \wedge (\bar\partial \partial |\zeta - z|^2)^{n-2-k}]. \end{aligned}$$

It is well-known that the kernel of Bochner-Martinelli, $B(\zeta, z)$, satisfies $|B(\zeta, z)| = O(|\zeta - z|^{-2n+1})$ and so it is sufficient to prove both theorems only for T_1.

There exists r_0 and δ_0 such that if $|q(z)| < r_0$ and $\partial q(z)/\partial z_j \neq 0$, say $j = 1$, then $t_1 = q(\zeta) - q(z)$, $t_2 = \operatorname{Im} F(\zeta, z)$; $t_{2j-1} = \operatorname{Re}(\zeta_j - z_j)$ and $t_{2j} = \operatorname{Im}(\zeta_j - z_j)$ for $j = 2, \cdots, n$ is a real coordinate system in the ball $B(z, \delta_0)$ such that $t(z) = 0$, $|t|^2 \simeq |\zeta - z|^2$ and $dm(\zeta) \simeq dt_1 \cdots dt_{2n}$.

Put $\alpha_i(\zeta) = \partial^2 q(\zeta)/\partial \bar\zeta_i \partial \zeta_i$. The coefficients of the kernels $G_k(\zeta, z)$ are functions of the type

$$\frac{h(\zeta)\alpha_{i_1}(\zeta) \cdots \alpha_{i_k}(\zeta)}{|\zeta - z|^\beta a(\zeta, z)^\gamma},$$

where h is a bounded function on $B(z, \delta_0)$, $i_1, \cdots, i_k$ are different indexes between 2 and n, and the pair (β, γ) is $(2n - 2k - 2, k + 1)$, $(2n - 2k - 3, k + 1)$ or $(2n - 2k - 3, k + 2)$.

We look for a lower bound for $|a(\zeta, z)|$. Put $c(x) := d(x)^{2n+1+\epsilon} + m(\pi(x))$. We have $2\operatorname{Re} F(\zeta, z) \geq q(\zeta) - q(z) + n(\zeta, z)|\zeta - z|^2$, where $n(\zeta, z) = \int_0^1 c(\zeta + t(z-\zeta))(1-t)dt$. So, $2\operatorname{Re} a(\zeta, z) \geq |q(\zeta)| + |q(z)| + n(\zeta, z)|\zeta - z|^2$.

If $|\zeta - z| \leq \epsilon m(\pi(z))$, then $|\pi(\zeta) - \pi(z)| \leq \epsilon m(\pi(z))$ and $|m(\pi(\zeta)) - m(\pi(z))| \leq c\epsilon m(\pi(z))$. So, $m(\pi(\zeta)) \geq cm(\pi(z))$ and $c(\zeta) \geq cm(\pi(z))$. On the other hand, $|\zeta + t(z-\zeta) - z| = |(1-t)(\zeta - z)| \leq \epsilon m(\pi(z))$, if $(1-t)\mathrm{diam(D)} \leq \epsilon m(\pi(z))$, that is, $1 - t \leq \epsilon m(\pi(z))/\mathrm{diam(D)}$. So,

$$n(\zeta, z) \geq cm(\pi(z)) \int_0^{\frac{\epsilon m(\pi(z))}{\mathrm{diam(D)}}} x dx \geq cm(\pi(z))^3.$$

When $|\zeta - z| \leq \epsilon d(z)$, we have, similarly, $d(\zeta) \geq cd(z)$ and $c(\zeta) \geq cd(z)^{2n+1+\epsilon}$. By the same anterior argument, we obtain $n(\zeta, z) \geq cd(z)^{2n+3+\epsilon}$.

If we put

$$s(z) := c[m(\pi(z))^3 + d(z)^{2n+3+\epsilon}],$$

we obtain $2\operatorname{Re} a(\zeta, z) \geq |q(\zeta)| + |q(z)| + s(z)|\zeta - z|^2$ and, finally,

$$|a(\zeta, z)| \geq |q(\zeta)| + |q(z)| + s(z)|\zeta - z|^2 + |\operatorname{Im} a(\zeta, z)|.$$

Lemma 4.

$$\textit{Put } g_k(z) := \begin{cases} \left[m(\pi(z))^3 + d(z)\right]^{-k}, & \textit{if } k = 1, \cdots, n-2. \\ |\log[m(\pi(z))d(z)]|, & \textit{if } k = 0. \end{cases}$$

$$\textit{Then } \int_D |G_k(\zeta, z)| dm(\zeta) = O(g_k(z)),\ k = 0, 1, \cdots, n-2.$$

Proof: We can suppose that $|q(z)| < r_0$ and estimate the integral only on $D \cap B(z, \delta_0)$. It is sufficient to consider, in terms of the coordinate system,

$$\int_{B(0,\delta_0)} \frac{dt_1 \cdots dt_{2n}}{|t|^\beta (2|q(z)| - t_1 + |t_2| + s(z)|t|^2)^\gamma}$$

for the worst case $(\beta, \gamma) = (2n-2k-3, k+2)$, $k = 0, 1, \cdots, n-2$. We take polar coordinates $t_1 = r\cos\theta$, $t_2 = r\sin\theta\cos\psi, \cdots$ and we distinguish the cases $k \geq 1$ and $k = 0$.

