Publicacions Matemàtiques, Vol 37 (1993), 317–334.

SHAPE THEORY INTRINSICALLY

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Abstract

We prove in this paper that the category $\mathcal{H}M$ whose objects are topological spaces and whose morphisms are homotopy classes of multi-nets is naturally equivalent to the shape category Sh. The description of the category $\mathcal{H}M$ was given earlier in the article "Shape via multi-nets". We have shown there that $\mathcal{H}M$ is naturally equivalent to Sh only on a rather restricted class of spaces. This class includes all compact metric spaces where a similar intrinsic description of the shape category using multi-valued functions was given by José M. R. Sanjurjo in [5] and [6].

1. Preliminaries for the description of the category $\mathcal{H}M$

In this section we shall collect definitions and results from [2] that are required for the description of the category $\mathcal{H}M$.

Normal covers.

Let $\hat{Y}$ denote the collection of all normal covers of a topological space Y [1]. With respect to the refinement relation $>$ the set $\hat{Y}$ is a directed set. Two normal covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a normal cover and it's equivalence class by the same symbol. Consequently, $\hat{Y}$ also stands for the associated quotient set.

Let $\tilde{Y}$ denote the collection of all finite subsets c of $\hat{Y}$ which have a unique (with respect to the refinement relation) maximal element $\overline{c} \in \hat{Y}$. We consider $\tilde{Y}$ ordered by the inclusion relation and regard $\hat{Y}$ as a subset of single-element subsets of $\hat{Y}$. Notice that $\tilde{Y}$ is a cofinite directed set.

Let $\sigma \in \hat{Y}$. Let σ^* denote the set of all normal covers τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, σ^{*n} denotes the set of all normal covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Multi-valued functions.

Let X and Y be topological spaces. By a *multi-valued function* $F : X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of X. Let $M(X, Y)$ denote all multi-valued functions from X into Y.

Let $F : X \to Y$ be a multi-valued function and let $\alpha \in \hat{X}$ and $\gamma \in \hat{Y}$. We shall say that F is an (α, γ)-*function* provided for every $A \in \alpha$ there is a $C_A \in \gamma$ with $F(A) \subset C_A$. On the other hand, F is γ-*small* provided there is an $\alpha \in \hat{X}$ such that F is an (α, γ)-function.

Let $F, G : X \to Y$ be multi-valued functions and let $\gamma \in \hat{Y}$. We shall say that F and G are γ-*close* and we write $F \overset{\gamma}{=} G$ provided for every $x \in X$ there is a $C_x \in \gamma$ with $F(x) \cup G(x) \subset C_x$. How one defines the notion "(α,γ)-*close*" is now obvious.

Let $F, G : X \to Y$ be multi-valued functions between topological spaces and let γ be a normal cover of the space Y. We shall say that F and G are γ-*homotopic* and write $F \overset{\gamma}{\simeq} G$ provided there is a γ-small multi-valued function H from the product $X \times I$ of X and the unit segment $I = [0, 1]$ into Y such that $F(x) \subset H(x, 0)$ and $G(x) \subset H(x, 1)$ for every $x \in X$. We shall say that H is a γ-*homotopy* that *joins* F and G or that it *realizes* the relation (or homotopy) $F \overset{\gamma}{\simeq} G$.

Lemma 1. *Let $F, G, H : X \to Y$ be multi-valued functions. Let $\sigma \in \hat{Y}$ and $\tau \in \sigma^*$. If $F \overset{\tau}{\simeq} G$ and $G \overset{\tau}{\simeq} H$, then $F \overset{\sigma}{\simeq} H$.*

Multi-nets.

Let X and Y be topological spaces. By a *multi-net* from X into Y we shall mean a collection $\varphi = \{F_c \mid c \in \hat{Y}\}$ of multi-valued functions $F_c : X \to Y$ such that for every $\gamma \in \hat{Y}$ there is a $c \in \tilde{Y}$ with $F_d \overset{\gamma}{\simeq} F_c$ for every $d > c$. We use functional notation $\varphi : X \to Y$ to indicate that φ is a multi-net from X into Y. Let $MN(X, Y)$ denote all multi-nets $\varphi : X \to Y$.

Two multi-nets $\varphi = \{F_c\}$ and $\psi = \{G_c\}$ between topological spaces X and Y are *homotopic* provided for every $\gamma \in \hat{Y}$ there is a $c \in \tilde{Y}$ such that $F_d \overset{\gamma}{\simeq} G_d$ for every $d > c$.

It follows from Lemma 1 that the relation of homotopy is an equivalence relation on the set $MN(X, Y)$. The homotopy class of a multi-net φ is denoted by $[\varphi]$ and the set of all homotopy classes by $\mathcal{H}M(X, Y)$.

2. Description of the category $\mathcal{H}M$

Composition of homotopy classes.

Our goal now is to define a composition for homotopy classes of multi-nets.

Let $\varphi = \{F_c\} : X \to Y$ be a multi-net. For every $c \in \tilde{Y}$ there is an $\bar{f}(c) \in \tilde{Y}$ such that for all $d,\ e > \bar{f}(c)$ there is a normal cover $\bar{f}(c,\ d,\ e)$ of $X \times I$ and an $(\bar{f}(c,\ d,\ e), \bar{c})$-map joining F_d and F_e.

Let $\mathcal{C} = \{(c,\ d,\ e) |\, c \in \tilde{Y},\ d,\ e > \bar{f}(c)\}$. Then $\mathcal{C}$ is a subset of $\tilde{Y} \times \tilde{Y} \times \tilde{Y}$ that becomes a cofinite directed set when we define that $(c,\ d,\ e) > (c',\ d',\ e')$ iff $c > c'$, $d > d'$, and $e > e'$.

