Publicacions Matemàtiques, Vol 37 (1993), 245–253.

SHADOWING FOR LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract

For a system of linear ordinary differential equations with constant coefficients a simple proof is given that hyperbolicity is equivalent to shadowing.

1. The notion of shadowing or the pseudo orbit tracing property (abbr.POTP) usually appears if one considers a dynamical system on a compact manifold. The famous Shadowing Lemma says, roughly speaking, that hyperbolicity implies the POTP. There are a number of proofs of this result: all of them rather complicated and tedious. In every case the compactness is essential. Morimoto, however, in **[4]** considered this property in $\mathbf{R}^n$ for discrete dynamical systems generated by linear homeomorphisms. He and Kakubari in **[3]** proved that hyperbolicity is equivalent to the POTP for such systems. In **[5]** we give a different proof which covers also infinite dimensional case. In this note we show the analogous statement for systems of linear ordinary differential equations with constant coefficients. A proof that hyperbolicity implies shadowing established for discrete case in **[4]** may be transformed to continuous case, **[7]**, yet we give here a different proof which is simpler and works also in discrete case. A proof of the converse statement mimics the discrete version from **[5]**. The concept of the POTP comes from Anosov and Bowen. For dynamical systems with continuous time it was examined by Franke and Selgrade in **[1]** and by Thomas in papers **[8]** **[9]** and others, see Thomas' papers for more details. For such systems the common definition of the POTP is as follows. Every δ-pseudo-orbit with sufficiently small $\delta > 0$ can be arbitrarily close uniformly approximated by a true orbit after some reparametrization of time on the true orbit. What we are going to show is that for a system of differential equations

*Supported by Polish scientific grant RP. I. 10.

of the form $x' = A \cdot x$, A is $n \times n$ matrix, the POTP is equivalent to hyperbolicity of the system, see below for a definition. Besides, we show that the POTP can be replaced by conditions which are slightly different from the original definition of the POTP, yet the above equivalence will still hold true.

2. Let (X, d) be a metric space and $\phi : X \times \mathbf{R} \to X$ be a flow i.e. ϕ is continuous, $\phi(x, 0) = x$, $\phi(\phi(x, t), s) = \phi(x, t + s)$ for every $x \in X$, t, $s \in \mathbf{R}$. An orbit of a point $x \in X$ is a set $\{\phi(x, t) : t \in \mathbf{R}\}$.

Let $\tau > 0$ and $\delta > 0$. A pair of sequences $(\{x^{\infty}_{n=-\infty}\}, \{t^{\infty}_{n=-\infty}\})$, $x_n \in X$, $t_n \in \mathbf{R}$, $t_n \geq \tau$, for all $n \in \mathbf{Z}$, is said to be a (δ, τ)-pseudo-orbit if for all $n \in \mathbf{Z}$

$$d(\phi(x_n, t_n), x_{n+1}) \leq \delta. \tag{1}$$

For a given (δ, τ)-pseudo-orbit we denote by $x_0 * t$ the point which is t units from x_0 along the pseudo-orbit. More precisely,

$$x_0 * t = \begin{cases} \phi(x_n, t - s_n), & \text{for } s_n \leq t < s_{n+1}, \quad \text{where } t \geq 0, \\ \phi(x_n, t + s_n), & \text{for } -s_n \leq t < -s_{n+1}, \quad \text{where } t \leq 0, \end{cases}$$

where $s_0 = 0$, $s_n = \sum_{i=0}^{n-1} t_i$, for $n = 1, 2, 3 \dots, s_n = \sum_{i=n}^{-1} t_i$, for $n = -1, -2, -3, \dots$.

A (δ, τ)-pseudo-orbit is ϵ-traced, $\epsilon > 0$ is given, by the orbit of a point x if there is a reparametrization of time i.e. an increasing homeomorphism $h : \mathbf{R} \to \mathbf{R}$, $h(0) = 0$, such that

$$d(\phi(x, h(t)), x_0 * t) \leq \epsilon, \text{ for all } t \in \mathbf{R}. \tag{2}$$

We say that the flow ϕ has the pseudo-orbits tracing property with respect to τ (POTP(τ)) if for every $\epsilon > 0$ there exists $\delta > 0$ such that any (δ, τ)-pseudo-orbit is ϵ-traced by some orbit. The flow has the POTP if it has the POTP(τ) for all $\tau > 0$.

