

DIFFEOMORPHISMS OF \mathbb{R}^n WITH OSCILLATORY JACOBIANS

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Abstract

The paper presents, mainly, two results: a new proof of the spectral properties of oscillatory matrices and a transversality theorem for diffeomorphisms of \mathbb{R}^n with oscillatory jacobian at every point and such that $N_M(f(x) - f(y)) \leq N_M(x - y)$ for all $x, y \in \mathbb{R}^n$, where $N_M(x) - 1$ denotes the maximum number of sign changes in the components z_i of $z \in \mathbb{R}^n$, where all z_i are non zero and z varies in a small neighborhood of x . An application to a semi-implicit discretization of the scalar heat equation with Dirichlet boundary conditions is also made.

I. Introduction

The present paper deals with diffeomorphisms $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f'(x)$ is an oscillatory matrix for all $x \in \mathbb{R}^n$. Oscillatory matrices were studied extensively by Gantmacher and Krein (see [G Kr]) and present very interesting spectral properties. They belong to the class of matrices that are variation-diminishing, i.e., transformations that decrease the number of sign changes in the components of a vector. These matrices have applications in mechanical systems, in approximation theory and in probability. Many results about oscillatory matrices as well as historical notes and a long list of references appear in the book “*Total Positivity*”, by S. Karlin (see [Ka]).

In Section II we decided to present another proof of the spectral properties of oscillatory matrices, summarized in Theorem 2.12; our approach is based on some ideas appearing in [FO-1] and [FO-2], where other kind of matrices were also studied. Theorem 2.13 shows that diffeomorphisms f with oscillatory jacobians satisfying an additional hypothesis have the

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property that given two hyperbolic fixed points, the corresponding unstable and stable manifolds are transversal. The proof follows some techniques and, as a matter of fact, was motivated by the main result of [FO-1] where it is studied a class of ordinary differential equations whose flow map at a fixed time is a diffeomorphism with oscillatory jacobian.

In Section III, after a semi-implicit double discretization of the heat equation with Dirichlet boundary condition, we constructed a class of Morse-Smale diffeomorphisms of \mathbf{R}^n with oscillatory jacobians.

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II. Definitions. Basic results. Main Theorem

All the matrices we will use are real.

2.1. Notation.

For a $m \times n$ matrix A , let's use the notations

$$A \begin{pmatrix} i_1 & i_2 & \dots & i_\nu \\ \kappa_1 & \kappa_2 & \dots & \kappa_\nu \end{pmatrix} = \det \begin{bmatrix} a_{i_1 \kappa_1} & a_{i_1 \kappa_2} & \dots & a_{i_1 \kappa_\nu} \\ a_{i_2 \kappa_1} & a_{i_2 \kappa_2} & \dots & a_{i_2 \kappa_\nu} \\ \vdots & \vdots & & \vdots \\ a_{i_\nu \kappa_1} & a_{i_\nu \kappa_2} & \dots & a_{i_\nu \kappa_\nu} \end{bmatrix}$$

$$1 \leq \nu \leq \min(m, n); \quad 1 \leq \begin{matrix} i_1 < i_2 < \dots < i_\nu \leq m \\ \kappa_1 < \kappa_2 < \dots < \kappa_\nu \leq n. \end{matrix}$$

2.2. Definition.

A $m \times n$ matrix A is called *totally positive* (*strictly totally positive*) if for all $1 \leq \nu \leq \min(m, n)$,

$$A \begin{pmatrix} i_1 & i_2 & \dots & i_\nu \\ \kappa_1 & \kappa_2 & \dots & \kappa_\nu \end{pmatrix} \geq 0 \quad (\text{resp. } > 0)$$

i.e., if all its minors are non-negative (resp. positive).

A square matrix A is called *oscillatory* if it is totally positive and if some of its power A^χ (χ a positive integer) is a strictly totally positive matrix.

2.3. The product C of two totally positive matrices A and B is a totally positive matrix.

The product C of two square matrices A and B , where one is strictly totally positive and the other is totally positive and non-singular is a strictly totally positive matrix.

2.4. The product $C = AB$ of two oscillatory matrices is an oscillatory matrix.

The product $A = A_1 A_2 \dots A_m$ of $m \geq n-1$ $n \times n$ oscillatory matrices is a strictly totally positive matrix.

Let A be a totally positive square matrix. Then, A is oscillatory if, and only if, A is non-singular and $a_{ij} > 0$ for $|i-j| = 1$.

2.5. For a matrix A , let A^* denote the matrix defined by $a_{ij}^* = (-1)^{i+j} a_{ij}$. Then, if A is an oscillatory matrix, the same is true for $(A^*)^{-1}$.