Now, let $f : \tilde{Y} \to \tilde{Y}$ be an increasing function such that $f(c) > \bar{f}(c),\ c$ for every $c \in \tilde{Y}$. We shall use the same notation f for an increasing function $f : \mathcal{C} \to X \hat{\times} I$ such that $f(c,\ d,\ e) > \bar{f}(c,\ d,\ e)$ for every $(c,\ d,\ e) \in \mathcal{C}$. Let $(c,\ d,\ e) \in \mathcal{C}$. For the normal cover $f(c,\ d,\ e)$ of $X \times I$, by [**3**, p. 358], there is a normal cover $\varepsilon = \hat{f}(c,\ d,\ e)$ of X and a function $r = \hat{f}(c,\ d,\ e) : \varepsilon \to \{2,\ 3,\ 4, \ldots\}$ such that every set $E \times [(i-1)/rE,\ (i+1)/rE]$, where $E \in \varepsilon$ and $i = 1,\ 2, \ldots,\ rE - 1$, is contained in a member of $f(c,\ d,\ e)$.

Let $\tilde{f} : \mathcal{C} \to \hat{X}$ be an increasing function with $\tilde{f}(c,\ d,\ e) > \hat{f}(c,\ d,\ e)$ for every $(c,\ d,\ e) \in \mathcal{C}$. We shall use the shorter notation $\tilde{f}(c)$ and $f(c)$ for the covers $\tilde{f}(c,\ f(c),\ f(c))$ and $f(c,\ f(c),\ f(c))$. In [**2**, Claim 1], the following lemma was proved.

Lemma 2. *There is an increasing function $f^* : \tilde{Y} \to \hat{X}$ such that*

(1) *$f^*(c) > \tilde{f}(c)$ for every $c \in \tilde{Y}$, and*
(2) *f^* is cofinal in $\tilde{f}$, i. e., for every $(c,\ d,\ e) \in \mathcal{C}$ there is an $m \in \tilde{Y}$ with $f^*(m) > \tilde{f}(c,\ d,\ e)$.*

The above discussion shows that every multi-net $\varphi : X \to Y$ determines eight functions denoted by $\bar{f}$, f, $\hat{f}$, $\tilde{f}$, and f^*. With the help of these functions we shall define the composition of homotopy classes of multi-nets as follows.

Let $\varphi = \{F_c\} : X \to Y$ and $\psi = \{G_u\} : Y \to Z$ be multi-nets. Let $\chi = \{H_u\}$, where $H_u = G_{g(u)} \circ F_{f(\{g^*(u)\})}$ for every $u \in \tilde{Z}$.

It was proved in [**2**] that the collection χ is a multi-net from X into Z.

We now define the composition of homotopy classes of multi-nets by the rule $[\{G_u\}] \circ [\{F_c\}] = [\{G_{g(u)} \circ F_{f(\{g^*(u)\})}\}]$. The composition of homotopy classes of multi-nets is well-defined and associative (see [**2**]).

The category $\mathcal{H}M$.

For a topological space X, let $\iota^X = \{I_a\} : X \to X$ be the identity multi-net defined by $I_a = id_X$ for every $a \in \tilde{X}$. It is easy to show that for every multi-net $\varphi : X \to Y$ the following relations hold:

$$[\varphi] \circ [\iota^X] = [\varphi] = [\iota^Y] \circ [\varphi].$$

It was shown in [**2**] that the topological spaces as objects, the homotopy classes of multi-nets as morphisms, the homotopy classes $[\iota^X]$ as identities, and the above composition of homotopy classes form the category $\mathcal{H}M$.

There is an obvious functor J from the category Top of topological spaces and continuous maps into the category $\mathcal{H}M$. On objects the functor J is the identity while on morphisms it associates to a map $f : X \to Y$ the homotopy class of a multi-net $\underline{f} = \{F_c\} : X \to Y$, where $F_c = f$ for every $c \in \tilde{Y}$.

3. Statement of the main theorem

Our main result can be stated as follows. Let Sh be the shape category of arbitrary topological spaces and let $S : Top \to Sh$ be the shape functor [**4**].

Theorem. *There is a functor θ from the category $\mathcal{H}M$ into the shape category Sh which is an isomorphism of categories and such that $S = \theta \circ J$.*

4. Preliminaries for the description of the functor θ

Cofinite Čech system.

With every space X one can associate an inverse system $\mathcal{X} = \{X_c, [p_d^c], \tilde{X}\}$, called the cofinite Čech system of X, where $X_c = |N(\overline{c})|$ is the nerve of $\overline{c}$ and $[p_d^c]$, for $d > c$ in $\tilde{X}$, is the unique homotopy class to which belong the projections $p_d^c : |N(\overline{d})| \to |N(\overline{c})|$. For a $c \in \tilde{X}$, let $[p^c] : X \to X_c$ be the unique homotopy class of the canonical mappings $p^c : X \to X_c$. Recall that $[p^c] = [p_d^c] \circ [p^d]$ whenever $d > c$ in $\tilde{X}$ so that $\mathbf{p} = \{[p^c]\} : X \to \mathcal{X}$ is a morphism of the pro-homotopy category pro-$\mathcal{H}Top$. Since the usual Čech system is an $\mathcal{H}Pol$-expansion (see [**4**, p. 328]), it is easy to show by direct verification of conditions (E1) and (E2), that $\mathbf{p}$ is also an $\mathcal{H}Pol$-expansion.

In the rest of this paper, let X, Y, and Z be topological spaces and let

$$\mathbf{p} = \{[p^a]\} : X \to \mathcal{X} = \{X_a, [p^a_b], \tilde{X}\},$$
$$\mathbf{q} = \{[q^c]\} : X \to \mathcal{Y} = \{Y_c, [q^c_d], \tilde{Y}\},$$

and

$$\mathbf{r} = \{[r^u]\} : Z \to \mathcal{Z} = \{Z_u, [r^u_v], \tilde{Z}\}$$

be cofinite Čech systems of X, Y, and Z, respectively.