The above definition was established by Franke and Selgrade and then used by many authors, see for example [6], [8] and references there in. Yet, for flows on $\mathbf{R}^n$ we will see that this definition may be weakened or strengthen in various ways and the new conditions such obtained are still equivalent to the original definition of the POTP.

We say that the flow has the strong POTP(τ) (SPOTP(τ)) if in the above definition of the POTP(τ) we take $h(t) = t$ for all $t \in \mathbf{R}$. We say that the flow has the normal POTP(τ) (NPOTP(τ)) if in the above definition of the POTP(τ) we may restrict ourselves to (δ, τ)-pseudo-orbits having all $t_n = \tau$. We say that the flow has the NSPOTP(τ) if it has both SPOTP(τ) and NPOTP(τ). At last, the flow has SPOTP, NPOTP or NSPOTP if it has the corresponding property with any $\tau > 0$.

A semi-orbit of a point $x \in X$ is a set $\{\phi(x,t) : t \geq 0\}$. A semi-(δ,τ)-pseudo-orbit is a pair of sequences $(\{x_{n=0}^{\infty}\}, \{t_{n=0}^{\infty}\})$, $x_n \in X$, $t_n \geq \tau$, such that (1) holds for all $n \in \mathbf{N}$. Now, any of the above definition may be reformulated in terms of semi-orbits and semi-pseudo-orbits. Corresponding concepts thus obtained will be denoted by semi-POTP(τ), semi-POTP, etc.

Let us note that any semi-(δ,τ)-pseudo-orbit may be extended to a (δ,τ)-pseudo-orbit by putting $x_n = \phi(x_0, n\tau)$ and $t_n = \tau$ for all $n = -1, -2, -3, \dots$. Hence any mutation of the POTP defined above implies the corresponding semi-property.

Let us note that in the definition of the POTP(τ) (and in all other definitions) we may assume that for all $n \in \mathbf{Z}$ we have $t_n \leq 2\tau$. In fact, if we have $t_n > 2\tau$ for some n then there is $k > 2$ such that $k\tau \leq t_n < (k+1)\tau$. We modify the (δ,τ)-pseudo-orbit by inserting between points x_n and x_{n+1} points $x_{ni} = \phi(x_n, i\tau)$, where $i = 0, \dots, k-1$ and by putting numbers $t_{ni} = \tau$ for $i = 0, \dots, k-2$, $t_{n(k-1)} = t_n - (k-1)\tau$ in place of t_n. It is clear that after such modifications the new (δ,τ)-pseudo-orbit shows the same $x_0 * t$ for all $t \in \mathbf{R}$ but now all $t_n \leq 2\tau$.

We also remark here that the above definitions do not depend on a particular metric used but rather on the uniform structure on the space X.

We consider a system of ordinary linear differential equations with constant coefficients

$$x' = A \cdot x \tag{3}$$

and its flow $\phi(x,t) = \exp(tA) \cdot x$, where A is certain $n \times n$ matrix. The system (or its flow) is said to be hyperbolic if all eigenvalues of the matrix A have non-zero real parts.

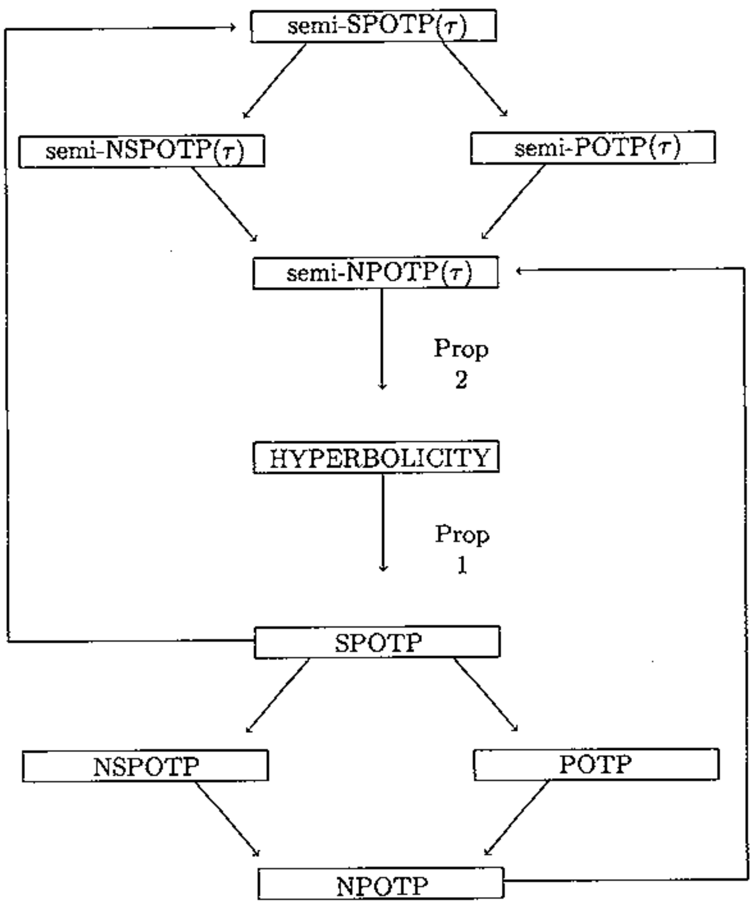
Our main results are established in the following two propositions and, in more complete form, as the theorem.