2.6. Let

$$J = \begin{pmatrix} a_1 & b_1 & & 0 \\ c_1 & a_2 & b_2 & \\ & & & b_{n-1} \\ 0 & & c_{n-1} & a_n \end{pmatrix}$$

be an $n \times n$ Jacobi matrix. Then, J is oscillatory if, and only if, the minors $J \begin{pmatrix} 1 & 2 & \dots & \nu \\ 1 & 2 & \dots & \nu \end{pmatrix}$, $1 \leq \nu \leq n$, a_j , $1 \leq j \leq n$, and the coefficients b_i and c_i , $1 \leq i \leq n-1$, are positive (> 0).

Following Fusco and Oliva, let us define now the discrete functional already considered in [FO-1]. (See also [K-L-S]).

For $x = (x^1, \dots, x^n) \in \mathbf{R}^n$ such that $x^i \neq 0$, $1 \leq i \leq n$, let $N(x) = 1$ be the number of sign changes in the sequence x^1, x^2, \dots, x^n . For an arbitrary $x \in \mathbf{R}^n$, define $N_m(x)$ and $N_M(x)$ as the minimum and maximum value of $N(x')$, for x' varying in a small neighbourhood of x with $(x')^i \neq 0$, $1 \leq i \leq n$. Extend N to the set $\mathcal{N} = \{x \in \mathbf{R}^n : N_m(x) = N_M(x)\}$ making $N(x) = N_m(x) = N_M(x)$ for all $x \in \mathcal{N}$. N is the functional used in [FO-1].

The next two results can be found in [Ka], and for completeness we reproduce the proofs.

2.7. Proposition. *Let A be an $n \times m$ matrix ($m < n$) such that all the minors $A \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ 1 & 2 & \dots & m \end{pmatrix}$ are nonzero and have the same sign, and let $x \in \mathbf{R}^m$, $x \neq 0$. Then $y = Ax$ satisfies $N_M(y) \leq m$.*

Proof:

Suppose, arguing by contradiction, that there is a nonzero vector $x \in \mathbf{R}^m$ such that $N_M(y) \geq m+1$ ($y = Ax$). Let $y = (y^1, y^2, \dots, y^n)$. Then there exist indices $i_1 < i_2 < \dots < i_{m+1}$ with the property that for $\nu = 1, 2, \dots, m+1$ the quantity $(-1)^\nu y^{i_\nu}$ is of constant sign, even in

the possible case where several of these numbers are zero. Now some of the $y^{i\nu}$'s are nonzero; otherwise, if $y^{i_1}, y^{i_2}, \dots, y^{i_m}$ are all zero, then the system of homogeneous equations

$$\sum_{j=1}^m a_{i\nu j} x^j = 0 \quad \nu = 1, 2, \dots, m$$

has a nontrivial solution, which implies

$$A \begin{pmatrix} i_1 & i_2 & \dots & i_m \\ 1 & 2 & \dots & m \end{pmatrix} = 0$$

in contradiction with the hypothesis. Consider now the determinant

$$\begin{vmatrix} a_{i_1 1} & a_{i_1 2} & \dots & a_{i_1 m} & y^{i_1} \\ a_{i_2 1} & a_{i_2 2} & \dots & a_{i_2 m} & y^{i_2} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{i_{m+1} 1} & a_{i_{m+1} 2} & \dots & a_{i_{m+1} m} & y^{i_{m+1}} \end{vmatrix}.$$

This determinant is zero, since $y^{i\nu} = \sum_{j=1}^m a_{i\nu j} x^j$. Expanding by the elements of the last column, we obtain

$$0 = \sum_{\nu=1}^{m+1} (-1)^{m+1+\nu} y^{i_\nu} A \begin{pmatrix} i_1 & i_2 & \dots & i_{\nu-1} & i_{\nu+1} & \dots & i_{m+1} \\ 1 & 2 & \dots & \nu-1 & \nu+1 & \dots & m \end{pmatrix}.$$

Now, since the y 's alternate in sign, some of them being nonzero, and since the A 's all have the same proper sign, the righthand side cannot be zero. This contradiction completes the proof. ■

2.8. Lemma. *Let A be an $n \times m$ strictly totally positive matrix and let $x \in \mathbb{R}^m$, $x \neq 0$. Then $y = Ax$ satisfies $N_M(y) \leq N_m(x)$.*

Proof:

Let $p+1 = N_m(x)$. Then the components of x can be divided into $p+1$ groups,

$$(x^1, x^2, \dots, x^{\nu_1}), (x^{\nu_1+1}, x^{\nu_1+2}, \dots, x^{\nu_p}), \dots, (x^{\nu_p+1}, x^{\nu_p+2}, \dots, x^m)$$

where each component in the i th group, say, either is zero or has a sign $(-1)^{i+1}$. Furthermore, there must be at least one nonzero component in

each group. Unless $n > p + 1$, which we assume to be the case, there is nothing to prove. We set $\nu_0 = 0$, $\nu_{p+1} = m$, and construct the vectors