It is well-known that shape morphisms from X into Y could be considered as equivalence classes of morphisms of inverse systems $\mathcal{X}$ and $\mathcal{Y}$ (see [4]). More precisely, the set $\mathcal{S}h(X, Y)$ of all shape morphisms between spaces X and Y can be identified with the set $pro\text{-}\mathcal{H}Pol(\mathcal{X}, \mathcal{Y})$ of all morphisms in the Grothendick's pro-category $pro\text{-}\mathcal{H}Pol$ of the homotopy category of polyhedra $\mathcal{H}Pol$ between the objects $\mathcal{X}$ and $\mathcal{Y}$. In our description of what θ does on morphisms of the category $\mathcal{H}M$ we shall view shape morphisms in this way.

Multi-valued functions B_σ.

We shall also be using the following natural multi-valued function B_σ from the nerve $|N(\sigma)|$ of a normal cover σ of X into X. The function B_σ associates to a point w of $|N(\sigma)|$ the intersection of members of the cover σ which span a simplex of $|N(\sigma)|$ that contains w. Hence, if w belongs to a simplex $\langle S_1, \ldots, S_n\rangle$ of $|N(\sigma)|$, where $S_1, \ldots, S_n$ are members of σ, then $B_\sigma(w) = \bigcap_{i=1}^n S_i$. Observe that B_σ is a $(*_\sigma, \sigma)$-function, where $*_\sigma$ denotes the (normal) cover of $|N(\sigma)|$ by open stars $*^S_\sigma$ of all vertices $S \in \sigma$ of $|N(\sigma)|$. Moreover, for every canonical mapping $p : X \to |N(\sigma)|$ (i. e., a map which satisfies $p^{-1}(*^S_\sigma) \subset S$ for every member S of σ) and every $x \in X$ there is a $V \in \sigma$ such that both x and the set $B_\sigma \circ p(x)$ lie in V while for every $x \in X$ and every $S \in \sigma$ with $x \in S$ the set $B_\sigma \circ p(x)$ is a subset of $st(S, \sigma)$. Hence, the composition $B_\sigma \circ p$ is $(\sigma, st(\sigma))$-homotopic and σ-close to the identity map id_X.

Lemma 3. *Let σ and τ be normal covers of a space X such that τ refines σ. Then $B_\tau \overset{\sigma}{\simeq} B_\sigma \circ p$ for every projection $p : |N(\tau)| \to |N(\sigma)|$.*

Proof: Let $T \in \tau$ and let $x \in *^T_\tau$. Then $B_\tau(x)$ is a subset of T while $p(x)$ lies in $*^S_\sigma$, where S is a member of σ which contains T. It follows that both $B_\tau(x)$ and $B_\sigma \circ p(x)$ are subsets of S. Hence, the function H from $|N(\tau)|$ into X defined by the rule $H(x) = B_\tau(x) \cup B_\sigma \circ p(x)$ for every $x \in |N(\tau)|$ is a σ-homotopy joining B_τ and $B_\sigma \circ p$. ■

Approximation of small functions with maps.

In the description of the functor θ we shall also need the following approximation result (see Lemma 2 in [**2**]).

Lemma 4. *For every open cover σ of a polyhedron Y there is an open cover τ of it such that for every τ-small multi-valued function $F : X \to Y$ from a space X into Y there is a single-valued continuous function $f : X \to Y$ with $F \overset{\sigma}{=} f$.*

5. Description of the functor θ

The functor θ will leave the objects unchanged. In order to explain how θ effects the morphisms we must work much harder.

Let $\varphi = \{F_s\}_{s\in\tilde{Y}} : X \to Y$ be a multi-net. For each $c \in \tilde{Y}$, pick an open cover σ_c of $Y_c = |N(\overline{c})|$ such that $st^3(\sigma_c)$-close maps into Y_c are homotopic. Since the set $\tilde{Y}$ is cofinite, we can select the covers σ_c so that σ_d refines $(q_d^c)^{-1}(\sigma_c)$ whenever $d > c$ in $\tilde{Y}$. Moreover, we can assume that σ_c refines the open cover $*_{\overline{c}}$ for every $c \in \tilde{Y}$.

By Lemma 4, we can find a $\tau_c \in {\sigma_c}^*$ so that every τ_c-small multi-valued function into Y_c is σ_c-close to a continuous single-valued function. Let $\xi_c \in \hat{Y}$ be a refinement of $(q^c)^{-1}(\tau_c)$. Since $\tilde{Y}$ is a cofinite directed set, we can assume that ξ_c refines ξ_e for every $e \in \tilde{Y}$ with $c > e$. Moreover, for every pair c, $e \in \tilde{Y}$ with $c > e$, the maps q^e and $q_c^e \circ q^c$ are joined by a homotopy K_c^e so that we can select a $\pi_c^e \in \hat{Y}$ and a stacked normal cover ϱ_c^e of $Y \times I$ over π_c^e which refines $(K_c^e)^{-1}(\tau_e)$. We shall assume that ξ_c also refines π_c^e for every index e with $c > e$.

Since φ is a multi-net, there is an index $\varphi_c \in \tilde{Y}$ so that

$$F_d \overset{\xi_c}{\simeq} F_e \qquad \text{for all } d,\, e > \varphi_c \tag{1}$$

Choose an increasing function $\varphi^* : \tilde{Y} \to \tilde{Y}$ such that $\varphi^*(c) > c, \varphi_c, \{\xi_c\}$ for every $c \in \tilde{Y}$.

Since the the function $F_{\varphi^*(c)}$ is ξ_c-small, there is an $\eta_c \in \hat{X}$ such that $F_{\varphi^*(c)}$ is an (η_c, ξ_c)-function. Let $\lambda_c \in {\eta_c}^*$. Choose an increasing function $\varphi : \tilde{Y} \to \tilde{X}$ such that $\varphi(c) > \{\lambda_c\}$ for every $c \in \tilde{Y}$. The composition $q^c \circ F_{\varphi^*(c)} \circ B_{\overline{\varphi(c)}}$ is a τ_c-small multi-valued function so that it is σ_c-close to a map $\varphi^c : X_{\varphi(c)} \to Y_c$.

