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P-LOCALIZATION OF SOME CLASSES OF GROUPS

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Abstract

The aim for the present paper is to study the theory of P-Localization of a group in a category $\mathcal{C}$ such that it contains the category of the nilpotent groups as a full sub-category. In the second section we present a number of results on P-localization of a group G, which is the semi-direct product of an abelian group A with a group X, in the category $\mathcal{G}$ of all groups. It turns out that the P-localized (G_P) is completely described by the P-localized X_P of X, A and the action ω of X on A. In the third section, we present the construction of the theory of P-localization in the category of all groups which are extensions of nilpotent groups by finite abelian groups. Our proof follows rather closely the one presented in [**2**, chapter I], and is based on the classical interpretation of the second cohomology group of a group.

Introduction

Since Sullivan first pointed out the availability and applicability of localization methods in homotopy theory, there has been considerable work done on further developments and refinements of the method and on the study of new areas of application. In [**2**] P. Hilton, G. Mislin and J. Roitberg constructed the theory of P-localization of nilpotent groups, where P is a set of primes. Some time later, P. Ribenboin in [**3**] showed that it was possible to localize any group. (There is another approach concerning P-localization in group theory developed by Bousfield in Topology 14 (1975) 133-150, and Mem. Amer. Math. Soc. 10 (1977) no. 186, but, in this work, we just use the concepts presented in [**2**], [**3**] and [**4**]). The construction presented in [**3**], however, seems to be quite abstract and this led us to try to obtain a more explicit construction of the P-localization of a group G in the category of all groups. We were successful when G is a semi-direct product of a finite abelian group A by some other group X. In addition, we managed to construct theory of P-localization of a group in the category $\mathcal{C}$ of groups which are extensions of nilpotent groups by finite abelian groups.

The question concerning semi-direct product is taken up in Section 2 and the main results are 2.1, 2.5 and 2.10 which could be stated as follows.

Let P' be the complementary of P in the set of all primes. Theorem (2.1) Let $N \xrightarrow{\mu} G \overset{\varepsilon}{\twoheadrightarrow} X$ be an exact sequence of groups, where N is a p-group and $p \in P'$.

Then, $e = e_0 o \varepsilon P$-localizes G, provided that $X \xrightarrow{e_0} X_P$ P-localizes X.

In this context, Theorem 2.5 says the following:

Let $X \xrightarrow{\omega} Aut(A)$ be an action, where A is a finite abelian p-group and $p \in P$.

Let $P_1 = \{q \in P' : q \mid |\omega(X)|\}$ and consider P_1^x the multiplicative set generated by P_1.

Set H the sub-group of X generated by all $x \in X$ such that the order of $\omega(x)$ belongs to P_1^x.

Let ω_H be the restriction of ω to H and $\Gamma = \Gamma^r_{\omega_H}$, where r is the smallest positive integer such that $\Gamma^r_{\omega_H} = \Gamma^{r+1}_{\omega_H}$ (Here Γ^j_ω has the ordinary meaning and its definition may be found in [**2**]). There is an action $X \xrightarrow{\overline{\omega}} Aut(A/_\Gamma)$ induced by ω, which can be factored as

$$\begin{array}{ccc} X & \xrightarrow{\omega} & Aut(A/_\Gamma) \\ e_0 \downarrow & \nearrow_{\omega'} & \\ X_P & & \end{array}$$

Let $G = A \rtimes_\omega X$ and $G' = A/_\Gamma \rtimes_{\omega'} X_P$

Then, $e : (a, x) \in G \longrightarrow (a + \Gamma, e_0(X)) \in G'$ P-localizes G.

Finally, Theorem 2.10 analyzes the situation in which A is a finite abelian group.

Let $X \xrightarrow{\omega} Aut(A)$ be an action, where A is a finite abelian group. Let $A_1, \dots, A_t$ be the p-primary components of A and $\omega_i : X \longrightarrow Aut(A_i)$ be the actions induced by ω.

Let $G = A \rtimes_\omega X, G_i = A_i \rtimes_{\omega_i} X$ and take $\overline{G} \overset{\varepsilon}{\twoheadrightarrow} X_P$ to be the pull-back of the arrows $(G_i)_P \overset{(\varepsilon_i)_P}{\twoheadrightarrow} X_P$. (Notice that $(G_i)_P$ are given by either 2.1 or 2.5. Then, we claim that the natural homomorphism $G \xrightarrow{f} \overline{G}$ P-localizes G.

We devote Section 3 to present our results concerning the construction of the theory of P-localization of a group in the category $\mathcal{C}$ of groups which are extensions of nilpotent groups by finite abelian groups. The matter could be described as follows:

Given $G \in \mid C \mid$, there exists a unique finite abelian sub-group U of G such that $G/_U$ is nilpotent and $\Gamma^2_\omega = U$ where ω is the action attached to the extension $\xi : U \rightarrowtail G \twoheadrightarrow G/_U$. Furthermore, there exists a unique group $\overline{U}$ and an epimorphism $\rho_v : U \twoheadrightarrow \overline{U}$ and a unique $\xi_P \in H^2((G/U)_P; \overline{U})$.

$(\xi_P : \overline{U} \rightarrowtail G_P \twoheadrightarrow (G/_U)_P)$ attached to ξ yielding commutativity in the diagram

$$\begin{array}{ccccccc} \xi : & & U & \rightarrowtail & G & \twoheadrightarrow & G/_U \\ & & \downarrow \rho_U & & \downarrow e & & \downarrow e_0 \\ \xi_P : & & \overline{U} & \rightarrowtail & G_P & \twoheadrightarrow & (G/_U)_P \end{array}$$

Under such conditions (3.12) states that $G \longrightarrow G_P$ is a functor and e is a natural transformation of functors. In addition, (3.13) also states that eP-localizes G.

In Section 1 we introduce some basic results needed in the following section. We believe, nevertheless, that Theorem (1.21) is important on its own accord; it states that if

$$\begin{array}{ccc} X & \xrightarrow{\omega} & Aut(A) \\ e_0 \downarrow & \nearrow_{\omega_P} & \\ X_P & & \end{array}$$

is a comutative diagram where $X \xrightarrow{e_0} X_P P$-localizes X and A is a P-local finite abelian group, then $e_0^* : H^n_{\omega_P}(X_P; A) \longrightarrow H^n_\omega(X; A)$ is an isomorphism.

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1. Preliminaries

In this section we introduce some general results on P-local groups, factoring of actions and some propositions concerning group cohomology.

We start by fixing the notations $P^x = \{n \in \mathbb{N}^* : p \mid n \Rightarrow p \in P\}$ = multiplicative set generated by P; P' is the complementary of P in the set of all primes.

We also recall that G is said to be a P-local group $\Longleftrightarrow$ $((\forall n \in P'^x) x \in G \mapsto x^n \in G$ is bijective).

Moreover, $G \xrightarrow{e} G_P$ P-localizes $G \in |\mathcal{C}|$ in the category $\mathcal{C} \iff (G_P \in |\mathcal{C}|$, G_P is P-local and $\forall H \in |\mathcal{C}|$, H P-local, $(\forall f \in Hom(G,H))(\exists\, !\, f_P \in Hom(G_P, H))$ which yields commutativity in the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ e\downarrow & \nearrow_{f_P} & \\ G_P & & \end{array}).$$

Proposition 1.1. *Let $E_1, \ldots, E_t, K$ be P-local groups, where P is a set of primes. Let $\varepsilon_i \in Hom(E_i, K)$, $i = 1, \ldots, t$. In these conditions, if $\varepsilon \in Hom(E, K)$ is the pull-back of the family $(\varepsilon_i)_{1<i<t}$, then E is P-local.*

Proof: Straightforward. ■

Proposition 1.2. *Let $\phi \in Hom(Y, F)$, where Y is a P-local group and F is a finite group. Then, $(\forall y \in Y)$ we have $o(\phi(y)) = n \in P^x$; $(o(\phi(y)) =$ order of $\phi(y))$.*

Proof: Let $y \in Y$ and suppose that $\exists\, q \in P'$ with $q \mid o(\phi(Y))$. Then we may consider $z = y^k$, where $o(\phi(y)) = q.k$. It follows that $o(\phi(z)) = q$. The fact that Y is P-local and $q \in P'$ enables us to state that $\forall r > 0, \exists\, z_r \in Y$ such that $z_r^{q^r} = z$. Thus, $\phi(z_r)^{q^r} = \phi(z) \neq 1$ and $\phi(z_r)^{q^{r+1}} = \phi(z)^q = 1$, so $o(\phi(z_r)) = q^{r+1}$, $\forall r > 0$. In particular $\{\phi(z_r) \in F : r > 0\}$ is infinite. However this is impossible, since F is finite. ■

Corollary 1.3. *Under the conditions of the previous proposition, we have $|\phi(Y)| \in P^x$ (ie, $\phi(Y)$ is a P-torsion (finite) sub-group of F).*

Remark 1.4. According to proposition (7.1) in [**4**] we have that a finite group F is P-local $\iff$ F is a P-torsion group.

The full subcategory of the category $\mathcal{G}$ of all groups consisting of all nilpotent groups is denoted by η.

Proposition 1.5. *Let $X \xrightarrow{\omega} Aut(N)$ be an action, where X and N are groups with $Aut(N)$ finite. Then $\exists\, !\omega_P$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\omega} & Aut(N) \\ e_0\downarrow & \nearrow_{\omega_P} & \\ X_P & & \end{array}$$

is commutative $\iff$ $\omega(X)$ is a P-torsion sub-group of $Aut(N)$.

Proof: It follows directly from Corollary 1.3 and Remark 1.4. ■

Now we take $A \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ an exact sequence of groups, where A is abelian.