Proposition 1. *If system (3) is hyperbolic, then its flow has the SPOTP.*

Proposition 2. *If the flow of system (3) has the semi-NPOTP(τ) for some $\tau > 0$, then the system is hyperbolic.*

Theorem. *For system (3) all definitions of the various types of the pseudo-orbits tracing property stated above are equivalent to each other and any of them is equivalent to hyperbolicity.*

The proofs of the propositions will be presented in the next section. The propositions easily imply the theorem, and a proof of the theorem is shown at the following figure. Let $\tau > 0$ be fixed. Then, all implications pointed out at the diagram are obvious.



3. In order to prove Propositions 1 and 2 we will need the three following lemmas. The first two have straightforward proofs.

Lemma 1. *Let (X_i, d_i), $i = 1,2$ be metric spaces and ϕ_i flows on X_i. Let $X = X_1 \times X_2$ be equipped with a metric compatible with the uniform product structure. Let ϕ be the product flow on X i.e. $\phi((x_1, x_2), t) = (\phi_1(x_1,t), \phi_2(x_2,t))$.*

(i) *If ϕ_1 and ϕ_2 have the SPOTP(τ) then ϕ does.*

(ii) *If ϕ has the semi-NPOTP(τ) then both ϕ_1 and ϕ_2 do.*

Lemma 2. *Let ϕ be a flow on a metric space X satisfying the following condition: For every $T > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that: $d(x,y) \leq \delta$ and $|t| \leq T$ imply $d(\phi(x,t), \phi(y,t)) \leq \epsilon$. (This condition is satisfied by the flow of system (3)). Then, if ϕ has the SPOTP(τ) then the reverse flow ψ, $\psi(x,t) = \phi(x,-t)$, has the same property.*

Lemma 3. *Let ϕ be a flow on $\mathbf{R}^n$. If ϕ has the semi-SPOTP(τ) then it has the SPOTP(τ).*

Proof: Let $\delta > 0$ be chosen to a given $\epsilon > 0$ by the semi-SPOTP(τ). Let $(\{x^{\infty}_{n=-\infty}\}, \{t^{\infty}_{n=-\infty}\})$ be a (δ,τ)-pseudo-orbit. Then for each $k \in \mathbf{N}$ $(\{x^{\infty}_{n=-k}\}, \{t^{\infty}_{n=-k}\})$ is a semi-(δ,τ)-pseudo-orbit starting from the point x_{-k}. So, there exists points $y_{-k} \in \mathbf{R}^n$ such that

$$d(\phi(y_{-k}, t), x_{-k} * t) \leq \epsilon, \text{ for all } t > 0.$$

It follows that the points $z_k = \phi(y_{-k}, s_{-k})$ belong to the ball $B(x_0, \epsilon)$. By the compactness of the closed ball we get a point $x \in \mathbf{R}^n$ and a sequence $k_i \to \infty$ such that $z_{k_i} \to x$. This point ϵ-traces the above (δ,τ)-pseudo-orbit. For it, fix $t \in \mathbf{R}$ and consider such $k_i's$ that $-s_{-k_i} \leq t$ (it is so for almost all $k_i's$ because $t_i \geq \tau$). We have:

$$\begin{aligned} d(\phi(z_{k_i}, t), x_0 * t) &= d(\phi(\phi(y_{-k_i}, s_{-k_i}), t), x_0 * t) = \\ &= d(\phi(y_{-k_i}, s_{-k_i} + t), x_{-k_i} * (s_{-k_i} + t)) \leq \epsilon. \end{aligned}$$