Claim 1. *The pair $\underline{\varphi} = (\varphi, \{\varphi^c | c \in \tilde{Y}\})$ is a morphism between cofinite Čech systems $\mathcal{X}$ and $\mathcal{Y}$.*

Proof: We must show that for every pair c, d of elements of $\tilde{Y}$ with $d > c$ it is possible to find an $a > x$, y so that

$$\varphi^c \circ p_a^x \simeq q_d^c \circ \varphi^d \circ p_a^y, \tag{2}$$

where $x = \varphi(c)$ and $y = \varphi(d)$.

Let $u = \varphi^*(c)$ and $v = \varphi^*(d)$. Since $v > u > \varphi_c$, by (1), there is a normal cover ϱ of $X \times I$ such that the functions F_u and F_v can be joined by a (ϱ, ξ_c)-homotopy $M : X \times I \to Y$. Pick a normal cover π of X and a stacked normal cover ω of $X \times I$ over π such that ω refines ϱ. For $a > \{\pi\}$, we see that $Q = q^c \circ M \circ (B_\alpha \times id_I)$ is a τ_c-homotopy joining $q^c \circ F_u \circ B_\alpha$ and $q^c \circ F_v \circ B_\alpha$, where $\alpha = \overline{a}$.

Since F_u is a (ξ, ξ_c)-function and $B_\alpha \overset{\xi}{\simeq} B_\xi \circ p_a^x$ by Lemma 3, we see that there is a τ_c-homotopy P with $P_0 = q^c \circ F_u \circ B_\xi \circ p_a^x$ and $P_1 = Q_0$, where $\xi = \overline{x}$. For a similar reason, there is a τ_c-homotopy R with $R_0 = Q_1$ and $R_1 = q^c \circ F_v \circ B_\lambda \circ p_a^y$, where $\lambda = \overline{y}$.

Since $F_v \circ B_\lambda \circ p_a^y$ is a ξ_d-function and ξ_d refines π_d^c, $S = K_d^c \circ ((F_v \circ B_\lambda \circ p_a^y) \times id_I)$ is a τ_c-homotopy with $S_0 = R_1$ and $S_1 = q_d^c \circ q^d \circ F_v \circ B_\lambda \circ p_a^y$.

Observe that our choices imply the existence of homotopies A, B, C, and D such that $A \overset{\sigma}{=} P$, $B \overset{\sigma}{=} Q$, $C \overset{\sigma}{=} R$, $D \overset{\sigma}{=} S$, $\varphi^c \circ p_a^x = P_0$, and $S_1 = q_d^c \circ \varphi^d \circ p_a^y$, where $\sigma = \sigma_c$. From here it follows that $\varphi^c \circ p_a^x \overset{\nu}{=} A_0$, $A_1 \overset{\nu}{=} B_0$, $B_1 \overset{\nu}{=} C_0$, $C_1 \overset{\nu}{=} D_0$, $D_1 \overset{\nu}{=} q_d^c \circ \varphi^d \circ p_a^y$, where $\nu = st(\sigma_c)$.

The way in which we selected the covers σ_c implies that the adjacent maps in the following long list are homotopic: $\varphi^c \circ p_a^x$, A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1, $q_d^c \circ \varphi^d \circ p_a^y$. Hence, the relation (2) holds. ■

Now we can define that θ acts on morphisms of the category $\mathcal{H}M$ (i. e., on homotopy classes of multi-nets) by the rule $\theta([\varphi]) = [\underline{\varphi}]$, where $[\underline{\varphi}]$ denotes the equivalence class of $\underline{\varphi}$ with respect to the equivalence relation $\sim$ (see [4, p. 6]).

Claim 2. *The function θ is well-defined, i. e., it does not depend on the choices of φ, φ^*, and φ^c in our description of $[\underline{\varphi}]$.*

Proof: Suppose that $\psi = \{G_c\} : X \to Y$ is multi-net homotopic to φ and let the morphism $\underline{\psi} = (\psi, \{\psi^c | c \in \tilde{Y}\})$ of inverse systems $\mathcal{X}$ and $\mathcal{Y}$ be constructed from ψ by the above procedure using in it ψ, ψ^*, and ψ^c instead of φ, φ^*, and φ^c, respectively. We must show that $\underline{\varphi}$ and $\underline{\psi}$ are equivalent, i. e., that for every $c \in \tilde{Y}$ there is an $a > x$, u with

$$\varphi^c \circ p_a^x \simeq \psi^c \circ p_a^u, \tag{3}$$

where $x = \varphi(c)$ and $u = \psi(c)$.

Let a $c \in \tilde{Y}$ be given. In order to prove (3), we shall argue the existence of indices a, $z \in \tilde{Y}$ such that

$$\varphi^c \circ p_a^x \overset{\sigma_c}{\equiv} q^c \circ F_y \circ B_\xi \circ p_a^x, \tag{4}$$

$$q^c \circ F_y \circ B_\xi \circ p_a^x \overset{\tau_c}{\simeq} q^c \circ F_y \circ B_\alpha, \tag{5}$$

$$q^c \circ F_y \circ B_\alpha \overset{\tau_c}{\simeq} q^c \circ F_z \circ B_\alpha, \tag{6}$$

$$q^c \circ F_z \circ B_\alpha \overset{\tau_c}{\simeq} q^c \circ G_z \circ B_\alpha, \tag{7}$$

$$q^c \circ G_z \circ B_\alpha \overset{\tau_c}{\simeq} q^c \circ G_v \circ B_\alpha, \tag{8}$$

$$q^c \circ G_v \circ B_\alpha \overset{\tau_c}{\simeq} q^c \circ G_v \circ B_\gamma \circ p_a^u, \tag{9}$$

$$q^c \circ G_v \circ B_\gamma \circ p_a^u \overset{\sigma_c}{\equiv} \psi_c \circ p_a^u, \tag{10}$$

where $\xi = \overline{x}$, $y = \varphi^*(c)$, $\gamma = \overline{u}$, $v = \psi^*(c)$, and $\alpha = \overline{a}$.