$$v_k = \sum_{j=\nu_{k-1}+1}^{\nu_k} |x^j| a_j \quad k = 1, 2, \dots, p+1$$

where a_j is the j th column vector of A . Then

$$y = \sum_{j=1}^m x^j a_j = \sum_{k=1}^{p+1} (-1)^{k+1} \sum_{j=\nu_{k-1}+1}^{\nu_k} |x^j| a_j = \sum_{k=1}^{p+1} (-1)^{k+1} v_k$$

and

$$\begin{aligned} V \begin{pmatrix} i_1 & i_2 & \dots & i_{p+1} \\ 1 & 2 & \dots & p+1 \end{pmatrix} &= \\ &= \begin{vmatrix} \sum_{j=1}^{\nu_1} |x^j| a_{i_1 j} & \dots & \sum_{j=\nu_{p-1}+1}^{\nu_p} |x^j| a_{i_1 j} \\ \vdots & & \vdots \\ \sum_{j=1}^{\nu_1} |x^j| a_{i_{p+1} j} & \dots & \sum_{j=\nu_{p-1}+1}^{\nu_p} |x^j| a_{i_{p+1} j} \end{vmatrix} = \\ &= \sum_{\ell_1=1}^{\nu_1} \sum_{\ell_2=\nu_1+1}^{\nu_2} \dots \sum_{\ell_{p+1}=\nu_{p-1}+1}^{\nu_p} \left[|x^{\ell_1}| |x^{\ell_2}| \dots |x^{\ell_{p+1}}| A \begin{pmatrix} i_1 & i_2 & \dots & i_{p+1} \\ \ell_1 & \ell_2 & \dots & \ell_{p+1} \end{pmatrix} \right] \end{aligned}$$

where V is the matrix whose k th column vector is v_k . By the nature of the construction of the blocks we know that there is at least one selection $(\ell_1, \ell_2, \dots, \ell_{p+1})$ for which $|x^{\ell_1}| |x^{\ell_2}| \dots |x^{\ell_{p+1}}| > 0$. Thus all the minors

$$V \begin{pmatrix} i_1 & i_2 & \dots & i_{p+1} \\ 1 & 2 & \dots & p+1 \end{pmatrix}$$

are positive. Now $y = \sum_{k=1}^{p+1} (-1)^{k+1} v_k$, and since V satisfies the hypothesis of Proposition 2.7, we obtain $N_M(y) \leq p+1 = N_m(x)$. ■

Lemma 2.8 will be crucial in the proof of the spectral theorem for oscillatory matrices. Before we go to this spectral theorem, let us prove another change-sign result, Lemma 2.9, useful to the transversality theorem.

2.9. Lemma. *Let A be a nonsingular totally positive $n \times n$ matrix. Then $N_M(Ax) \leq N_M(x)$ for any $x \in \mathbb{R}^n$.*

Proof:

Let $J(\varepsilon)$ be the $n \times n$ Jacobi matrix

$$J(\varepsilon) = \begin{pmatrix} 1 & \varepsilon & 0 \\ \varepsilon & 1 & \varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}.$$

By continuity and by 2.6 there exists $\varepsilon_0 > 0$ such that $J(\varepsilon)$ is oscillatory for any $0 < \varepsilon < \varepsilon_0$. Hence, by 2.4 $B(\varepsilon) = J(\varepsilon)^{n-1}$ is a strictly totally positive matrix for any $0 < \varepsilon < \varepsilon_0$ and using 2.3 and continuity we conclude that $C(\varepsilon) = B(\varepsilon)A$ is strictly totally positive, $0 < \varepsilon < \varepsilon_0$, and $\lim_{\varepsilon \rightarrow 0} C(\varepsilon) = A$. Thus, for ε sufficiently small,

$$N_m(Ax) \leq N_m(C(\varepsilon)x) \leq N_M(C(\varepsilon)x)$$

and by Lemma 2.8, applied to $C(\varepsilon)$, we obtain for $x \neq 0$

$$(*) \quad N_m(Ax) \leq N_m(x).$$

Now, since A is nonsingular, there exists a neighbourhood U of x such that $N_m(x) = \max_{x' \in U \cap \mathcal{N}} N(x')$ and $N_M(Ax) = \max_{y \in AU \cap \mathcal{N}} N(y)$.

Thus $N_M(Ax) = N(A\bar{x})$ for some $\bar{x} \in U$ and by $(*)$

$$N(A\bar{x}) \leq N_m(\bar{x}) \leq N_M(\bar{x}).$$

Since $\bar{x} \in U$, $N_M(\bar{x}) \leq N_M(x)$ and the lemma is proved. ■

Following [FO-1] we introduce now a family of "cones" in \mathbf{R}^n which plays an important role in the spectral and transversality theorems.