Let $X \overset{\omega}{\longrightarrow} Aut(A)$ be the action defined by $\mu(\omega((x).a) = g.\mu(a).g^{-1}$ (where $\varepsilon(g) = x$).

Fix a collection of primes P, $n \in \mathbb{N}$ and $x \in X$, and define:

$\theta_n(x) = 1_A + \omega(x) + \cdots + \omega(x^{n-1}) \in End(A)$.

For this endomorphism we have:

Lemma 1.6. $(\mu(a).g)^n = \mu(\theta(\varepsilon(g)).a)g^n$; $\forall g \in G, \forall a \in A, \forall n \in \mathbb{N}$.

Proof: It is easy by induction on n. ■

Proposition 1.7. *Let P be a set of primes and let $A \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ be an exact sequence of groups, where A is abelian and ω is the action attached to the extension. Fix the conditions: (i) G is P-local; (ii) X is P-local; (iii) $\theta_n(x) \in Aut(A), \forall x \in X, \forall n \in P'^x$.*

Then, if two of (i); (ii); (iii); hold so does the third.

Proof: (ii) + (iii) $\Longrightarrow$ (i).

Fix $n \in P'^x$. Let g, $h \in G$ and suppose that $g^n = h^n$. Then, $\varepsilon(g)^n = \varepsilon(h)^n \Rightarrow \varepsilon(g) = \varepsilon(h)$ since X is P-local. So, $g = \mu(a).h$ and $h^n = g^n = (\mu(a).h)^n = \mu(\theta_n(\varepsilon(h)).a)h^n$ (Lemma 1.6). Hence $\theta_n(\varepsilon(h)).a = 0$. So $a = 0$ and $g = h$.

Likewise, let $g \in G.\exists x \in X$ such that $\varepsilon(g) = x^n$ ($X P$-local). Therefore, $\varepsilon(g) = x^n = \varepsilon(h^n)$. Therefore $g = \mu(a).h^n$. Take $b \in A$ such that $a = \theta_n(\varepsilon(h)).b$. Thus $g = \mu(\theta_n(\varepsilon(h)).b).h^n = (\mu(b).h)^n$ due to 1.6. So $g \in G \longrightarrow g^n \in G$ is bijective.

(The other implications are similar). ■

Proposition 1.8. *Let P be a set of primes and let $A \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ be an exact sequence of groups, where A is finite abelian. Then, $\theta_n(x) \in Aut(A)$, $\forall x \in X$, $\forall n \in P'^x$, provided that either G is P-local or A and X are P-local.*

Proof: (I) G is P-local.

Fix $n \in P'^x$ and $x \in X$. Suppose that $\theta_n(x).a = 0$. Let $g \in G$ such that $\varepsilon(g) = x$. Then $(\mu(a).g)^n = \mu(\theta_n(x).a)g^n = g^n$. So $\mu(a).g = g$ and $a = 0$. Thus $\theta_n(x) \in Aut(A)$ since A is finite.

(II) A and X are P-local.

Fix $n \in P'^x$ and $x \in X$, and suppose $\theta_n(x).a = 0$. Thus, $(\omega(x^n) - 1_A).a = (\omega(x) - 1_A)o\theta_n(x).a = 0$ therefore $\omega(x^n).a = a$. On the other hand, $o(\omega(x)) = m \in P^x$ (Prop. 1.2). Therefore $\omega(x)^m.a = a$. As $\gcd(m,n) = 1$, it follows that $\omega(x).a = a$, whence $0 = \theta_n(x).a = n.a$. Thus $a = 0$ since A is P-local. So $\theta_n(x) \in Aut(A)$. ■

Corollary 1.9. *In the conditions of the proposition above (1.8), G is P-local $\Longleftrightarrow$ A and X are P-local.*

Next we consider a split extension $N \overset{\mu}{\rightarrowtail} G \underset{\sigma}{\overset{\varepsilon}{\rightleftarrows}} X$, where N is a finite group. Let $X \xrightarrow{\omega} Aut(N)$ be the action given by $\mu(\omega(x).a) = \sigma(x).\mu(a).\sigma(x)^{-1}$ and take $\theta_n(x) : N \longrightarrow N$ defined by $\theta_n(x).a = 1_A(a).(\omega(x).a)\dots(\omega(x)^{n-1}.a)$, $x \in X; n \in \mathbb{N}^*$:

In this slightly different context we now describe properties which are quite similar to Prop. 1.6, 1.7, 1.8 and 1.9.

Lemma 1.10. $(\mu(a).(x))^n = \mu(\sigma_n(x).a).\sigma(x)^n$; $\forall x \in X$; $\forall n \in \mathbb{N}^*$; $\forall a \in A$.

Proof: See Prop. 1.6. ■

Proposition 1.11. *Let $N \overset{\mu}{\rightarrowtail} G \underset{\sigma}{\overset{\varepsilon}{\rightleftarrows}} X$ be a split short exact sequence of groups and let ω be the action defined by the splitting σ. Fix the statements:*

(i) G is P-local; (ii) X is P-local; (iii) $\theta_n(x)$ is a bijection, $\forall x \in X$; $\forall n \in P'^x$.

Then, if two of (i); (ii); (iii); hold, so does the third.

Proof: See Prop. 1.7. ■

Proposition 1.12. *Let $N \rightarrowtail G \underset{\sigma}{\overset{\epsilon}{\rightleftarrows}} X$ be a split short exact sequence of groups, where N is finite. Then $\theta_n(x)$ is a bijection, $\forall n \in P'^x$, $\forall x \in X$ provided that either G is P-local or N and X are P-local.*

Proof: If G is P-local, then the proof follows as (I) Prop. 1.8.

So let's suppose that N and X are P-local. Let $\omega(X) \xrightarrow{i} Aut(N)$ and $\overline{G} = N \rtimes_i \omega(X)$. $\omega(X)$ is a P-torsion group (Cor. 1.3) and N is a P-torsion group (Remark 1.4). Thus $\overline{G}$ is a P-torsion group. So $\overline{G}$ is P-local (Remark 1.4).

Then taking the sequence $N \overset{\overline{\mu}}{\rightarrowtail} \overline{G} \underset{\overline{\sigma}}{\overset{\overline{\varepsilon}}{\rightleftarrows}} \omega(X)$ and invoking the first statement of this proposition we conclude that $\overline{\theta_n}(\tau) : N \longrightarrow N$ given by $\overline{\theta_n}(\tau) = 1_N.i(\tau)\dots i(\tau^{n-1}) = 1_N\tau\dots\tau^{n-1}$ is a bijection, $\forall\tau \in \omega(X)$, $\forall n \in P'^x$. So, $\forall x \in X$ $\forall n \in P'^x$ we have that $\theta_n(x)$ is a bijection, since $\theta_n(x) = 1_N.\omega(x)\dots\omega(x^{n-1}) = \overline{\theta_n}(\tau)$, where $\tau = \omega(x) \in \omega(X)$. ■

Corollary 1.13. *Under the condition of the previous proposition (1.12), G is P-local $\Longleftrightarrow$ N and X are P-local.*

Proof: See Prop. 1.9. ■

Proposition 1.14. *Let $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ be an exact sequence of groups. Then, G and X P-local $\Longrightarrow$ N P-local.*

Proof: It follows directly from the definitions. ■

From now on we establish some results which play an important role in Section 3.

Let $X \xrightarrow{\omega} Aut(A)$ and $X \xrightarrow{\theta} Aut(B)$ be actions, where A and B are abelian. Let also $\alpha \in Hom_{\mathbb{Z}[X]}(A, B)$ (ie $\alpha(\omega(x).a) = \theta(x).\alpha(a)$).

The reader interested in more details about the constructions involved in the propositions below should collect material in [**5**, chapter II, Proposition 4.3.], for instance.

The proofs of the next three propositions follow easily from the definitions according the usual techniques.

Proposition 1.15. *Consider the diagram*

$$\begin{array}{rccccc} \xi: & A & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & X \\ & \alpha\downarrow & & & & \downarrow\gamma \\ \zeta: & B & \overset{\nu}{\rightarrowtail} & Q & \overset{\pi}{\twoheadrightarrow} & Y \end{array}$$

where A and B are abelian and the rows are exact.

If there exists $\beta \in Hom(G, Q)$ making the diagram commutative, then $\alpha \in Hom_{\mathbb{Z}[X]}(A, B)$ and $\alpha_*\xi = \gamma^*\zeta$.

Conversely, if $\alpha \in Hom_{\mathbb{Z}[X]}(A, B)$ and $\alpha_*\xi = \gamma^*\zeta$, then there does exist $\beta \in Hom(G, Q)$ making the diagram commutative.

Proposition 1.16. *In the diagram*

$$\begin{array}{ccccc} A & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & X \\ \alpha\downarrow & & \tau\downarrow\downarrow\beta & & \downarrow\gamma, \\ B & \overset{\nu}{\rightarrowtail} & Q & \overset{\pi}{\twoheadrightarrow} & Y \end{array}$$

the rows are exact, A and B are abelian and τ and β yield commutativity. Then, there exists a cross homomorphism $\kappa : X \longrightarrow B$ such that $\forall g \in G$, $\beta(g) = \nu\kappa\varepsilon(g).\tau(g)$.

Proposition 1.17.

$$\begin{array}{ccccc} A & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & X \\ \alpha\downarrow & & \downarrow\tau & & \downarrow\gamma \\ B & \overset{\nu}{\rightarrowtail} & Q & \overset{\pi}{\twoheadrightarrow} & Y, \end{array}$$

In the commutative diagram the rows are exact and A and B abelian. Let $X \xrightarrow{\kappa} B$ be a cross homomorphism. In these conditions the function $G \xrightarrow{\beta} Q$ given by $\beta(g) = \nu\kappa\varepsilon(g).\tau(g)$, $\forall g \in G$ is a group homomorphism.