Once we have the relations (4)–(10), we approximate each of the τ_c-homotopies by a σ_c-close homotopy and conclude from here as in the proof of Claim 1 that (3) holds.

Add (4) and (10). This follows from the fact that φ^c and ψ^c are σ_c-close to functions $q^c \circ F_y \circ B_\xi$ and $q^c \circ G_v \circ B_\gamma$, respectively.

Add (5). It suffices to observe that F_y is a (ξ, ξ_c)-function and that $B_\alpha \overset{\xi}{\simeq} B_\xi \circ p_a^x$ for every $a > x$.

Add (6). Let $z > y$. Then there is a normal cover λ of $X \times I$ and a (λ, ξ_c)-homotopy L joining F_y and F_z. Pick a normal cover π of X and a stacked normal cover ϱ of $X \times I$ over π such that ϱ refines λ. Let $a > \{\pi\}$. The composition $q^c \circ L \circ (B_\alpha \times id_I)$ is a τ_c-homotopy which realizes the relation (6).

Add (7). Since φ and ψ are homotopic multi-nets, there is a $b \in \tilde{Y}$ such that $F_z \overset{\xi_c}{\simeq} G_z$ for every $z > b$. Let $z > b$ and let M be a ξ_c-homotopy joining F_z and G_z. Choose normal covers λ and ϱ of $X \times I$ and π of X such that M is a (λ, ξ_c)-function, ϱ refines λ and ϱ is stacked over π. Let $a > \{\pi\}$. Then the composition $q^c \circ M \circ (B_\alpha \times id_I)$ is a τ_c-homotopy which realizes the relation (7).

Add (8) and (9). These are analogous to relations (6) and (5), respectively. ∎

Claim 3. *Let $\iota = \{I_a\}_{a \in \tilde{X}}$ be the identity multi-net on a space X, where $I_a = id_X$ for every $a \in \tilde{X}$. Then the homotopy class $[\underline{\iota}]$ associated to the homotopy class $[\iota]$ by the function θ is the identity homotopy class $[(id_{\tilde{X}}, \{I_a \,|\, a \in \tilde{X}\})]$.*

Proof: We must show that for every $a \in \tilde{X}$ there is a $b > a, \iota(a)$ such that

$$p^a_b \simeq \iota^a \circ p^{\iota(a)}_b. \tag{11}$$

In order to prove the above statement, observe that we can assume that the function $\iota : \tilde{X} \to \tilde{X}$ (which corresponds to the function φ in the above description) has the property that the cover ξ is a star-refinement of the cover $(p^a)^{-1}(\tau_a)$ for every $a \in \tilde{X}$, where $\xi = \overline{x}$ and $x = \iota(a)$.

Let an $a \in \tilde{X}$ be given. By construction, we have $\iota^a \circ p^x \overset{\sigma_a}{=} p^a \circ B_\xi \circ p^x$.

Since $B_\xi \circ p^x$ is $st(\xi)$-homotopic to the identity map id_X on X and $st(\xi)$ refines $(p^a)^{-1}(\tau_a)$, there is a τ_a-homotopy H with $H_0 = p^a \circ B_\xi \circ p^x$ and $H_1 = p^a$.

Let K be a single-valued continuous function σ_a-close to H. Then $\iota^a \circ p^x$ is $st(\sigma_a)$-close to K_0 and K_1 is σ_a-close to p^a. It follows that $\iota^a \circ p^x$ and p^a are homotopic. Finally, we can use the property (E2) of the cofinite Čech system $\mathcal{X}$ to get a $b > a, \iota(a)$ such that (11) holds. ■

Claim 4. *For multi-nets $\varphi = \{F_c\}_{c \in \tilde{Y}} : X \to Y$ and $\psi = \{G_u\}_{u \in \tilde{Z}} : Y \to Z$,*

$$\theta([\psi] \circ [\varphi]) = \theta([\psi]) \circ \theta([\varphi]).$$

Proof: Let $\chi = \{H_u\} : X \to Z$, where $H_u = G_{g(u)} \circ F_{f(\{g^*(u)\})}$ for every $u \in \tilde{Z}$. Then $[\chi] = [\psi] \circ [\varphi]$. Let $\underline{\varphi} = (\varphi, \{\varphi^c\}_{c \in \tilde{Y}})$, $\underline{\psi} = (\psi, \{\psi^u\}_{u \in \tilde{Z}})$, and $\underline{\chi} = (\chi, \{\chi^u\}_{u \in \tilde{Z}})$ be obtained from φ, ψ, and χ by the above procedure. We must show that $\underline{\chi}$ and $\underline{\psi} \circ \underline{\varphi}$ are homotopic. Since $\underline{\psi} \circ \underline{\varphi} = (\varphi \circ \psi, \{\psi^u \circ \varphi^{\psi(u)}\}_{u \in \tilde{Z}})$, this amounts to show that for every $u \in \tilde{Z}$ there is an $a > t, x$ such that

$$\chi^u \circ p^b_a \simeq \psi^u \circ \varphi^k \circ p^t_a, \tag{12}$$

where $t = \varphi \circ \psi(u)$, $k = \psi(u)$, and $b = \chi(u)$.