Letting $k_i \to \infty$ we have $d(\phi(x,t), x_0 * t) \leq \epsilon$. ■

Proof of Proposition 1: Fix $\tau > 0$. First we show that any system of the form (3) with a matrix A which all eigenvalues have negative real parts shares the semi-SPOTP(τ). Then, by Lemma 3 such a system has the SPOTP(τ). Hence, any system of the form (3) with a matrix A which all eigenvalues have positive real parts has, by Lemma 2, the SPOTP (τ). Now, any hyperbolic system is, by the Jordan Decomposition, a product of two systems; one having all eigenvalues with positive and the other with negative real parts. Lemma 1(i) will complete the proof.

So we assume that all eigenvalues of the matrix A have negative real parts. It is known, see for example [**2**], that there are a norm on $\mathbf{R}^n$ and a constant $c > 0$ such that

$$\|\exp(tA)\cdot x\| \leq e^{-ct}\cdot\|x\|, \text{ for all } t \geq 0 \text{ and } x \in \mathbf{R}^n.$$

Fix $\epsilon > 0$ and let $\delta > 0$ be small enough to be determined later. Let $(\{x_{n=0}^{\infty}\}, \{t_{n=0}^{\infty}\})$ be a semi-(δ,τ)-pseudo-orbit. We show that this semi-orbit is ϵ'-traced by the point x_0. Recall that $s_0 = 0$, $s_n = \sum_{i=0}^{n-1} t_i$ for $n = 1, 2, 3, \ldots$.

First we have:

$$\begin{aligned}\|\phi(x_0, s_{n+1}) - x_{n+1}\| &\leq \|\phi(x_0, s_{n+1}) - \phi(x_n, t_n)\| + \|\phi(x_n, t_n) - x_{n+1}\| \leq \\ &\leq \|\phi(\phi(x_0, s_n), t_n) - \phi(x_n, t_n)\| + \delta \leq \\ &\leq \|\exp(t_n A)\cdot(\phi(x_0, s_n) - x_n)\| + \delta \leq \\ &\leq e^{-c\tau}\|\phi(x_0, s_n) - x_n\| + \delta.\end{aligned}$$

Hence by induction

$$\|\phi(x_0, s_n) - x_n\| \leq \sum_{i=0}^{n-1}\delta e^{-ci\tau} \leq \frac{\delta}{1 - e^{-c\tau}} \leq \epsilon'$$

for any ϵ', if δ is sufficiently small.

Fix $t > 0$. There exists $n \in \mathbf{N}$ with $s_n \leq t < s_{n+1}$. By the remark that we made after the definitions of the POTP we may assume: $s_{n+1} - s_n \leq 2\tau$. We have:

$$\begin{aligned}\|\phi(x_0, t) - x_0 * t\| &= \|\phi(\phi(x_0, s_n), t - s_n) - \phi(x_n, t - s_n)\| = \\ &= \|\exp(t - s_n)A\cdot(\exp(s_n A)\cdot x_0 - x_n)\| \leq \\ &e^{(t-s_n)\|A\|}\cdot\|\phi(x_0, s_n) - x_n\| \leq e^{2\tau\|A\|}\cdot\epsilon' \leq \epsilon\end{aligned}$$

for small $\epsilon' > 0$. The proof is complete. ∎

Proof of Proposition 2: Assume that the flow of system (3) has the semi-NPOTP(τ) with some positive τ. Assume that the matrix A is not hyperbolic and let $\lambda = i\beta$, $\beta \in \mathbf{R}$, be an eigenvalue of A with zero real part. Let B be the real Jordan block of A corresponding to λ. By Lemma 1(ii) the flow of the system $x' = B\cdot x$ has the NPOTP(τ).

Matrix B may have one of the following forms: $B = 0$,

$$B = \begin{pmatrix} 000 & \ldots & 0 \\ 010 & \ldots & 0 \\ 001 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 010 \end{pmatrix}, \; B = C, \; B = \begin{pmatrix} C0 & \ldots & 0 \\ IC0 & \ldots & 0 \\ 0IC & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0IC \end{pmatrix},$$

where

$$C = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \text{ and } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our goal is to construct a semi-(δ, τ)-pseudo-orbit, with $t_n = \tau$, for any small δ which is not traced by any point. It will complete the proof. In fact, we construct such a semi-(δ, τ)-pseudo-orbit in all the above cases yet a proof is presented only for the last, more interesting one.