Lemma 1.18. *Let Q be a P-torsion abelian group. Then $H_q(Q)$ is a P-torsion abelian group, $= \forall q > 0$. (Here $H_q(Q)$ means the homology of the group Q with integer coefficients).*

Proof: The assertion is readily checked, since, according the theory in [**2**] we have $H_n(Q)_{P'} \cong H_n(Q_{P'}) = H_n((0)) = (0)$; $n \geq 1$ (once Q is P-torsion abelian). ∎

Lemma 1.19. *Let $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ be a central exact sequence of groups. If G acts P-locally on an abelian group A, then Q acts P-locally on $H^*(N; A)$.*

Proof: We recall that if G acts on A by means of ω, then the action ω is P-local if and only if $(\forall n \in P'^x)(\forall x \in G)\theta_n(x) = 1_A + \omega(x) + \cdots + \omega(x^{n-1}) \in Aut(A)$.

Moreover, taking $z \in G$ such that $\varepsilon(z) = x$, it is known that the induced action of Q on $H^s(N; A)$ is given by: $Q \xrightarrow{\Omega} Aut(H^s(N; A))$, where $\Omega(x) = \omega(z)_*$ (remember that the extension $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ is central).

Thus, fixing $x \in Q$ and putting $\Theta_n(x) = 1_{H^s(N;A)} + \Omega(x) + \cdots + \Omega(x^{n-1})$, we get: $\Theta_n(x) = (1_A)_* + \omega(z)_* + \cdots + \omega(z^{n-1})_* = [1_A + \omega(z) + \cdots + \omega(z^{n-1})]_* = \Theta_n(z)_*$. So $\Theta_n(x)$ is an isomorphism. ∎

Lemma 1.20. *Suppose that the action $X \xrightarrow{\omega} Aut(A)$ is P-local and X is a P'-torsion group (A an abelian group). Then, ω is trivial.*

Proof: Set $x \in X$. By hypothesis, $\exists\ n \in P'^x$ such that $x^n = 1$. So, $0 = \omega(x^n) - 1_A = \theta_n(x) o[\omega(x) - 1_A]$.

Then, $\omega(x) = 1_A$ (for $\theta_n(x) \in Aut(A)$). ■

The next theorem is stated in the category η.

Theorem 1.21. *Consider the commutative diagram*

$$\begin{tikzcd} X \arrow[r, "\omega"] \arrow[d, "e_0"'] & Aut(A) \\ X_P \arrow[ur, "\omega_P"'] & \end{tikzcd},$$

where X is a nilpotent group, A is a P-local finite abelian group and ω, ω_P are actions.

Then we have $H^n_{\omega_P}(X_P; A) \xrightarrow[e_0^*]{\cong} H^n_\omega(X; A)$.

Proof: (Induction on $c = \text{nil}\ X$.)

If X is abelian, we take the following short exact sequences:

$$0 \longrightarrow Ker(e_0) \longrightarrow X \xrightarrow{e_0'} e_0(X) \longrightarrow 0 \ldots (1)$$
$$0 \longrightarrow e_0(X) \xrightarrow{e_0''} X_P \longrightarrow Coker(e_0) \longrightarrow 0 \ldots (2)$$

(1) yields a spectral sequence (Lyndon-Hochschild-Serre) where $E_2^{r,s} = H^r(e_0(X); H^s(Ker(e_0); A))$.

Noticing that $Ker(e_0)$ acts trivially on A we are allowed to say that the following sequence is exact $0 \longrightarrow Ext(H_{s-1}(Ker(e_0)); A) \longrightarrow H^s(Ker(e_0; A)) \longrightarrow Hom(H_s(Ker(e_0); A) \longrightarrow 0$.

Since $Hom(P'$-torsion, P-local$) = (0) = Ext(P'$-torsion, P-local$)$ and $(\forall s > 0) H_s(Ker(e_0))$ is P'-torsion (Lemma 1.18) we conclude that $E_2^{r,s} = (0), \forall s > 0$, whence the spectral sequence collapses.

Thus, $H^r_{\omega_P}(e_0(X); A) \cong E_2^{r,0} = E_\infty^{r,0} \cong H^r_\omega(X; A)$. Therefore, we have got that $e_0'^*$ is an isomorphism.

Likewise, (2) yields another spectral sequence where $E_2^{r,s} = H^r(Coker(e_0); H^s(e_0(X); A))$. Here $Coker(e_0)$ acts trivially on $H^s(e_0(X); A)$.

This may be seen from Lemmas 1.19 and 1.20 according to the following argument: due to Proposition 1.5 $\omega_P(X_P)$ is a (finite) P-torsion group. So, $Y = A \rtimes_i \omega_P(X_P)$ (where $\omega_P(X_P) \overset{i}{\hookrightarrow} Aut(A)$) is a finite P-group. So Y is P-local, and then it follows that ω_P acts P-locally on A (use the same argument that the one in 1.12).

Now, by Lemma 1.19, we have that $Coker(e_0)$ acts P-locally on $H^s(e_0(X); A)$. As $Coker(e_0)$ is P'-torsion, our statement now follows from Lemma 1.20.

Thus, taking into account that the sequence $0 \longrightarrow Ext(H_{r-1}(Coker(e_0));$ $H^s(e_0(X);A)) \longrightarrow E_2^{r,s} \longrightarrow Hom(H_r(Coker(e_0);H^s(e_0(X);A)) \longrightarrow 0$ is exact, $Coker(e_0)$is P'-torsion and $H^s(e_0(X);A))$ is P-local we obtain $E_2^{r,s} = (0)\ \forall r > 0$.

Thus, $H^s(e_0(X);A) \cong E_2^{0,s} = E_\infty^{0,s} \cong H^s(X_P;A)$, whence $e_0''^*$ is an isomorphism.

Remebering that the diagram commutes

$$\begin{tikzcd} H^n_\omega(X;A) & & H^n_{\omega_P}(X_P;A) \arrow[ll, "e_0^*"'] \arrow[dl, "\cong\ e_0''^*"] \\ & H^n_{\omega_P}(e_0(X);A) \arrow[ul, "\cong\ e_0'^*"] & \end{tikzcd}$$

we conclude that e_0^* is an isomorphism, so that the proof is ended if $c = \text{nil } X = 1$.

Now let $X \in |\eta|$ with nil $X = c > 1$ and set $\Gamma = \Gamma^c X \neq \{1\}$.

Then, the commutative diagram with exact rows

$$\begin{tikzcd} \Gamma \arrow[r, tail] \arrow[d, "e_0''"] & X \arrow[r, two heads] \arrow[d, "e_0"] & X/\Gamma \arrow[d, "e_0'"] \\ \Gamma_P \arrow[r, tail] & X_p \arrow[r, two heads] & (X/\Gamma)_P \end{tikzcd}$$

induces a map of the spectral sequences. In particular we obtain the commutative diagram

$$\begin{tikzcd} E_2^{r,s} = H^r(X/_\Gamma; H^s(\Gamma;A)) & \\ & H^r((X/_\Gamma)_P; H^s(\Gamma;A)) \arrow[ul, "e_0'^*"'] \arrow[dl, "(e_0''^*)_*"] \\ \overline{E}_2^{r,s} = H^r((X/_\Gamma)_P; H^s(\Gamma_P;A)) \arrow[uu, "(e_0', e_0'')^*"] & \end{tikzcd}$$

It now follows from the first step that $H^s(\Gamma_P;A) \xrightarrow[\cong]{e_0''^*} H^s(\Gamma;A)$, whence $(e_0''^*)_*$ is an isomorphism. On the other hand, from [**2**, theorem 4.14, pg.40], we are allowed to conclude that $e_0'^*$ is an isomorphism, since $H^s(\Gamma;A)$ is P-local.

So $E_2^{r,s} \xrightarrow[\cong]{(e_0',e_0'')^*} \overline{E}_2^{r,s}$ and we may infer that $H^n(X_P;A) \xrightarrow[\cong]{e_0^*} H^n(X;A)$ finishing the proof. ■

Finally, in order to finish this section, we are going to consider the following situation (which will appear again in the next two sections):

Let $X \xrightarrow{\omega} Aut(A)$ be an action, where A is a finite abelian p-group $(p \in P)$.

Let $P_1 = \{q \in P' : q \mid |\omega(x)|\}$ and set $H = < x \in X : o(\omega(x)) \in P_1^x >$ = (sub-group generated by $x \in X$ such that $o(\omega(x)) \in P_1^x$). Noticing that $H \lhd X$ and setting $\omega_H = \omega|_H$: $H \longrightarrow X \xrightarrow{\omega} Aut(A)$ we get:

Proposition 1.22. *$\Gamma^j_{\omega_H}$ is a ω-sub-module of A, $\forall j \geq 1$.*

Proof: It follows, by induction on j, from the fact that $\omega(x).(\omega(h).a - a) = \omega(xhx^{-1}).(\omega(x).a) - \omega(x).a$, together with the fact that $H \lhd X$ ($a \in \Gamma^{j-1}_{\omega_H}; h \in H, x \in X$). ■

Next set $\Gamma = \Gamma^r_{\omega_H}$, where r is the smallest positive integer such that $\Gamma^r_{\omega_H} = \Gamma^{r+1}_{\omega_H}$.

Proposition 1.22 enables us to take $X \xrightarrow{\overline{\omega}} Aut(A/_\Gamma)$, where $\overline{\omega}(x)(a + \Gamma) = \omega(x).a + \Gamma$. Then it is plain that $\overline{\omega \mid_H} = \overline{\omega} \mid_H$ and we have:

Lemma 1.23. *$\overline{\omega} \mid_H$ is trivial.*

Proof: Recall that $\Gamma^j_{\overline{\omega}|_H} = \Gamma^j_{\omega_H}/\Gamma^r_{\omega_H}$. Thus $\overline{\omega}_H$ is nilpotent.