Let a $u \in \tilde{Z}$ be given. In order to prove the above statement, we shall argue the existence of large enough indices $w \in \tilde{Z}$, $z \in \tilde{Y}$, and $a \in \tilde{X}$ such that

$$\chi^u \circ p^b_a \overset{\alpha_u}{=} r^u \circ H_x \circ B_\beta \circ p^b_a \tag{13}$$

$$r^u \circ H_x \circ B_\beta \circ p_a^b \overset{\beta_u}{\simeq} r^u \circ H_x \circ B_\alpha, \tag{14}$$

$$r^u \circ H_x \circ B_\alpha \overset{\beta_u}{\simeq} r^u \circ G_y \circ F_z \circ B_\alpha, \tag{15}$$

$$r^u \circ G_y \circ F_z \circ B_\alpha \overset{\beta_u}{\simeq} r^u \circ G_w \circ F_z \circ B_\alpha, \tag{16}$$

$$r^u \circ G_w \circ F_z \circ B_\alpha \overset{\beta_u}{\simeq} r^u \circ G_e \circ F_z \circ B_\alpha, \tag{17}$$

$$r^u \circ G_e \circ F_z \circ B_\alpha \overset{\beta_u}{\simeq} r^u \circ G_e \circ F_i \circ B_\alpha, \tag{18}$$

$$r^u \circ G_e \circ F_i \circ B_\alpha \overset{\beta_u}{\simeq} r^u \circ G_e \circ F_i \circ B_\tau \circ p_a^t, \tag{19}$$

$$r^u \circ G_e \circ F_i \circ B_\tau \circ p_a^t \overset{\beta_u}{\simeq} r^u \circ G_e \circ B_\kappa \circ q^k \circ F_i \circ B_\tau \circ p_a^t, \tag{20}$$

$$r^u \circ G_e \circ B_\kappa \circ q^k \circ F_i \circ B_\tau \circ p_a^t \overset{\alpha_u}{\cong} r^u \circ G_e \circ B_\kappa \circ \varphi^k, \tag{21}$$

and

$$r^u \circ G_e \circ B_\kappa \circ \varphi^k \overset{\alpha_u}{\cong} \psi^u \circ \varphi^k \circ p_a^t, \tag{22}$$

where $\alpha = \overline{a}$, $x = \chi^*(u)$, $\beta = \overline{b}$, $y = \psi(u)$, $e = \psi^*(u)$, $i = \varphi^*(k)$, $\tau = \overline{t}$, $\kappa = \overline{k}$, and α_u and β_u are covers of Z_u analogous to covers σ_c and τ_c of Y_c, respectively.

Once we have the relations (13)–(22), we approximate each of the β_u-homotopies by an α_u-close homotopy and conclude from here as in the proof of Claim 1 that (12) holds.

Add (13) and (22). This follows from the fact that χ^u and ψ^u are α_u-close to functions $r^u \circ H_x \circ B_\beta$ and $r^u \circ G_e \circ B_\kappa$, respectively.

Add (21). Observe that $r^u \circ G_e$ is a (κ, α_u)-function, B_κ is a $(*_k, \kappa)$-function, the cover σ_k refines the cover $*_k$, and the map φ^k is σ_k-close to the composition $q^k \circ F_i \circ B_\tau$, where $*_k$ denotes the normal cover of Y_k by the open stars of all vertices.

Add (14). Notice that $r^u \circ H_x$ is a (β, β_u)-function and $B_\alpha \overset{\beta}{\simeq} B_\beta \circ p_a^b$ for every $a > b$.

Add (15). Recall that $H_x = G_y \circ F_s$, where $s = f(\{g^*(x)\})$. Since $r^u \circ G_u$ is a $(g^*(x), \beta_u)$-function, it suffices to take $z > s$ because then F_s and F_z are joined by the $g^*(u)$-homotopy L so that the composition $r^u \circ G_y \circ L \circ (B_\alpha \times id_I)$ will be a β_u-homotopy which realizes (15) whenever a is sufficiently large.

Add (16). Let $w > y$. Then G_y and G_w are joined by a ξ-homotopy M, where $\xi = \overline{x}$. But, the cover ξ refines the cover $\gamma_u = (r^u)^{-1}(\beta_u)$. It follows that $r^u \circ M$ is a β_u-homotopy joining $r^u \circ G_y$ and $r^u \circ G_w$. Since for every normal cover λ of Y we can find indices a and z such that the composition $F_z \circ B_\alpha$ is a λ-small function, it is clear that there are indices z and a such that $r^u \circ M \circ (F_z \circ B_\alpha \times id_I)$ realizes the relation (16).

Add (17). Let $w > e$. Then G_w and G_e are joined by a γ_u-small homotopy N. As in the proof of (16), we get the existence of large enough indices z and a such that $r^u \circ N \circ (F_z \circ B_\alpha \times id_I)$ realizes the relation (17).

Add (18). Observe that G_e is a (κ, γ_u)-function. Also, we can always assume that the cover ξ_k was selected so that it refines the cover κ. Then for $z > i$ the functions F_z and F_i are joined by the ξ_k-homotopy P. Hence, for a large index a, the composition $r^u \circ G_e \circ P \circ (B_\alpha \times id_I)$ realizes the relation (18).

Add (19). The argument for this relation is similar to the one given for the relation (14).