2.10. Definition.

For any given integer $1 \leq i \leq n$ let K_i be the set

$$K_i = \overline{\{x \in \mathbf{R}^n : N_M(x) \leq i\}}$$

and let $\overset{\circ}{K}_1$ be the set

$$\overset{\circ}{K}_1 = \{0\} \cup \text{int } K_1 = \{0\} \cup \{x \in \mathbf{R}^n : N_M(x) \leq i\}.$$

Remark. Note that by Lemma 2.9, if A is totally positive $n \times n$ matrix, the sets $\overset{\circ}{K}_i$ and $\{0\} \cup (\mathbf{R}^n \setminus K_i)$ are invariant under A and A^{-1} , respectively. Also, $\overset{\circ}{K}_1 \cap \{0\} \cup (\mathbf{R}^n \setminus K_1) = \{0\}$ and $C\ell[\overset{\circ}{K}_1 \cup \{0\} \cup (\mathbf{R}^n \setminus K_1)] = \mathbf{R}^n$. As we will see, these invariant and transversality properties will be useful to prove the transversality theorem.

We state now, without proof, the generalization of Perron's Theorem, with the version due to C. Fusco and W. M. Oliva, which, together with Lemma 2.9, will be the tools to prove the spectral theorem for oscillatory matrices.

2.11. Theorem ([FO-2]). *Let E be an n -dimensional real vectors space, let $K \subset E$ be a closed set with non empty interior and let T be a linear transformation of E into E . Assume that*

- h1) $x \in K, \alpha \in \mathbf{R} \Rightarrow \alpha x \in K$;
- h2) $\max\{\dim W : W \text{ a subspace, } W \subset K\} = d, 1 \leq d < n$; and
- h3) $T(K \setminus \{0\}) \subset \text{int } K$.

Then there exist (unique) subspaces W_1, W_2 such that

- 1) $W_1 \cap W_2 = \{0\}, \dim W_1 = d, \dim W_2 = n - d$;
- 2) $TW_j \subset W_j, j = 1, 2$; and
- 3) $W_1 \subset \{0\} \cup \text{int } K, W_2 \cap K = \{0\}$.

Moreover, if $\sigma_1(T), \sigma_2(T)$ are the spectra of T restricted to W_1, W_2 , then, between $\sigma_1(T)$ and $\sigma_2(T)$ there is a gap

$$\lambda \in \sigma_1(T), \mu \in \sigma_2(T) \Rightarrow |\lambda| > |\mu|.$$

2.12. Theorem ([G Kr]). *Let A be an $n \times n$ oscillatory matrix. Then,*

- (a) *all the eigenvalues of A are real, simple and positive;*
- (b) *If $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$ denote the eigenvalues of A and u_i is an eigenvector corresponding to λ_i , then $u_i \in \mathcal{N}$ and $N(u_i) = i$; and*
- (c) *If $w = \sum_{j=h}^k c_j u_j$, where $1 \leq h \leq k \leq n$, then $w \neq 0$ implies*

$$h \leq N_m(w) \leq N_M(w) \leq k.$$

Proof:

It suffices to prove the theorem for a strictly totally positive matrix since by 2.4 A^{n-1} and A^n are strictly totally positive matrices.

We refer the reader to [FO-2, Theorem 2] for the proof, including the facts that the sets $K_i, 1 \leq i < n$, satisfy h1) and h2), with $d = i$.

Lemma 2.8 implies that $A(K_i \setminus \{0\}) \subset \text{int } K_i, 1 \leq i < n$, since if $x \in K_i \setminus \{0\}$ then $N_m(x) \leq i$ and hence $N_M(Ax) \leq N_m(x) \leq i$ (remember that $\text{int } K_i = \{x \in \mathbf{R}^n : N_M(x) \leq i\}$). Therefore Theorem 2.11 implies that, for each $1 \leq i < n$, there is an A -invariant i -dimensional subspace $W_1^i \subset K_i$ and an A -invariant $(n-i)$ -dimensional subspace $W_2^i, W_2^i \cap K_i = \{0\}$ with corresponding spectral gap. Clearly (with $W_1^n = W_2^0 = \mathbf{R}^n$) we have $W_1^i \subset W_1^{i+1}, W_2^i \subset W_2^{i-1}, 1 \leq i < n$. For each $1 \leq i \leq n$ let $V_i = W_1^i \cap W_2^{i-1}$. V_i is an A -invariant subspace and $\dim V_i = 1$ because $W_1^i \cap W_2^i = \{0\}$. It is also clear that $\text{span}\{V_1, \dots, V_n\} = \mathbf{R}^n$, therefore the spectrum of A is the set of the

(real) eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ corresponding to V_1, \dots, V_n . Since A is strictly totally positive, for each $1 \leq r \leq n$ the r th exterior power of A is a matrix with positive coefficients and by the classical Perron Theorem (which is a particular case of Theorem 2.11) one easily concludes that all the λ_i 's are positive. By the spectral gap we have $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$. Let $u_i \in V_i$ be a nonzero vector, then $u_i \in \text{int } K_i$ and therefore $N_M(u_i) \leq i$. Also, $u_i \notin K_{i-1}$ and therefore $N_m(u_i) > d - 1$. It follows that $u_i \in \mathcal{N}$ and $N(u_i) = i$. The last statement is proved similarly, just remembering that $w = \sum_{j=h}^k c_j u_j \neq 0$ belongs to the subspaces $W_1^k \cap W_2^{h-1}$. ■

Remark.