Remembering that $A/_\Gamma$ is a finite abelian p-group and invoking the Prop.7, pg. 7, [1] we conclude that $\overline{\omega}(H) = \overline{\omega}_H(H)$ is a (finite) p-group.

Taking, on the other hand, h a generator of H, we have $h \in H$ and $o(\omega(h)) \in P_1^x$. It follows that $o(\overline{\omega}(h)) \in P_1^x$, since $o(\overline{\omega}(h)) \mid o(\omega(h))$. Taking into account that $P_1 \subset P'$, it follows, at last, that $o(\overline{\omega}(h)) = 1$ and therefore $\overline{\omega}(h) = 1_{A/_\Gamma}, \forall h \in H$. ■

Proposition 1.24. *$\overline{\omega}(X)$ is a P-torsion sub-group of $Aut(A/_\Gamma)$.*

Proof: Suppose that $\exists\ q \in P'$ with $q \mid |\overline{\omega}(X)|$. Thus $\exists\ y \in X$ such that $o(\overline{\omega}(y)) = q$. Then we have $o(\omega(y)) = q^\ell.m$, $\gcd(q, m) = 1$ (for $o(\overline{\omega}(y)) \mid o(\omega(y))$). Hence, $o(\omega(y^m)) = q^\ell$ and therefore $o(\overline{\omega}(y^m)) = q$. It follows that $\overline{\omega}(y^m) \neq 1_{A/_\Gamma}$, which contradicts the statement of the preceding lemma (1.23).

Corollary 1.25. *$\exists\,!$ action $X_P \xrightarrow{\omega'} Aut(A/_\Gamma)$, with $\omega'(X_P) = \overline{\omega}(X)$, which yields commutativity in the following diagram*

$$\begin{tikzcd} X \arrow[r, "\overline{\omega}"] \arrow[d, "e_0"'] & Aut(A/_\Gamma) \\ X_P \arrow[ur, dotted, "\omega'"'] & \end{tikzcd}$$

Proof: It follows directly from the Prop. 1.5. ■

Now consider the commutative diagram

$$\begin{array}{ccccc} A & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & X \\ \alpha \downarrow & & \downarrow \beta & & \downarrow \gamma \\ B & \overset{\nu}{\rightarrowtail} & Q & \overset{\pi}{\twoheadrightarrow} & Y \end{array}$$

where the rows are exact and A and B are finite abelian p-groups. Let $P_1(A) = \{q \in P' : q \mid |\omega_1(X)|\}$; $P_1(B) = \{q \in P' : q \mid |\omega_2(Y)|\}$.

Proposition 1.26. *In the conditions above,* $\alpha(\Gamma^j_{\omega_{1|H_A}}) \subset \Gamma^j_{\omega_{2|H_B}}$, $\forall j \geq 1$ *(In particular* $\alpha(\Gamma(H_A)) \subset \Gamma(H_B)$*).*

Proof: We argue by induction on j. It is clear for $j = 1$. Next, take $a \in \Gamma^{j-1}_{\omega_{1|H_A}}$ and x a generator of H_A. Thus, $o(\omega_1(x)) = n \in P_1(A)^x$. Let's suppose that $o(\omega_2(\gamma(x))) = r.s$ where $r \in P^x$ and $s \in P_1(B)^x$. Let $n_1 = o(\omega_1(x^s)) \mid o(\omega_1(x)) = n$. Then, $n_1 \in P_1(A)^x$. Now, $\omega_2(\gamma(x^s))^r.\alpha(a) = \alpha(a)$ and $\omega_2(\gamma(x^s))^{n_1}.\alpha(a) = \alpha(\omega_1(x^s)^{n_1}.a) = \alpha(a)$. It follows that $\omega_2(\gamma(x^s)).\alpha(a) = \alpha(a)$, due to the fact that $\gcd(r, n_1) = 1$. Also, $\gcd(r, s) = 1 \iff \exists\ k, \ell \in \mathbb{Z}$ such that $kr + \ell s = 1$. Therefore $\omega_2(\gamma(x)).\alpha(a) = \omega_2(\gamma(x)^r)^k.\omega_2(\gamma(x^s))^\ell.\alpha(a) = \omega_2(\gamma(x)^r)^k.\alpha(a)$. But $o(\omega_2(\gamma(x^r)) = s \in P_1(B)^x$. Therefore $\gamma(x^r) \in H_B$, whence $\gamma(x^r)^k \in H_B$.

It follows that $\alpha(\omega_1(x).a - a) = \omega_2(\gamma(x)).\alpha(a) - \alpha(a) = \omega_2(\gamma(x^r))^k.\alpha(a) - \alpha(a) \in \Gamma^j_{\omega_{2|H_B}}$, since $\alpha(a) \in \Gamma^{j-1}_{\omega_{2|H_B}}$ by the inductive hypothesis. Now it is easy to finish the proof. ■

2. P-Localization of semi-direct products

Throughout this section we just work with the theory of localization in the category $\mathcal{G}$, as developed in [3].

Let P be a set of primes.

Proposition 2.1. *Let* $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ *be an exact sequence of groups, where N is a p-group ($p \in P'$). Then,* $e = e_0\varepsilon$ *P-localizes G, where* $X \xrightarrow{e_0} X_P$ *P-localizes X.*

Proof: It is enough to notice that if N_P is trivial, then every homomorphism from G into a P-local group vanishes on N, and hence ε is a P-equivalence. ■

Next we prove a proposition which plays a fundamental role in the proof of the main theorems in this section.

Proposition 2.2. *Let* $X \xrightarrow{\omega} Aut(K)$ *and* $X_P \xrightarrow{\overline{\omega}} Aut(\overline{K})$ *be actions, where K and $\overline{K}$ are finite groups, $\overline{K}$ P-local. Let* $G = K \rtimes_\omega X$ *and* $\overline{G} = \overline{K} \rtimes_{\overline{\omega}} X_P$*. Consider also the commutative diagram*

$$\begin{tikzcd}
 & K \arrow[rr, tail, "\mu"] \arrow[d, "\pi"] \arrow[ddl, "e'"'] & & G \arrow[rr, two heads, shift left, "\varepsilon"] \arrow[d, "\bar{e}"] \arrow[ddl, "e"'] & & X \arrow[ll, shift left, "\sigma"] \arrow[d, "e"] \arrow[ddl, "e"'] \\
 & \overline{K} \arrow[rr, tail, "\overline{\mu}"] \arrow[dl, "\tilde{e}"] & & \overline{G} \arrow[rr, two heads, shift left, "\overline{\varepsilon}"] & & X_P \arrow[ll, shift left, "\overline{\sigma}"] \arrow[dl, equal] \\
N \arrow[rr, tail, "\mu'"] & & G_P \arrow[rr, two heads, shift left, "\varepsilon_P"] & & X_P \arrow[ll, shift left, "\sigma_P"] &
\end{tikzcd}$$

In the diagram $\mu, \varepsilon, \sigma, \overline{\mu}, \overline{\varepsilon}, \overline{\sigma}, \varepsilon_P, \sigma_P$ are defined as usual, eP-localizes G, $N = ker\ \varepsilon_P, \mu' : N \longrightarrow G_P, e'$ is defined by the restriction of e, π is surjective and $\tilde{e}_\pi = e'$. Let's also suppose that $\forall x \in X, \forall a \in K$ we have: (i) $\pi(\omega(x).a) = \overline{\omega}(e_0(x)).\pi(a)$ and (ii) $e'(\omega(x).a) = \omega_P(e_0(x)).e'(a)$. Under these conditions we claim that $\tilde{e}(\overline{\omega}(z).a) = \omega_P(z).\tilde{e}(a), \forall z \in X_P, \forall \overline{a} \in \overline{K}$.

Proof: $\overline{e} \in Hom(G, \overline{G})$ due to the hypothesis about π, as well as $\overline{G}$ is P-local by corollary 1.13. So $\exists\ !\ \phi \in Hom(G_P, \overline{G})$ such that $\phi e = \overline{e}$ and it is plain that $\overline{\varepsilon}\phi = \phi_P$ and $\phi\sigma_P = \overline{\sigma}$.

ϕ defines ϕ' by restriction, whence we have $N \xrightarrow{\phi'} K$ with $\overline{\mu}\phi' = \phi\mu'$ and $\phi' e' = \pi$. So $\phi'\tilde{e} = 1_K$. Then we get the split short exact sequence $B \rightarrowtail N \underset{\tilde{e}}{\overset{\phi'}{\rightleftarrows}} \overline{K}$, $B = ker\ \phi'$.

Use the proposition 6 [**3**] shows that

$$X_P = \bigcup_{i=0}^{\infty} \langle I_{P',i}(X_P, e_0(X)) \rangle.$$

Therefore, it is enough to show that our formula holds $\forall z \in \langle I_{P',i}(X_P, e_0(X)) \rangle$, $\forall i \geq 0$. We argue by induction on i.

$\langle I_{P',0}(X_P, e_0(X)) \rangle = e_0(X)$ and $\tilde{e}(\overline{\omega}(e_0(x)).\overline{a}) = \omega_P(e_0(x)).\tilde{e}(\overline{a})$ easily.

Now suppose $z \in I_{P',i}(X_P, e_0(X)) = I_{P',1}(X_P, \langle I_{P',i-1}(X_P, e_0(X)) \rangle)$.

Thus, $\exists\ n \in P'^x$ such that $z^n \in \langle I_{P',i-1}(X_P, e_0(X)) \rangle$, whence $\tilde{e}(\overline{\omega}(z^n).\overline{a}) = \omega_P(z^n).\tilde{e}(\overline{a})$ by inductive hypothesis.

Next we take $m = o(\overline{\omega}(z^n)) \in P^x$ by Prop. 1.2.