Add (20). Notice that $r^u \circ G_e$ is a (μ, γ_u)-function, where μ is a normal cover of Y such that $st(\kappa)$ refines μ. Also, there is a $(\kappa, st(\kappa))$-homotopy R joining id_X with the composition $B_\kappa \circ q^k$. Once again, if we choose the cover ξ_k so that it refines the cover κ, then $r^u \circ G_e \circ R \circ (F_i \circ B_\tau \circ p^t_a \times id_I)$ will be a β_u-homotopy that realizes the relation (20). ■

Claim 5. *θ is a functor and the relation $S = \theta \circ J$ holds.*

Proof: That θ is a functor follows from the Claims 3 and 4. It remains to see that $S = \theta \circ J$. Let $f : X \to Y$ be a map, i. e., a morphism of the category *Top*. Then $J(f)$ is represented by a multi-net $\varphi = \{F_c\}_{c \in \tilde{Y}} : X \to Y$, where $F_c = f$ for every $c \in \tilde{Y}$. It follows that $\theta \circ J(f)$ is represented by a morphism $\underline{\varphi} = (\varphi, \{\varphi^c\}_{c \in \tilde{Y}})$ between inverse systems $\mathcal{X}$ and $\mathcal{Y}$, where φ^c is a map which is σ_c-close to $q^c \circ f \circ B_\alpha$ for a suitable index $a \in \tilde{X}$ and the cover $\alpha = \overline{a}$ of X. We shall now prove that

$$\varphi^c \circ p^a \simeq q^c \circ f. \tag{23}$$

Observe that

$$\varphi^c \circ p^a \overset{\sigma_c}{=} q^c \circ f \circ B_\alpha \circ p^a. \tag{24}$$

But, $B_\alpha \circ p^a$ is α-close to id_X. Hence, had we chosen a so that α refines the cover $(q^c \circ f)^{-1}(\sigma_c)$, we would get

$$q^c \circ f \circ B_\alpha \circ p^a \overset{\sigma_c}{\cong} q^c \circ f. \tag{25}$$

From (24) and (25) it follows that $\varphi^c \circ p^a$ and $q^c \circ f$ are $st(\sigma_c)$-close and we get the relation (23).

In order to conclude now that $S(f) = \theta \circ J(f)$ we must recall (see [**4**]) that $S(f)$ is a unique morphism $\underline{f}$ such that $\mathbf{q} \circ f = \underline{f} \circ \mathbf{p}$. ∎

We shall now prove that θ is a category isomorphism by constructing for every pair of objects X and Y of the shape category a function

$$\zeta : Mor_{Sh}(X, Y) \to Mor_{HM}(X, Y)$$

such that $\theta \circ \zeta = id$ and $\zeta \circ \theta = id$.

6. Preliminaries for the description of the functor ζ

Factorization through canonical mappings.

Lemma 5. *Let X be an arbitrary space, let Y be a polyhedron (endowed with the CW topology), let σ be an open covering of Y, and let $f : X \to Y$ be a map. Then there exist a normal cover τ of X such that for every normal cover ϱ of X which refines τ there exist a map $k : |N(\varrho)| \to Y$ with the following properties:*

(i) *For any canonical map $p : X \to |N(\varrho)|$ the maps $k \circ p$ and f are σ-close.*

(ii) *Each $R \in \varrho$ admits an $S_R \in \sigma$ such that*

(a) *$k(*_\varrho^R) \subset S_R$, and*

(b) *$f(R) \subset S_R$.*

Proof: Let $\delta \in \sigma^*$. By [4], there is an ANR space M and maps $u : Y \to M$ and $d : M \to Y$ such that

$$d \circ u \overset{\delta}{=} id_Y. \tag{26}$$

Let $\beta = d^{-1}(\delta)$. Let $h = u \circ f$. By Lemma 2 on p. 316 of [4], there is a normal cover τ of X and a map $g : |N(\tau)| \to M$ with the following properties:

(i') For any canonical map $r : X \to |N(\tau)|$ the maps $g \circ r$ and h are β-close.

(ii') Each $T \in \tau$ admits a $B_T \in \beta$ such that

(a') $g(*_\tau^T) \subset S_T$, and

(b') $h(T) \subset S_T$.

Let $\varrho \in \hat{X}$ be a refinement of τ and let $q : |N(\varrho)| \to |N(\tau)|$ be a simplicial projection mapping induced by the selection of a member $\nu(R)$ of τ with $R \subset \nu(R)$ for every member R of ϱ. Let $k = d \circ g \circ q$.

In order to check (i), let $p : X \to |N(\varrho)|$ be a canonical map. By Theorem 6 on p. 326 of [4], the composition $r = q \circ p$ is also a canonical map. By (i'), the maps $g \circ r$ and h are β-close. Composing with the map d we see that $k \circ p$ and $d \circ u \circ f$ are δ-close. But, by (26), $d \circ u \circ f$ is δ-close to f so that $f \overset{\sigma}{=} k \circ p$.

Finally, to verify (ii), let $R \in \varrho$. By (ii'), there is a $D_{\nu(R)} \in \delta$ such that

(c') $g(*_\tau^{\nu(R)}) \subset d^{-1}(D_{\nu(R)})$, and
(d') $h(\nu(R) \subset d^{-1}(D_{\nu(R)})$.

For each $R \in \varrho$ choose an $S_R \in \sigma$ such that S_R contains the star $st(D_{\nu(R)}, \delta)$ of $D_{\nu(R)}$ with respect to the cover δ. Then

$$k(*_\varrho^R) = d \circ g \circ q(*_\varrho^R) \subset d \circ g(*_\tau^{\nu(R)}) \subset D_{\nu(R)} \subset S_R.$$

On the other hand, from (d') we get $d \circ u \circ f(R) \subset D_{\nu(R)}$ for every $R \in \varrho$. But, by (26), for each $y \in f(R)$, some member of δ contains both y and $d \circ u(y)$. Hence, $f(R) \subset S_R$. ■

Hooked and small implies homotopic.

The following notion and lemma are from [**2**]. Let σ be a normal cover of a space Y. Two multi-valued functions $F, G : X \to Y$ are *σ-hooked* provided for every $x \in X$ there is an $S_x \in \sigma$ such that S_x has non-empty intersection with both $F(x)$ and $G(x)$. Observe that σ-close multi-valued functions are σ-hooked.