Let A be an $n \times m$ matrix. We say that A is strictly sign regular (see [Ka]) if for each $1 \leq \nu \leq \min(m, n)$, all the minors $A \begin{pmatrix} i_1 & \dots & i_\nu \\ k_1 & \dots & k_\nu \end{pmatrix}$ are nonzero and have the same sign. Lemma 2.8 is true for these matrices and hence we have the same spectral theorem for strictly sign regular square matrices if we disregard the positiveness of the eigenvalues and consider them in decreasing order of their moduli.

We now prove the transversality theorem:

2.13. Theorem. *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^k diffeomorphism, $k \geq 1$, such that:*

- (H1) *For all $x, y \in \mathbb{R}^n$, $N_M(f(x) - f(y)) \leq N_M(x - y)$; and*
- (H2) *For all $x \in \mathbb{R}^n$, $f'(x)$ is an oscillatory matrix.*

Then, if e^- and e^+ are two hyperbolic fixed points of f , the unstable manifold $W^u(e^-)$ and the stable manifold $W^s(e^+)$ are transversal.

Proof:

Let $x \in W^u(e^-) \cap W^s(e^+)$. Consider the sequence

$$u_k = \frac{f^{k+1}(x) - f^k(x)}{\|f^{k+1}(x) - f^k(x)\|}, \quad k \in \mathbb{Z}.$$

Taking a subsequence if necessary we have:

$$\begin{aligned} \lim_{k \rightarrow +\infty} u_k &= u^+ \in T_{e^+} W^s(e^+) \\ \lim_{k \rightarrow -\infty} u_k &= u^- \in T_{e^-} W^u(e^-). \end{aligned}$$

By (H1), $N_M(u^+) \leq N_M(u^-)$.

Let $\{u_i^\pm\}_{i=1,\dots,n}$ be a basis formed by eigenvectors of $f'(e^\pm)$ as in Theorem 2.12 and let m^\pm be the dimension of $W^u(e^\pm)$. Then,

$$u^+ = \sum_{i=m^++1}^n \alpha_i^+ u_i^+$$

$$u^- = \sum_{i=1}^{m^-} \alpha_i^- u_i^-.$$

From Theorem 2.12 $N_M(u^+) \geq m^+ + 1$ and $N_M(u^-) \leq m^-$; since $N_M(u^+) \leq N_M(u^-)$, we have

$$m^+ \leq m^- - 1.$$

Now we proceed as in the proof of the main result of [FO-1]. In fact, the last inequality implies that the $n - m^-$ eigenvectors $u_{m^-+1}^+, \dots, u_n^+$ of $f'(e^+)$ belong to $T_{e^+} W^s(e^+)$ and by Theorem 2.12 we conclude that $\Sigma = \text{span}\{u_{m^-+1}^+, \dots, u_n^+\}$ is contained in $\{0\} \cup (\mathbf{R}^n \setminus K_{m^-})$. Since $\mathbf{R}^n \setminus K_{m^-}$ is an open set and $W^s(e^+)$ is a smooth manifold, there exists an integer $n_0 > 0$ such that $T_{f^{n_0}(x)} W^s(e^+)$ contains a $(n - m^-)$ -dimensional vector subspace $\tilde{\Sigma}$ contained in $\{0\} \cup (\mathbf{R}^n \setminus K_{m^-})$. Let Σ^0 be the vector space $\Sigma^0 = (f^{-n_0})'_{(f^{n_0}(x))} \tilde{\Sigma}$. Then, $\dim \Sigma^0 = n - m^-$ and by (H2) we have $\Sigma^0 \subset T_x W^s(e^+) \cap (\{0\} \cup (\mathbf{R}^n \setminus K_{m^-}))$. A similar argument shows that $T_x W^u(e^-) \subset \overset{\circ}{K}_{m^-}$. Since $\dim(T_x W^u(e^-)) = m^-$ and since $\overset{\circ}{K}_{m^-} \cap (\{0\} \cup (\mathbf{R}^n \setminus K_{m^-})) = \{0\}$ we have $\Sigma^0 \oplus T_x W^u(e^-) = \mathbf{R}^n$ and therefore the theorem is proved. ■

2.14. Example. Let A be a strictly totally positive matrix and λ be an eigenvalue of A with eigenvector u .

Define $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ by

$$g(x) = \frac{1}{2\lambda} \left(\frac{x_1^3}{u_1^2} + x_1, \dots, \frac{x_n^3}{u_n^2} + x_n \right)$$

where $x = (x_1, \dots, x_n)$ and $u = (u_1, \dots, u_n)$.