So, $\omega_P(z^{nm}).\tilde{e}(\overline{a}) = \tilde{e}(\overline{\omega}(z^n)^m.\overline{a}) = \tilde{e}(\overline{a})$, whence $\omega_P(z^{nm}) \mid_{\tilde{e}(\overline{K})} = 1_{\tilde{e}(\overline{K})}$.

On the other hand, taking $\tau = \phi' \ o \ \omega_P(z^m) \ o \ \tilde{e}$ we have $\tau = \overline{\omega}(z^m)$ and $o(\overline{\omega}(z^m)) \in P^x$ by Prop. 1.2 again.

But, $\omega(z^m)^n = 1_{\overline{K}} \Longrightarrow o(\overline{\omega}(z^m)) \mid n$, so $\overline{\omega}(z^m) = l_{\overline{K}}$.

So, $\phi'\omega_P(z^m)\tilde{e} = l_{\overline{K}}$. Now recalling that $B \rightarrowtail N \overset{\phi}{\underset{\leftarrow}{\twoheadrightarrow}} K$ splits we have $\omega_P(z^m).\tilde{e}(\overline{a}) = b.\tilde{e}(\overline{a}_1); b \in B; \overline{a}_1 \in \overline{K}$.

Use of the $\phi'.\omega_P(z^m).\tilde{e} = 1_K$ shows that $\overline{a}_1 = \overline{a}$. Therefore $\omega_P(z^m).\tilde{e}(\overline{a}) = b.\tilde{e}(\overline{a})$.

Applying successively $\omega_P(z^m)$ to the expression above we obtain: $\tilde{e}(\overline{a}) = \omega_P(z^m)^n.\tilde{e}(\overline{a}) = [(\omega_P(z^m)^{n-1}.b)\dots(\omega_P(z^m).b).b]\tilde{e}(\overline{a}) = (\overline{\theta}_n(z^m).b^{-1})^{-1}.\tilde{e}(\overline{a})$, (where $\overline{\theta}_n(z^m).u = u(\omega_P(z^m).u)\dots(\omega_P(z^m)^{n-1}.u)$).

Use of the proposition 1.11 together with the fact that X_P and G_P are P-local shows that $\overline{\theta}_n(z^m)$ is bijective, whence $b = 1$.

So, $\omega_P(z^m) \mid_{\tilde{e}(\overline{K})} = 1_{\tilde{e}(\overline{K})}$. Also, $gcd(m,n) = 1 \Longrightarrow \exists \ r,s \in \mathbb{Z}$ such that $rm + sn = 1$. Thus, $\omega_P(z).\tilde{e}(\overline{a}) = \omega_P(z^n)^s \ o \ \omega_P(z^m)^r.\tilde{e}(\overline{a}) = \omega_P(z^n)^s.\tilde{e}(\overline{a}) = \tilde{e}(\overline{\omega}(z^n)^s.\overline{a}) = \tilde{e}(\overline{\omega}(z^n)^s \ o \ \overline{\omega}(z^m)^r.\overline{a}) = \tilde{e}(\overline{\omega}(z).\overline{a}), \forall \overline{a} \in K, \forall z \in I_{P',i}(X_P, e_0(X))$.

Now it is easy to complete the argument. ∎

Next let $X \xrightarrow{\omega} Aut(A)$ be an action, where A is a finite abelian p-group ($p \in P$). Let $P_1 = \{q \in P' : q \mid |\omega(X)|\}$.

Take $\Gamma = \Gamma(H)$ as it was defined just after Prop. 1.22.

Thus, corollary 1.25 yields

$$\begin{array}{ccc} X & \xrightarrow{\overline{\omega}} & Aut(A/_\Gamma) \\ e_0 \downarrow & \nearrow_{\omega'} & \\ X_P & & \end{array}$$

Let $G = A \ \rceil \ _\omega X$ and $G' = {}^A/_\Gamma \ \rceil \ _{\omega} X_P$.

$$
\begin{tikzcd}
& \Gamma \arrow[d] \\
& A \arrow[r, tail, "\mu"] \arrow[d, "\pi"] \arrow[dd, bend right=60, "\bar e"'] & G \arrow[r, two heads, shift left, "\varepsilon"] \arrow[d, "e'"] \arrow[dd, bend right=60, "e"'] & X \arrow[l, tail, shift left, "\sigma"] \arrow[d, "e_0"] \arrow[dd, bend right=60, "e_0"'] \\
& A/\Gamma \arrow[r, tail, "\mu'"] \arrow[d, shift right, "\tilde e"'] & G' \arrow[r, two heads, shift left, "\varepsilon'"] & X_P \arrow[l, tail, shift left, "\sigma'"] \arrow[d, equal] \\
& N \arrow[r, tail, "\bar\mu"] \arrow[u, shift right, two heads, "\bar f"'] & G_P \arrow[u, "f"'] \arrow[r, two heads, shift left, "\varepsilon_P"] & X_P \arrow[l, tail, shift left, "\sigma_P"] \\
B \arrow[ur]
\end{tikzcd}
$$

Diagram 2.3

In the diagram π is the natural projection (therefore $\pi \in Hom_{\mathbb{Z}[X]}(A, A/\Gamma))$, $e'(a,x) = (\pi(a), e_0(x))$, therefore $e' \in Hom(G, G')$.

Thus $\exists\,!\ f \in Hom(G_P, G')$ such that $fe = e'$.

The other functions are defined as usual except for $\tilde{e}$, which is going to be defined just after the next lemma.

Lemma 2.4. $\overline{e} \mid \Gamma = 0$.

Proof: Let x be a generator of H and $m = o(\omega(x)) \in P_1^x$. Let $b = \omega(x).a - a; a \in \Gamma$.

Then, $(b,x)^m = (\theta_m(x).b, x^m) = ((\omega(x)^m - 1_A).a, x^m) = (0, x^m) = (0,x)^m$. Thus $e(b,x)^m = e(0,x)^m \Longrightarrow e(b,x) = e(0,x)$ for G_P is P-local. Therefore, $\overline{e}(b) = 1$. Now it is plain that $\overline{e}(\Gamma) = \{1\}$. ∎

Define $\tilde{e} \in Hom({}^{A}/_{\Gamma}, N)$ such that $\tilde{e}\pi = \overline{e}$. Then, $\overline{f}\tilde{e} = 1_{A/_\Gamma}$, whence $B \rightarrowtail N \underset{\tilde{e}}{\overset{\overline{f}}{\rightleftarrows}} {}^{A}/_{\Gamma}$ splits.

Theorem 2.5. *In the conditions above, $e' : G \longrightarrow G'$ P-localizes G.*

Proof: Set $\phi' : G' \longrightarrow G_P$, where $\phi'(a,z) = \overline{\mu}\tilde{e}(a).\sigma_P(z)$.

Owing to prop. 2.2, we have $\phi' \in Hom(G', G_P)$, and then it is readily checked that $\phi' = f^{-1}$. ∎

Finally, we analyse the situation in which A is (only) a finite abelian group.

Let $X \xrightarrow{\omega} Aut(A)$ be an action, where A is a finite abelian group.

If $|A| = p_1^{\beta_1} \dots p_t^{\beta_t}$, then

$$A = \bigoplus_{i=1}^{t} A_i,$$

$A_i = p_i$-primary component and

$$Aut(A) \cong \prod_{i=1}^{t} Aut(A_i).$$

Thus, there is (uniquely determined) $\omega_i : X \longrightarrow Aut(A_i); i = 1, \dots, t$.

Let $G = A \rtimes_{\omega} X; G_i = A_i \rtimes_{\omega_i} X; G \underset{\sigma}{\overset{\varepsilon}{\rightleftarrows}} X; G_i \underset{\sigma_i}{\overset{\varepsilon_i}{\rightleftarrows}} X$ as usual.

It is well-known that ε is the pull-back of $(\varepsilon_i)_{1 \le i \le t}$.

Let $G \xrightarrow{\pi_i} G_i$ be the projection and $\overline{G} \xrightarrow{\overline{\varepsilon}} X_P$ be the pull-back of the arrows $(G_i)_P \xrightarrow{(\varepsilon_i)_P} X_P; i = 1, \dots, t$.

Since $(\varepsilon_i)_P \; o \; (\sigma_i)_P = 1_{X_P}$, there does exist (only one) $\overline{\sigma} \in Hom(X_P, \overline{G})$ such that $\overline{\pi}_i \; o \; \overline{\sigma} = (\sigma_i)_P, \forall i$ (here $\overline{\pi}_i$ is the usual projection).

Likewise, it is plain that $\exists \, ! \, f \in Hom(G, \overline{G})$ such that $\overline{\pi}_i o f = e_i \pi_i (e_i : G_i \longrightarrow (G_i)_P$ P-localizes $G_i)$.

It follows that $\overline{\varepsilon} f = e_0 \varepsilon$ and $f\sigma = \overline{\sigma} e_0$. We recall that $\overline{G}$ is P-local by prop. 1.1.

Moreover, $\exists \, ! \, \phi \in Hom(G_P, \overline{G})$ such that $\overline{\pi_i} o \phi = (\pi_i)_P$, since $(\varepsilon_i)_P o (\pi_i)_P = \varepsilon_P, \forall i$ (In particular $\overline{\pi}_i$ is an isomorphism). By uniqueness we have got $f = \phi e; \overline{\varepsilon}\phi = \varepsilon_P; \phi\sigma_P = \sigma$ and $\overline{\varepsilon\sigma} = 1_{X_P}$ as well. (So $\overline{G} = C \rtimes X_P$).