If $k \geq 1$, after integrating respect to the angles, we arrive to

$$\int_0^{\delta_0} \frac{r^{2k} dr}{(2|q(z)| + s(z)r^2)^k}.$$

Now we put $r := |q(z)|^{\frac{1}{2}}x/s(z)^{\frac{1}{2}}$ and $b(z) := \delta_0 s(z)^{\frac{1}{2}}/|q(z)|^{\frac{1}{2}}$, then the integral is:

$$\frac{|q(z)|^{\frac{1}{2}}}{s(z)^{k+\frac{1}{2}}}\int_0^{b(z)}\frac{x^{2k}dx}{(1+x^2)^k}$$

If $b(z) \leq 1$, it is $O(|q(z)|^{-k})$ and if $b(z) \geq 1$, it is $O(s(z)^{-k})$.

If $k = 0$, after integrating respect to the angle ψ and putting $\nu = -\cos\theta$, we arrive to

$$\int_0^{\delta_0}\int_0^1 \frac{r\,d\nu dr}{2|q(z)| + r\nu + s(z)r^2} \leq \int_0^{\delta_0} |\log[2|q(z)| + s(z)r^2]|dr.$$

With the same anterior change, the integral is:

$$\frac{|q(z)|^{\frac{1}{2}}}{s(z)^{\frac{1}{2}}}\int_0^{b(z)} |\log[|q(z)|(1+x^2)]|dx.$$

If $b(z) \leq 1$, it is $O(|\log|q(z)||)$ and if $b(z) \geq 1$, it is

$$\frac{|q(z)|^{\frac{1}{2}}}{s(z)^{\frac{1}{2}}}\left(\int_0^1 |\log|q(z)||dx + \int_1^{b(z)} |\log(|q(z)|x^2)|dx\right) \leq$$

$$\leq \frac{|q(z)|^{\frac{1}{2}}}{s(z)^{\frac{1}{2}}}\left(|\log|q(z)|| + b(z)|\log[(\delta_0)^2 s(z)]| + 2b(z)\right) = O(|\log s(z)|). \blacksquare$$

Observe that the case of a strictly convex domain is obtained when the function $\inf_{\zeta\in\bar{D}} n(\zeta, z)$ is bounded below.

Proof of theorem 1: Observe that the worst estimate in lemma 4 is for $k = n-2$ and that for $n = 2$ we have only $k = 0$.

Proof of theorem 2: Take also into account that lemma 4 gives $\int_D |G_k(\zeta,z)|dm(z) = O(g_k(\zeta))$, $k = 0,\cdots,n-2$, by symmetry. Using Fubini's Theorem, we have

$$\int_D |T_1 f(z)|dm(z) \leq \sum_{k=0}^{n-2} c_{n,k}\int_D |f(\zeta)|\int_D |G_k(\zeta,z)|dm(z)dm(\zeta) \leq$$

$$\leq \sum_{k=0}^{n-2} c_{n,k}\int_D |f(\zeta)|g_k(\zeta)dm(\zeta). \blacksquare$$

Proof of theorem 3: For $0 < \epsilon < 1$, i) $\int_D |G_0(\zeta,z)||q(\zeta)|^{-\epsilon}dm(\zeta) = O(|q(z)|^{-\epsilon})$ and ii) $\int_D |G_0(\zeta,z)||q(z)|^{-\epsilon}dm(z) = O(|q(\zeta)|^{-\epsilon})$ will give, as it is well known, the L^p-estimates, $1 < p < \infty$.

By symmetry, it is sufficient to prove i) only.

Using $|G_0(\zeta,z)| = O(1/|\zeta - z||a(\zeta,z)|^2)$ and the coordinate system, we must consider:

$$\int_{B(0,\delta_0)} \frac{(|q(z)| - t_1)^{-\epsilon} dt_1 dt_2 \cdots dt_{2n}}{|t|(2|q(z)| - t_1 + |t_2| + s(z)|t|^2)^2}.$$

Now, taking polar coordinates $t_1 = r\cos\theta$, $t_2 = r\sin\theta\cos\psi, \cdots$, integrating respect to the angle ψ and putting $\nu = -\cos\theta$, we obtain:

$$\int_0^{\delta_0}\int_0^1 \frac{r(|q(z)| + r\nu)^{-\epsilon} d\nu dr}{2|q(z)| + r\nu + s(z)r^2} \leq \int_0^{\delta_0}\int_0^1 \frac{r d\nu dr}{(|q(z)| + r\nu)^{1+\epsilon}} \leq c|q(z)|^{-\epsilon}. \quad \blacksquare$$

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