g is a C^∞ -diffeomorphism with derivative

$$g'(x) = \frac{1}{2\lambda} \begin{pmatrix} 3 \left(\frac{x_1}{u_1} \right)^2 + 1 & & 0 \\ & \ddots & \\ 0 & & 3 \left(\frac{x_n}{u_n} \right)^2 + 1 \end{pmatrix}, \text{ for all } x \in \mathbf{R}^n.$$

The map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $f = A \circ g$ satisfies the following properties:

- (i) f is a C^∞ -diffeomorphism.
- (ii) $f(x) - f(y) = A(g(x) - g(y))$ and by Lemma 2.8 we have $N_M(f(x) - f(y)) \leq N_m(g(x) - g(y))$. On the other side $[g(x) - g(y)]_i = \frac{1}{2\lambda}(x_i - y_i) \left[\frac{x_i^2 + x_i y_i + y_i^2}{u_i^2} \right]$, $1 \leq i \leq n$. Since $x_i^2 + x_i y_i + y_i^2 \geq 0$ one can conclude that $N_m(g(x) - g(y)) \leq N_m(x - y)$ and finally

$$N_M(f(x) - f(y)) \leq N_m(x - y) \leq N_M(x - y).$$

- (iii) $f'(x) = Ag'(x) \forall x \in \mathbb{R}^n$. Since A is strictly totally positive and $g'(x)$ is totally positive and non-singular, by 2.3 $f'(x)$ is strictly totally positive.
- (iv) By i), ii) and iii), f satisfies the hypothesis of Theorem 2.13. Moreover, $f(0) = Ag(0) = A \cdot 0 = 0$ and $f(u) = Ag(u) = A \cdot \frac{1}{\lambda}u = \frac{1}{\lambda}Au = u$. Then 0 and u are fixed points of f . We have $f(\alpha u) = \frac{\alpha^2 + \alpha}{2}u$; this shows that αu belongs to $W^u(u) \cap W^s(0)$ for $0 < \alpha < 1$.

We also have

$$\begin{aligned} g'(0) &= \frac{1}{2\lambda}I \Rightarrow f'(0) = \frac{1}{2\lambda}A \\ g'(u) &= \frac{1}{\lambda}I \Rightarrow f'(u) = \frac{1}{\lambda}A. \end{aligned}$$

Denoting by $\mu_i(0)$ (resp. $\mu_i(u)$) $i = 1, \dots, n$, the eigenvalues of $f'(0)$ (resp. $f'(u)$) we have $\mu_i(0) = \frac{\lambda_i}{2\lambda}$ and $\mu_i(u) = \frac{2\lambda_i}{\lambda}$, where $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$ are the eigenvalues of A . The next Lemma 2.15 shows that it is possible to choose A such that 0 and u are hyperbolic fixed points, neither sink nor source, obtaining this way a non trivial application of Theorem 2.13.

Lemma 2.15. *Let $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$ be n given real numbers. Then, there exists a strictly totally positive matrix A with eigenvalues λ_i , $i = 1, \dots, n$.*

Proof:

Let $\mu_i = \ell n \lambda_i$, $1 \leq i \leq n$. By Lemma 14 of [FO-1], there exists a positive Jacobi matrix J with eigenvalues μ_i , $1 \leq i \leq n$. Let $A = e^J$ that has eigenvalues λ_i . For a proof that A is strictly totally positive see [L] and [Ka]. ■

III. Application

Consider the following problem:

$$(3-I) \quad \begin{cases} u_t = u_{xx} + f(u); 0 \leq x \leq L; t > 0 \\ u(0, t) = u(L, t) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

where $f: \mathbf{R} \rightarrow \mathbf{R}$ is C^1 -function satisfying

$$(3-II) \quad |f'(x)| \leq k, \quad \forall x \in \mathbf{R}.$$

For the discretization of this problem, denote by $h > 0$ the time spacement and divide the interval $[0, L]$ in $N + 1$ subintervals of length $D = \frac{L}{N+1}$, choosing the points $x_i = i * D$, $0 \leq i \leq N + 1$, as the knots. A semi-implicit discretization of the PDE above gives us the following system of equations:

$$\begin{aligned} \frac{u(x_i, t+h) - u(x_i, t)}{h} = \\ = \frac{u(x_{i-1}, t+h) - 2u(x_i, t+h) + u(x_{i+1}, t+h)}{D^2} + f(u(x_i, t)) \end{aligned}$$

or

$$\begin{aligned} -\frac{1}{D^2}u(x_{i-1}, t+h) + \left(\frac{1}{h} + \frac{2}{D^2}\right)u(x_i, t+h) - \frac{1}{D^2}u(x_{i+1}, t+h) = \\ = \frac{1}{h}u(x_i, t) + f(u(x_i, t)). \end{aligned}$$