Finally, let

$$C = ker \; \overline{\varepsilon} \cong \bigoplus_{i=1}^{t} ker(\varepsilon_i)_P;$$

$\overline{\mu} : C \longrightarrow \overline{G}, N = ker \; \varepsilon_P; \mu' : N \longrightarrow G_P, e, f, \phi$, define $\overline{e} : A \longrightarrow N, \overline{f} : A \longrightarrow C$ and $\overline{\phi} : N \longrightarrow C$ by restriction. Let $B = ker \; \phi$.

Soon we are going to show that $\exists \, ! \, e' \in Hom(C, N)$ such that $e'\overline{f} = \overline{e}$.

We are able, at last, to construct the following commutative diagram:

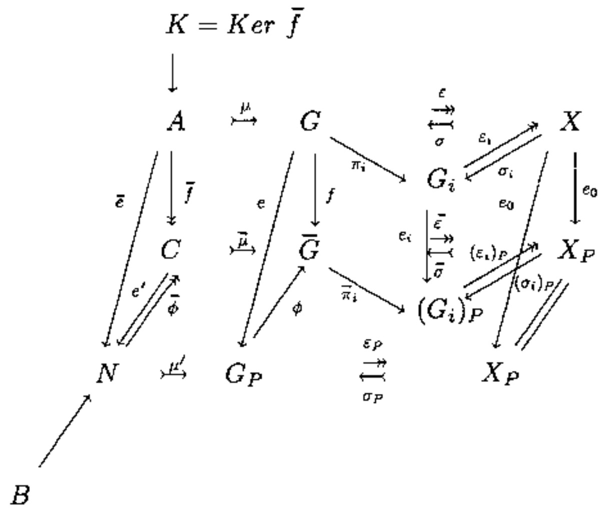

Diagram 2.6

In order to justify all the indications in the diagram, we still need two lemmas.

Lemma 2.7. $\overline{f}$ *is an epimorphism.*

Proof: This follows from the fact that

$$\overline{f} = \bigoplus_{i=1}^{t} e_i,$$

where

$$\begin{array}{ccccc} A & \overset{\mu_i}{\rightarrowtail} & G_i & \overset{\varepsilon_i}{\twoheadrightarrow} & X \\ \downarrow \overline{e_i} & & \downarrow e_i & & \downarrow e_0 \\ ker(\varepsilon_i)_P & \overset{\mu_i'}{\rightarrowtail} & (G_i)_P & \overset{(\varepsilon_i)_P}{\twoheadrightarrow} & X_P \end{array}$$

taken in the conjunction with the cases previously analysed. ■

Lemma 2.8. $\overline{e}\,|_K = 0.$

Proof: Since

$$K = \bigoplus_{i=1}^{t} K_i (K_i = ker\ \overline{e_i}),$$

we just have to show that $\overline{e}\mid_{K_i} = 0, \forall i$.

This follows from the diagram (ν_i is a splitting attached to π_i)

$$\begin{tikzcd}
 & & A \arrow[dr, "\mu"] & \\
K_i \arrow[r, tail] & A_i \arrow[ur] \arrow[r, tail, "\mu_i"] \arrow[d, "\overline{e_i}"] & G_i \arrow[r, two heads, "\nu_i"] \arrow[d, "e_i"] & G \arrow[d, "e"] \\
 & ker(\varepsilon_i)_P \arrow[r, tail, "\mu_i'"] & (G_i)_P \arrow[r, two heads, "(\nu_i)_P"] & G_P
\end{tikzcd}$$

So we have $e' \in Hom(C, N)$ with $e'f = \overline{e}$. ■

Lemma 2.9.

$$\left.\begin{array}{l} (i)\overline{f}(\omega(x).a) = \overline{\omega}(e_0(x)).\overline{f}(a) \\ (ii)\overline{e}(\omega(x).a) = \omega_P(e_0(x)).\overline{e}(a) \end{array}\right\} ; \forall x \in X; \forall a \in A$$

Proof: Both statements are readily checked from the definitions. ■

Theorem 2.10. *In the conditions above, $G \xrightarrow{f} \overline{G}$ P-localizes G.*

Proof: Let $\psi : \overline{G} \longrightarrow G_P$ defined by $\psi(\overline{\mu}(c).\overline{\sigma}(z)) = \mu' e'(c).\sigma_P(z); c \in C; z \in X_P$.

$\psi \in Hom(\overline{G}, G_P)$ by prop. 2.2, so that it is plain that $\psi = \phi^{-1}$. ■

3. P-localization on the category $\mathcal{C}$

Throughout this section we construct the theory of P-localization of a group in the category $\mathcal{C}$ of groups which are extensions of nilpotent groups by finite abelian groups. (althought we still use the same notation $G \xrightarrow{\varepsilon} G_P$ for P-localization in the category $\mathcal{C}$).

Proposition 3.1. *Let $A \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ be an exact sequence of groups, where A is abelian finite and X is nilpotent. Let $X \xrightarrow{\omega} Aut(A)$ be the action attached to the extension and suppose $\Gamma_\omega^2 = A$.*

Let also $\beta \in Hom(G, K)$ and $B \overset{\nu}{\rightarrowtail} K \overset{\kappa}{\twoheadrightarrow} Y$ be an exact sequence, where B is finite abelian and Y nilpotent. Then, there exist $\alpha \in Hom(A, B)$ and $\gamma \in Hom(X, Y)$ which yield commutativity in the diagram

$$\begin{tikzcd}
A \arrow[r, tail, "\mu"] \arrow[d, "\alpha"'] & G \arrow[r, two heads, "\varepsilon"] \arrow[d, "\beta"'] & X \arrow[d, "\gamma"] \\
B \arrow[r, tail, "\nu"] & K \arrow[r, two heads, "\kappa"] & Y
\end{tikzcd}$$

Proof: Let $H = \kappa\beta\mu(A) < Y$. It follows from $\Gamma^2_\omega = A$ that $H \subset [Y, H]$. So $H \subset [Y, H] \subset \Gamma^2 Y$ and then, by induction, $H \subset \Gamma^k Y, \forall k \geq 2$; whence $H = \{1\}$ since Y is nilpotent. This completes the proof. ■

Proposition 3.2. *$\forall G \in |\mathcal{C}|, \exists\,!\, U = U(G) \triangleleft G, U$ finite abelian with G/U nilpotent such that $\Gamma^2_\omega = U$, provided that ω is the action attached to the extension $U \rightarrowtail G \twoheadrightarrow G/U$.*

Proof: Let $A \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} X$ be an extension where A is finite abelian and X is nilpotent. Let $\Omega : X \longrightarrow Aut(A)$ be the action attached to this extension, and set $\Gamma = \Gamma^r_\Omega$, where r is the smallest positive integer such that $\Gamma^r_\Omega = \Gamma^{r+1}_\Omega$. Let $U = \mu(\Gamma) \triangleleft G$. So U is finite abelian. Furthermore, $A/\Gamma \rightarrowtail G/U \twoheadrightarrow X$ is exact and X acts nilpotently on A/Γ, so that G/U is nilpotent. It's also plain that $\Gamma^2_\omega = U$, if $\omega(gU)u = gug^{-1}$. Finally we point out that the uniqueness follows in a straightforward way from proposition 3.1. ■

Now let p be a prime and $\mathcal{C}_p$ be the full sub-category of $\mathcal{C}$ of all groups, which are extensions of X by A, where A is a finite abelian p-group.

Corollary 3.3. *$G \in |\mathcal{C}_p| \Rightarrow U = U(G)$ is a finite abelian p-group.*

Proof: In fact, $U = \mu(\Gamma)$ and Γ is a sub-group of A. ■

Corollary 3.4. *$G \in |\mathcal{C}|; G$ is nilpotent $\Longleftrightarrow U = U(G) = \{1\}$.*

Proof: $(\Longrightarrow) G \in |\,\eta\,| \Longrightarrow \omega : G/U \longrightarrow Aut(A)$ is nilpotent $\Longrightarrow U = \Gamma^2_\omega = \ldots = \Gamma^{c+1}_\omega = \{1\}.(c = nil\ \omega)$.

$(\Longleftarrow)$ It is obvious. ■

Corollary 3.5. *$G \in |\,\eta\,| \Longleftrightarrow \exists\ p, q$ primes, $p \neq q$, such that $G \in |\mathcal{C}_p| \cap |\mathcal{C}_q|$.*

Proof: It follows from cor. 3.3 and cor. 3.4. ■

We now define $G_P \in |\,\mathcal{C}\,|$ provided $G \in |\,\mathcal{C}\,|$. Fix $\xi : U \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} G/U$ where $U = U(G)$ is defined by prop. 3.2; $\omega(gU)u = gug^{-1}$ and $G/U \xrightarrow{e_0} (G/U)_P$ P-localizes G/U in η.

We consider 3 cases:

Let p be a prime and suppose firstly $G \in |\mathcal{C}_p|$.

I) $p \in P'$.

Set $e = e_0\ o\ \varepsilon, G \overset{\varepsilon}{\twoheadrightarrow} G/U \xrightarrow{\rho_0} (G/U)_P$.

Then we have:

$$\begin{array}{cccccc} \xi: & U & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & G/_U \\ & \pi\downarrow & & \downarrow e & & \downarrow e_0 \\ \xi_P: & (O) & \twoheadrightarrow & (G/_U)_P & = & (G/_U)_P \end{array}$$

We should point out that $\pi_*\xi = e_0^*\xi_P = 0$.

II) $p \in P$.