In the case $B = 0$ the flow acts on some one-dimensional subspace which can be identified with the real axis $\mathbf{R}$. We define a semi-(δ, τ)-pseudo-orbit by $x_n = n\delta$, $t_n = \tau$ for $n \in \mathbf{N}$. In the second case the flow acts on certain k-dimensional subspace, say $\mathbf{R}^k$. We define a semi-(δ, τ)-pseudo-orbit as $x_0 = 0 \in \mathbf{R}^k$ and

$$x_{n+1} = \begin{pmatrix} (n+1)\delta \\ \phi(x_n)_{(2)} \\ \vdots \\ \phi(x_n)_{(k)} \end{pmatrix}.$$

Here, $a_{(i)}$ denotes the i-th coordinate of a vector a from $\mathbf{R}^k$. In the third case the flow acts on a plane, say $\mathbf{R}^2$. We define a semi-(δ, τ)-pseudo-orbit by

$$x_n = \begin{pmatrix} n\delta \cdot \cos(n\tau\beta) \\ n\delta \cdot \sin(n\tau\beta) \end{pmatrix}, \quad t_n = \tau.$$

In the last case the flow acts on some $2k$-dimensional subspace, say $\mathbf{R}^{2k}$. A semi-(δ, τ)-pseudo-orbit is defined by: $x_0 = 0 \in \mathbf{R}^{2k}$,

$$x_{n+1} = \begin{pmatrix} (n+1)\delta \cdot \cos(n+1)\tau\beta \\ (n+1)\delta \cdot \sin(n+1)\tau\beta \\ \phi(x_n)_{(3)} \\ \vdots \\ \phi(x_n)_{(2k)} \end{pmatrix}, \quad t_n = \tau.$$

To see it is in fact a semi-(δ, τ)-pseudo-orbit note that:

$$\exp(tB) = \begin{pmatrix} R_t & 0 & \dots & 0 \\ & R_t & 0 & \\ \vdots & & & 0 \\ \vdots & \vdots & & R_t \end{pmatrix}, \text{ where } R_t = \begin{pmatrix} \cos t\beta & -\sin t\beta \\ \sin t\beta & \cos t\beta \end{pmatrix}.$$

Hence

$$\begin{pmatrix} (\exp(tB) \cdot x)_{(1)} \\ (\exp(tB) \cdot x)_{(2)} \end{pmatrix} = R_t \cdot \begin{pmatrix} x_{(1)} \\ x_{(2)} \end{pmatrix}.$$

Hence

$$
\begin{aligned}
&d(\phi(x_n,t_n),x_{n+1}) = \|\phi(x_n,\tau) - x_{n+1}\| = \\
&= \left\| R_t \cdot \begin{pmatrix} (x_n)_{(1)} \\ (x_n)_{(2)} \end{pmatrix} - \begin{pmatrix} (x_{n+1})_{(1)} \\ (x_{n+1})_{(2)} \end{pmatrix} \right\| = \delta \left\| \begin{pmatrix} \cos(n+1)\tau\beta \\ \sin(n+1)\tau\beta \end{pmatrix} \right\| = \delta.
\end{aligned}
$$

This semi-(δ,τ)-pseudo-orbit is not traced by any point of $\mathbf{R}^{2k}$. In fact, given $x \in \mathbf{R}^{2k}$ and an increasing homeomorphism $h : \mathbf{R} \to \mathbf{R}$, $h(0) = 0$, we have:

$$
\begin{aligned}
&d(\phi(x,h(n\tau)), x_0 * n\tau) = \|\exp(h(n\tau)B)\cdot x - x_n\| \geq \\
&\geq \left\| \begin{pmatrix} (\exp(h(n\tau)B)\cdot x)_{(1)} \\ (\exp(h(n\tau)B)\cdot x)_{(2)} \end{pmatrix} - \begin{pmatrix} (x_{n+1})_{(1)} \\ (x_{n+1})_{(2)} \end{pmatrix} \right\| \geq \left| \left\| \begin{pmatrix} x_{(1)} \\ x_{(2)} \end{pmatrix} \right\| - n\delta \right| \to \infty,
\end{aligned}
$$

as $n \to \infty$.

One could ask why we did not use the complexification method as it was done in [**5**] for the discrete case. The reason is that the complexification method would require that the converse statement to that in Lemma 1(ii) holds true and this is not so obvious. ■

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Rebut el 17 de Gener de 1992