Using the Dirichlet boundary conditions $u(x_0, t) = u(x_{N+1}, t) = 0$ we can rewrite these equations in next vectorial form (3-III):

$$\begin{aligned} \begin{bmatrix} \frac{1}{h} + \frac{2}{D^2} & -\frac{1}{D^2} & \cdots & 0 \\ -\frac{1}{D^2} & \frac{1}{h} + \frac{2}{D^2} & -\frac{1}{D^2} & 0 \\ & \vdots & -\frac{1}{D^2} & \\ 0 & \cdots & -\frac{1}{D^2} & \frac{1}{h} + \frac{2}{D^2} \end{bmatrix} \begin{bmatrix} u(x_1, t+h) \\ u(x_2, t+h) \\ \vdots \\ u(x_{N-1}, t+h) \\ u(x_N, t+h) \end{bmatrix} = \\ = \begin{bmatrix} \frac{1}{h}u(x_1, t) + f(u(x_1, t)) \\ \frac{1}{h}u(x_2, t) + f(u(x_2, t)) \\ \vdots \\ \frac{1}{h}u(x_{N-1}, t) + f(u(x_{N-1}, t)) \\ \frac{1}{h}u(x_N, t) + f(u(x_N, t)) \end{bmatrix}. \end{aligned}$$

Denote by $J_{h,D}$ the tri-diagonal matrix of the system (3-III). We can prove the following lemma:

3.1. Lemma. *For any $h > 0$ and $D > 0$, $J_{h,D}$ is non-singular and $J_{h,D}^{-1}$ is an oscillatory matrix.*

Proof.

Using the notation introduced in 2.5, $J_{h,D}^*$ is a Jacobi matrix with positive non-diagonal coefficients, and for any $1 \leq \nu \leq N$,

$$J_{h,D}^* \begin{pmatrix} 1 & 2 & \dots & \nu \\ 1 & 2 & \dots & \nu \end{pmatrix} = \prod_{k=1}^{\nu} \left\{ \frac{1}{h} + \frac{2}{D^2} \left[1 + \cos \left(\frac{k\pi}{\nu+1} \right) \right] \right\} > 0.$$

From 2.6 it follows that $J_{h,D}^*$ is an oscillatory matrix. Hence, $J_{h,D}$ is non-singular and by 2.5 $J_{h,D}^{-1}$ is an oscillatory matrix. ■

Remark. One can prove that the power χ which makes $(J_{h,D}^{-1})^\chi$ strictly totally positive is exactly $N - 1$ and hence, for $N \geq 3$, $J_{h,D}^{-1}$ is not a strictly totally positive matrix.

Since $J_{h,D}$ is non-singular, equation (3-III) induces a map $\phi_{h,D} : \mathbf{R}^N \rightarrow \mathbf{R}^N$. Denoting by x a vector in \mathbf{R}^N and by $\underline{f} : \mathbf{R}^N \rightarrow \mathbf{R}^N$ the extension of f to \mathbf{R}^N , i.e., $[\underline{f}(x)]^i = f(x^i)$, $1 \leq i \leq n$, we can write $\phi_{h,D}$ in the form:

$$\phi_{h,D}(x) = J_{h,D}^{-1} \left[\frac{1}{h}x + \underline{f}(x) \right].$$

Let us introduce now a functional V which is a discrete version of a functional used in the heat equation problem 3-I):

$$V(x) = \sum_{i=0}^N \left\{ \frac{1}{2D^2} [x^{i+1} - x^i]^2 - F(x^i) \right\}$$

where $x^{N+1} = x^0 = 0$ and $F(t) = \int_0^t f(s) ds$.

We have the following theorem:

3.2. Theorem. *If $hk < 1$, then:*

(a) $\phi_{h,D}$ is a diffeomorphism of \mathbf{R}^N satisfying

(a-1) For any $x, y \in \mathbf{R}^N$, $N_M(\phi_{h,D}(x) - \phi_{h,D}(y)) \leq N_M(x - y)$;
and

(a-2) For any $x \in \mathbf{R}^N$, $\phi'_{h,D}(x)$ is an oscillatory matrix;

(b) For any $x \in \mathbf{R}^N$, $V(\phi_{h,D}(x)) \leq V(x)$ and the equality occurs if and only if x is a fixed point of $\phi_{h,D}$.

Proof:

(a) For any $y \in \mathbf{R}^N$, the equation $y = \frac{1}{h}x + \underline{f}(x)$ has a unique solution, since it is equivalent to the fixed-point equation $x = hy - hf(x)$ and $hk < 1$ implies that the map $x \mapsto hy - hf(x)$ is a contraction, $\forall y \in \mathbf{R}^N$.

Hence, $\phi_{h,D}(x) = J_{h,D}^{-1} \left[\frac{1}{h}x + \underline{f}(x) \right]$ is one-to-one and onto. But, $\phi'_{h,D}(x) = J_{h,D}^{-1} \left[\frac{1}{h}I + \underline{f}'(x) \right]$ and $hk < 1$ implies that $\phi'_{h,D}(x)$ is non-singular for any $x \in \mathbf{R}^N$. By the inverse function theorem we conclude that $\phi_{h,D}$ is a diffeomorphism.