Let $P_1 = \{q \in P' : q \mid |\omega(G/U)|\}, H = < x \in G/U : o(\omega(x))P_1^x >, \Gamma = \Gamma(H)$ and $\overline{\omega} : G/U \longrightarrow Aut(U/\Gamma)$ as defined just after theorem 1.21. Corollary 1.25 allows us to claim that $\exists$! action ω_P making commutative the diagram

$$\begin{array}{ccc} G/_U & \longrightarrow & Aut(U/_\Gamma) \\ \downarrow & \nearrow_{\omega_P} & \\ (G/_U)_P & & \end{array}$$

Taking the natural projection $U \twoheadrightarrow U/\Gamma$, we have that $\exists$! ξ_P such that $e_0^*\xi_P = \pi_*\xi$ where

$$H^2_\omega(G/U;U) \xrightarrow{\pi^*} H^2_{\overline{\omega}}(G/U;U/\Gamma) \underset{\cong}{\xleftarrow{e_0^*}} H^2_{\omega_P}((G/U)_P;U/\Gamma)$$

Once more it is shown by prop. 1.15 that there is a commutative diagram

$$\begin{array}{cccccc} \xi: & U & \overset{\mu}{\rightarrowtail} & G & \overset{\epsilon}{\twoheadrightarrow} & G/_U \\ & \pi\downarrow & & \downarrow e & & \downarrow e_0 \\ \xi_P: & U/_\Gamma & \overset{\overline{\mu}}{\rightarrowtail} & G_P & \overset{\epsilon_P}{\twoheadrightarrow} & (G/_U)_P \end{array}$$

At this point it is important to point out that we have defined $G \in \bigcup_p | \mathcal{C} | \longrightarrow G_P \in | \mathcal{C} |$ and this definition is "good" since $G \in | \mathcal{C}_p | \cap | \mathcal{C}_q | \Longrightarrow G \in \eta$ (cor. 3.5) and then $U = \{1\}$ (cor. 3.4)

In particular, this construction extends the one made in [**2**].

Example 3.6. Let $\omega : \mathbb{Z} \longrightarrow Aut(\mathbb{Z}/3 \oplus \mathbb{Z}/5)$ given by $\omega(1).a = 2a$ and $\omega(1).b = 2b$. Let $G = (\mathbb{Z}/3 \oplus \mathbb{Z}/5) \rtimes_\omega \mathbb{Z}$.

Then $\Gamma^2_\omega = \mathbb{Z}/3 \oplus \mathbb{Z}/5 = A$, whence $G \notin | \eta |$. However $G \in | \mathcal{C} |$ and since $U = \mu(A)$, it follows that $G \notin \bigcup_p |\mathcal{C}_p|$.

This example shows that

$$|\mathcal{C}| \setminus \bigcup_p |\mathcal{C}_p| \neq \emptyset.$$

So we must consider

III) $G \in |\mathcal{C}| \setminus \bigcup_p |\mathcal{C}_p|$.

Now U is no longer a P-group. Neverthless,

$$U = \bigoplus_{i=1}^{t} U_i$$

where U_i is the p_i-primary component of U.

Also,

$$G/U \xrightarrow{\omega} Aut(U) \cong \prod_{i=1}^{t} Aut(U_i)$$

and $\omega = (\omega_1, \dots, \omega_t)$.

We have

$$H^2_\omega(G/U; U) \underset{(\pi_{1*}, \dots, \pi_{t*})}{\xrightarrow{\cong}} \bigoplus_{i=1}^{t} H^2_{\omega_i}(G/U; U_i);$$

where $U \overset{\pi_i}{\twoheadrightarrow} U_i$ is the usual projection. Notice that $\Gamma^2_\omega = U \longrightarrow \Gamma^2_{\omega_i} = U_i; \forall i = 1, \dots, t$. Let $\xi_i = \pi_{i*}\xi$ and consider the commutative diagram

$$\begin{array}{ccc}
\xi \in H^2(G/_U; U) & \underset{\cong}{\xrightarrow{(\pi_{1*}, \dots, \pi_{t*})}} & (\xi_i)_i \in \bigoplus_{i=1}^t H^2_{\omega_i}(G/_U; U_i) \\
e_0^{*-1} o^{\rho_*} \downarrow & & \downarrow (\bigoplus_{i=1}^t e_o^{*-1})(\bigoplus_{i=1}^t \rho_{i*}) \\
\xi_P \in H^2_{\omega_P}(G/_U)_P; \overline{U}) & \underset{\cong}{\xrightarrow{(\pi_{1*}, \dots, \pi_{t*})}} & (\xi_i)_P \in \bigoplus_{i=1}^t H^2_{(\omega_i)_P}((G/_U)_P; \overline{U}_i
\end{array}$$

where

$$\begin{array}{ccccccc}
\xi_i : & U_i & \overset{\mu_i}{\rightarrowtail} & G_i & \overset{\varepsilon_i}{\twoheadrightarrow} & G/_U \\
& \downarrow \rho_i & & \downarrow \rho_i & & \downarrow \rho_o \\
(\xi_i)P : & \overline{U}_i & \overset{\overline{\mu}_i}{\rightarrowtail} & (G_i)_P & \overset{(\varepsilon_i)_P}{\twoheadrightarrow} & (G/_U)_P
\end{array}$$

is defined by (I) or (II). Also,

$$U = \bigoplus_{i=1}^{t} U_i$$

and $\exists\, !\ \xi_P$ such that $(\overline{\pi_{1*}}, \dots, \overline{\pi_{t*}})\xi_P = ((\xi_i)_P)_i$ provided that $\overline{\pi_i}$ is the usual projection and $\rho = \oplus_i \rho_i$.

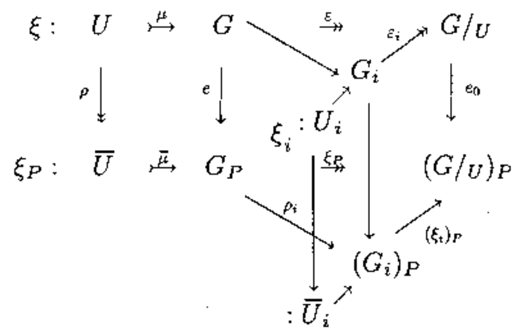

Diagram 3.7

By definition, we have that ξ_P is the pull-back of the arrows $\{(\xi_i)_P\}_i$ since $\xi_P = \rho_* \xi$.

As for $G \longrightarrow G_P$ defined by (I);(II);(III) we have the next two propositions.

Proposition 3.8. *e is P-surjective.*

Proof: It follows directly from the definitions. ■

Corollary 3.9. $G \xrightarrow{e} G_P \overset{f}{\underset{g}{\rightrightarrows}} K$, *with K P-local. Then $fe = ge \Longrightarrow f = g$.*

Proof: Obvious. ■

Proposition 3.10. *G P-local $\Longrightarrow$ e is an isomorphism.*

Proof: We have 3 cases to analyse. The only one which is not obvious is (II).

Suppose we have

$$\begin{array}{ccccc} U & \rightarrowtail & G & \twoheadrightarrow & G/_U \\ \pi \downarrow & & \downarrow e & & \downarrow e_0 \\ U/_\Gamma & \rightarrowtail & G_P & \twoheadrightarrow & (G/_U)_P \end{array}$$

G P-local $\Rightarrow G/U$ P-local (cor. 1.9). So $\omega(G/U)$ is a P-torsion sub-group of $Aut(U)$, since $Aut(U)$ is finite and G/U is P-local (cor. 1.3). Therefore $H = \{1\}$ and $\Gamma = \{1\}$. Therefore $\pi = 1_U$. So e is an isomorphism. ■

Next we consider a commutative diagram

$$\begin{array}{ccccccc} U(G) = & U & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & G/_U \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ U(K) = & V & \overset{\nu}{\rightarrowtail} & K & \overset{\kappa}{\twoheadrightarrow} & K/_V \end{array}$$

Let us define $\overline{\alpha} \in Hom(\overline{U}, \overline{V})$ induced by $(\overline{\alpha}\rho_U = \rho_V\alpha; U \overset{\rho_U}{\twoheadrightarrow} \overline{U})$.

We take $U = \oplus U(p)$ and $V = \oplus V(p)$; p-primary decompositions.

Proposition 1.26 assures that $\alpha(\Gamma_{U(p)}) \subset \Gamma_{V(p)}$, since $\alpha(U(p)) \subset V(p)$.

Actually, we consider

$$\begin{array}{cccccc} \xi(p): & U(p) & \rightarrowtail & G(p) & \twoheadrightarrow & G/_U \\ & \downarrow \alpha|_{U(p)} & & \downarrow & & \downarrow \\ \zeta(p): & V(p) & \rightarrowtail & K(p) & \twoheadrightarrow & K_V \end{array}$$

and then use prop. 1.26 to $\xi(p) = \pi(p) * \xi$ and $\zeta(p) = \pi(p) * \zeta$.

We define, by restriction, $\zeta(p) : U(p) \longrightarrow V(p)$, whence we have

$$\begin{array}{cccc} & U(p) & \overset{\alpha(p)}{\longrightarrow} & V(p) \\ & \rho_{U(p)} \downarrow & & \downarrow \rho_{V(p)} \\ \overline{U}(p) = & \overline{U}(p)/_{\Gamma_{U(p)}} & \overset{\overline{\alpha}(p)}{\longrightarrow} & V(p)/_{\Gamma_{V(p)}} \equiv \overline{V}(p) \end{array}$$

and finally $\overline{\alpha} = \Theta_p\overline{\alpha}(p)$.

At this point we state a fundamental proposition.

Proposition 3.11. *$\overline{\alpha}$ is an homomorphism of modules.*

Proof: Let us consider the comutative diagrams.

$$\begin{array}{ccc} G_U & \overset{\overline{\omega}}{\longrightarrow} & Aut\overline{U}) \\ e_0 \downarrow & \nearrow_{\omega_P} & \\ (G/_U)_P & & \end{array}$$

$$\begin{array}{ccc} K_V & \overset{\overline{\Omega}}{\longrightarrow} & Aut(\overline{\Omega}) \\ e_0 \downarrow & \nearrow_{\Omega_P} & \\ (K/_V)_P & & \end{array}$$

where ω_P and Ω_P are the actions given by the extensions

$$\begin{array}{cccccc} \xi_P: & U & \overset{\overline{\mu}}{\rightarrowtail} & G_P & \overset{\varepsilon_P}{\twoheadrightarrow} & (G/_U)_P \\ & \downarrow \overline{\alpha} & & & & \downarrow \gamma_P \\ \zeta^P: & \overline{V} & \overset{\overline{\nu}}{\rightarrowtail} & K_P & \overset{\kappa_P}{\twoheadrightarrow} & (K_V)_P \end{array}$$

We ougth to get that $\overline{\alpha}(\omega_P(z).\overline{a}) = \Omega_P(\gamma_P(z)).\overline{\alpha}(a), \forall z \in (G/U)_P$ and $\forall \overline{a} \in \overline{U}$.

Fix $z \in (G/U)_P$ and $\overline{a} \in \overline{U}$.

G/U nilpotent $\Rightarrow$ $\exists$ n $\in$ P'^x such that $z^n = e_0(x)$. Then, $\overline{\alpha}(\omega_P(z^n).\overline{a}) = \overline{\alpha}(\omega_P(e_0(x)).\overline{a}) = \overline{\alpha}(\overline{\omega}(x).\overline{a}) = \overline{\alpha}(\overline{\omega(x).a})$ (by definition) $= \rho_V\alpha(\omega(x).a) = \rho_V(\Omega(\gamma(x)).\alpha(a))$ (α is a homomorphism of modules) $= \overline{\Omega}(\gamma(x)).\overline{\alpha(a)} = \Omega_P(e_0(\gamma(x))).\overline{\alpha}(\overline{a}) = \Omega_P\gamma_P(z^n).\overline{\alpha}(\overline{a}) \ldots (*)$.

On the other hand $o(\omega_P(z)) = m \in P^x$, (prop. 1.2), since $(G/U)_P$ is P-local and $Aut(\overline{U})$ is finite.

So, $\Omega_P(\gamma_P(z^n)^m).\overline{\alpha}(\overline{a}) = \overline{\alpha}(\omega_P(z^n)^m.\overline{a}) = \overline{\alpha}(\overline{a}), \forall \overline{a} \in \overline{U}$. Therefore, $\Omega_P(\gamma_P(z^m)^n)\mid_{\overline{\alpha}(\overline{U})} = 1_{\overline{\alpha}(\overline{U})}$.

Still, taking into account that in the exact sequence $\zeta_P : \overline{V} \rightarrowtail K_P \twoheadrightarrow (K/V)_P$; K_P and $(K/V)_P$ are P-local we can state that $\tilde{\theta}_n(\gamma_P(z^m)) = 1_{\overline{V}} + \Omega_P(\gamma_P(z^m)) + \cdots + \Omega_P(\gamma_P(z^m)^{n-1}) \in Aut(\overline{V}), \forall n \in P'^x$ and $\gamma_P(z^m) \in (K/V)_P$.

Thus $\forall \overline{a} \in \overline{U}$ we have: $0 = \overline{\alpha}(\overline{a}) - \Omega_P(\gamma_P(z^m)^n).\overline{\alpha}(\overline{a}) = [1_{\overline{V}} - \Omega_P(\gamma_P(z^m))^n]\overline{\alpha}(\overline{a}) = \tilde{\theta}_n(\gamma_P(z^m))o[1_{\overline{V}} - \Omega_P(\gamma_P(z^m))].\overline{\alpha}(\overline{a})$. Therefore, $\overline{\alpha}(\overline{a}) - \Omega_P(\gamma_P(z^m)).\overline{\alpha}(\overline{a}) = 0$, whence $\Omega_P(\gamma_P(z^m))\mid_{\overline{\alpha}(\overline{U})} = 1_{\overline{\alpha}(\overline{U})}$.

Finally $\exists$ r, s $\in$ $\mathbb{Z}$ such that $rm + sn = 1$ $(gcd(m,n) = 1)$. Therefore $\overline{\alpha}(\omega_P(z).\overline{a}) = \overline{\alpha}(\omega_P(z^n)^s o\omega_P(z^m)^r.\overline{a}) = \overline{\alpha}(\omega_P(z^n)^s.\overline{a}) = \Omega_P(\gamma_P(z^n)^s).\overline{\alpha}(\overline{a}) = \Omega_P(\gamma_P(z^n)^s).\Omega_P(\gamma_P(z^m)^r).\overline{\alpha}(\overline{a}) = \Omega_P(\gamma_P(z)).\overline{\alpha}(\overline{a})$. ∎

Theorem 3.12. *$G, K \in |C|; \exists\, !\, \beta_P \in Hom(G_P, K_P)$ yelding commutativity in the diagram :*

$$\begin{tikzcd}
\xi : U \arrow[rr, tail, "\mu"] \arrow[dr, "\rho_U"] \arrow[dd, "\alpha"'] & & G \arrow[rr, two heads, "\varepsilon"] \arrow[dr] \arrow[dd, "\beta"] & & G/U \arrow[dr, "e_0"] \arrow[dd, "\gamma"] & \\
& \xi_P : \overline{U} \arrow[rr, tail, "\overline{\mu}"] & & G_P \arrow[rr, two heads, "\varepsilon_P"] & & (G/U)_P \arrow[dd, "\gamma_P"] \\
\zeta : V \arrow[rr, tail, "\nu"] \arrow[dr] & & K \arrow[rr, two heads, "\kappa"] \arrow[dr, "e"] & & K/V \arrow[dr, "e_0"] & \\
& \zeta_P : \overline{V} \arrow[rr, tail, "\overline{\nu}"] & & K_P \arrow[rr, two heads, "\kappa_P"] & & (K/V)_P
\end{tikzcd}$$

Proof: The uniqueness follows from the corollary 3.9:

For the existence we observe that $e_0^*\gamma_P^*\zeta_P = \gamma^* e_0^*\zeta_P = \gamma^*\rho_{V*}\zeta$ (definition of ζ_P) $= \rho_{V*}\gamma^*\zeta = \rho_{V*}\alpha_*(\xi)$ (prop. 1.16) $= \overline{\alpha} * \rho_{U*}\xi = \overline{\alpha} * e_0^*\xi_P$ (def. of ξ_P) $= e_0^*\overline{\alpha}_*\xi_P$.

It follows that $\gamma_P^*\zeta_P = \overline{\alpha_*}\xi_P$ due to the fact that $H^2((G/U)_P; \overline{V}) \xrightarrow[\cong]{e_0^*}$

$H^2(G/U;\overline{V})$ (Th. 1.21).

So by proposition 1.16, $\exists\ \tau \in Hom(G_P, K_P)$ yielding commutativity in the "front face" of the diagram.

Thus τe and $e\beta$ make commutative the diagram

$$\begin{array}{ccccc} U & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & G/_U \\ \rho_V o\alpha \downarrow & & \tau e \downarrow\downarrow e\beta & & \downarrow e_0 \\ \overline{V} & \overset{\overline{\nu}}{\rightarrowtail} & K_P & \overset{\kappa_P}{\twoheadrightarrow} & (K/_V)_P \end{array}$$

Use of the proposition 1.17 shows that $\theta : G/U \longrightarrow \overline{V}$ a cross homomorphism such that $e\beta(g) = \overline{\nu}\theta\varepsilon(g).\tau e(g), \forall g \in G$. However, $H^1((G/U)_P;\overline{V}) \xrightarrow[\cong]{e_0^\bullet} H^1(G/U;V)$ (Th. 1.21).

So $\theta = \theta'_P e_0 + \delta'_v$, where $\delta'_v(x) = v - x.v, v \in V$.

Setting $\delta_v : (G/U)_P \longrightarrow \overline{V}, \delta_V(z) = v - z.v$, it follows that $\delta_v\ o\ e_0 = \delta'_v$ and therefore $\theta = \theta_P\ o\ e_0$, where $\theta_P = \theta'_P + \delta_v$.

Now $e\beta(g) = \overline{\nu}\theta_P e_0\varepsilon(g).\tau e(g) = \overline{\nu}\theta_P\varepsilon_P e(g).\tau e(g), \forall g \in G$.

Thus defining $\beta_P : G_P \longrightarrow K_P$ by $\beta_P(z) = \overline{\nu}\theta_P\varepsilon_P(z).\tau(z), \forall z \in G_P$, it follows from prop. 1.18 that $\beta_P \in Hom(G_P, K_P)$ and $\beta_P e = e\beta$.

Besides, $\kappa_P\beta_P = \gamma_P\varepsilon_P$ and $\beta_P\overline{\mu} = \overline{\nu\alpha}$. ■

Remark. The theorem above shows us that $G \longrightarrow G_P$ is a functor and e is a natural transformation of functors.

Theorem 3.13. $G \xrightarrow{e} G_P$ *P-localizes G in $\mathcal{C}$.*

Proof: Let $G, K \in |\mathcal{C}|$, with K P-local, and $\beta \in Hom(G, K)$.

Owing to proposition 3.1, there exists a commutative diagram

$$\begin{array}{cccccc} U(G) = & U & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & G/_U \\ & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma \\ U(K) = & V & \overset{\nu}{\rightarrowtail} & K & \overset{\kappa}{\twoheadrightarrow} & K/_V \end{array}$$

Now using th. 3.12 we conclude that $\exists\ !\ \beta_P \in Hom(G_P, K_P)$ such that $\beta_P e = e\beta$.

So it is enough to take $\overline{\beta} = e^{-1}\ o\ \beta_P$

$$\begin{tikzcd} G \arrow[r, "\beta"] \arrow[d, "e"'] & K \arrow[d, "\cong", "e"'] \\ G_P \arrow[r, "\beta_P"'] \arrow[ur, dashed, "\overline{\beta}"] & K_P \end{tikzcd}$$

(prop. 3.10)

The uniqueness follows from cor. 3.9. ■

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