To prove (a1) note that from Lemma 2.9 we have

$$N_M(\phi_{h,D}(x) - \phi_{h,D}(y)) \leq N_M \left(\frac{1}{h}x + \underline{f}(x) - \frac{1}{h}y - \underline{f}(y) \right),$$

because $J_{h,D}^{-1}$ is an oscillatory matrix. Also, $hk < 1$ implies that

$$\text{sign} \left[\frac{1}{h}x^i + f(x^i) - \frac{1}{h}y^i - f(y^i) \right] = \text{sign}[x^i - y^i]$$

and hence (a1).

For (a2), let us write again $\phi'_{h,D}(x)$:

$$\begin{aligned} \phi'_{h,D}(x) &= J_{h,D}^{-1} \left[\frac{1}{h}I + \underline{f}'(x) \right] = \\ &= J_{h,D}^{-1} \cdot \begin{bmatrix} \frac{1}{h} + f'(x^1) & & & \\ & \frac{1}{h} + f'(x^2) & & 0 \\ & & \ddots & \\ 0 & & & \frac{1}{h} + f'(x^N) \end{bmatrix}. \end{aligned}$$

Since $hk < 1$, $\frac{1}{h} + f'(x^i) > 0$, $1 \leq i \leq N$, and the result follows from 2.3, 2.4 and by the fact that $J_{h,D}^{-1}$ is an oscillatory matrix.

(b) Note that $V(x)$ can be written as:

$$V(x) = \frac{1}{2} \langle J_{h,D}x, x \rangle - \frac{1}{2h} \langle x, x \rangle - \sum_{i=1}^N F(x^i),$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product of \mathbf{R}^N . Take $y = \phi_{h,D}(x)$. Then,

$$\begin{aligned}
 (*) \quad V(x) - \frac{1}{2h} \langle x - y, x - y \rangle &= \\
 &= \frac{1}{2} \langle J_{h,D} x, x \rangle - \frac{1}{h} \langle x, x \rangle + \frac{1}{h} \langle x, y \rangle - \frac{1}{2h} \langle y, y \rangle - \sum_{i=1}^N F(x^i) = \\
 &= \frac{1}{2} \langle J_{h,D} x, x \rangle - \frac{1}{h} \langle x, x - y \rangle - \frac{1}{2h} \langle x, x \rangle - \sum_{i=1}^N F(x^i) = \\
 &= \frac{1}{2} \langle J_{h,D} y, y \rangle - \frac{1}{2h} \langle y, y \rangle + \langle J_{h,D} y - \frac{1}{h} x, x - y \rangle + \\
 &\quad + \frac{1}{2} \langle J_{h,D}(x - y), x - y \rangle - \sum_{i=1}^N F(x^i).
 \end{aligned}$$

From Taylor's Theorem:

$$\begin{aligned}
 F(y^i) &= F(x^i) + f(x^i)(y^i - x^i) + \frac{1}{2} f'(\xi^i)(x^i - y^i)^2 \Rightarrow \\
 &\Rightarrow F(x^i) \leq F(y^i) + f(x^i)(x^i - y^i) + \frac{k}{2} (x^i - y^i)^2 \Rightarrow \\
 &\Rightarrow \sum_{i=1}^N F(x^i) \leq \sum_{i=1}^N F(y^i) + \langle f(x), x - y \rangle + \frac{k}{2} \langle x - y, x - y \rangle.
 \end{aligned}$$

Substituting the inequality in (*) we get

$$\begin{aligned}
 V(x) - \frac{1}{2h} \langle x - y, x - y \rangle &\geq V(y) + \langle J_{h,D} y - \frac{1}{h} x - f(x), x - y \rangle \\
 &\quad - \frac{k}{2} \langle x - y, x - y \rangle + \frac{1}{2} \langle J_{h,D}(x - y), x - y \rangle \\
 \Rightarrow V(y) &\leq V(x) - \frac{1}{2} \left[\frac{1 - hk}{h} \langle x - y, x - y \rangle + \langle J_{h,D}(x - y), x - y \rangle \right]
 \end{aligned}$$

and since $hk < 1$ and $J_{h,D}$ is symmetric positive definite, (b) is proved. ■

Remark. The fixed points of $\phi_{h,D}$ are exactly the critical points of V .

3.3. Corollary. If $hk < 1$, the unstable and stable manifolds $W^u(e^-)$ and $W^s(e^+)$ of two hyperbolic fixed point e^- and e^+ of $\phi_{h,D}$ are transversal, and the set of non-wandering points of $\phi_{h,D}$ coincides with the set of its fixed points.

Proof:

Theorem 3.2 (a) states that $\phi_{h,D}$ satisfies the hypothesis of Theorem 2.13 and the transversality is proved.

For the result about the set of non-wandering points, note that Theorem 3.2 (b) implies that V is a Liapunov functional for the dynamical system generated by $\phi_{h,D}$. ■

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