Lemma 6. *Let $F, G : X \to Y$ be multi-valued functions and let σ be a normal cover of Y. If F and G are σ-small and σ-hooked, then $F \overset{st(\sigma)}{\simeq} G$.*

7. Description of the functor ζ

Let $\underline{f} = (f, \{ f^c \}_{c \in \tilde{Y}})$ be a morphism between cofinite Čech systems $\mathcal{X}$ and $\mathcal{Y}$ associated to spaces X and Y, respectively. For every $c \in \tilde{Y}$, define a multi-valued function $f_c : X \to Y$ to be the composition $B_\gamma \circ f^c \circ p^x$, where γ denotes the normal cover $\bar{c}$ and $x = f(c)$. Let $\overline{f} = \{f_c\}_{c \in \tilde{Y}}$.

Claim 6. *The family $\overline{f}$ is a multi-net from X into Y.*

Proof: Let a $\sigma \in \hat{Y}$ be given. We must find an index $c \in \tilde{Y}$ such that

$$f_c \overset{\sigma}{\simeq} f_d \qquad \text{for every } d > c. \tag{27}$$

Let $\tau \in \sigma^{*2}$. Put $c = \{\tau\}$. Let $d > c$. Since $\underline{f}$ is a morphism between $\mathcal{X}$ and $\mathcal{Y}$, there is an index $a > x, y$ and a homotopy H with

$$H_0 = f^c \circ p^x_a \quad \text{and} \quad H_1 = q^c_d \circ f^d \circ p^y_a, \tag{28}$$

where $x = f(c)$ and $y = f(d)$. Moreover, there are homotopies G and K with

$$G_0 = p^x, \qquad G_1 = p^x_a \circ p^a, \qquad K_0 = p^y_a \circ p^a, \qquad K_1 = p^y. \tag{29}$$

Let $\gamma = \overline{c}$ and $\delta = \overline{d}$. Let L be a τ-homotopy joining $B_\gamma \circ q^c_d$ and B_δ (for this see Lemma 3). The compositions $A = B_\gamma \circ f^c \circ G$, $C = L \circ (f^d \circ p^y_a \circ p^a \times id_I)$, $B = B_\gamma \circ H \circ (p^a \times id_I)$, and $D = B_\delta \circ f^d \circ K$ are τ-homotopies such that $f_c = A_0$, $A_1 = B_0$, $B_1 = C_0$, $C_1 = D_0$, and $D_1 = f_d$. Hence, $f_c \overset{\sigma}{\simeq} f_d$. ■

Now we can define the function ζ by the rule $\zeta([\underline{f}]) = [\overline{f}]$.

Claim 7. *The function ζ is well-defined, i. e., the value $\zeta([\underline{f}])$ does not depend on the choice of the representative $\underline{f}$ of the equivalence class $[\underline{f}]$.*

Proof: Let $\underline{g} = (g, \{g^c\}_{c \in \tilde{Y}})$ be another morphism from $\mathcal{X}$ into $\mathcal{Y}$ equivalent to $\underline{f}$ and let $\overline{g} = \{g_c\}_{c \in \tilde{Y}}$ be a multi-net constructed from $\underline{g}$ by the above procedure. We must show that $\overline{f}$ and $\overline{g}$ are homotopic, i. e., that for every $\sigma \in \hat{Y}$ there is an index $c \in \tilde{Y}$ such that

$$f_d \overset{\sigma}{\simeq} g_d \qquad \text{for every } d > c. \tag{30}$$

Let a $\sigma \in \hat{Y}$ be given. Let $\tau \in \sigma^*$. Put $c = \{\tau\} \in \tilde{Y}$. Let $d > c$. Since morphisms $\underline{f}$ and $\underline{g}$ are equivalent, there is an index $a > x, y$ and a homotopy H with

$$H_0 = f^d \circ p^x_a \quad \text{and} \quad H_1 = g^d \circ p^y_a. \tag{31}$$

Moreover, there are homotopies G and K with

$$G_0 = p^x, \quad G_1 = p^x_a \circ p^a, \quad K_0 = p^y_a \circ p^a \quad \text{and} \quad K_1 = p^y, \tag{31}$$

where $x = f(d)$ and $y = g(d)$. Let $\delta = \overline{d}$.

The compositions $A = B_\delta \circ f^d \circ G$, $B = B_\delta \circ H \circ (p^a \times id_I)$, and $C = B_\delta \circ g^d \circ K$ are τ-homotopies such that $f_d = A_0$, $A_1 = B_0$, $B_1 = C_0$, and $C_1 = g_d$. Hence, $f_d \overset{\sigma}{\simeq} g_d$. ■

8. Verification that ζ is an inverse of θ

Claim 8. *For every morphism* $\underline{f} = (f, \{f^c\}_{c\in\check{Y}}) : \mathcal{X} \to \mathcal{Y}$ *we have* $[\underline{f}] = \theta \circ \zeta([\underline{f}])$.

Proof: Let $\zeta([\underline{f}]) = [\varphi]$, where $\varphi = \{f_c\}_{c\in\check{Y}}$, $f_c = B_\gamma \circ f^c \circ p^x$, $\gamma = \overline{c}$, and $x = f(c)$ for every $c \in \check{Y}$. Let

$$\theta(\zeta([\underline{f}])) = \theta([\varphi]) = [(\varphi, \{\varphi^c\}_{c\in\check{Y}})],$$

where φ^c is a map which is σ_c-close to $q^c \circ f_b \circ B_\varepsilon$, $b = \varphi^*(c)$, $e = \varphi(c)$, and $\varepsilon = \overline{e}$. Hence, φ^c is a map which is σ_c-close to $q^c \circ B_\beta \circ f^b \circ p^d \circ B_\varepsilon$, where $\beta = \overline{b}$ and $d = f(b)$.

Let us apply Lemma 5 in the case $X = Y$, $M = Y_c$, $\sigma = \tau_c$, and $h = q^c$ to get a cover μ_c of Y such that for every refinement ϱ of μ_c there exist a map $g^c_\varrho : |N(\varrho)| \to Y_c$ with the following properties:

(32) For any canonical map $p : Y \to |N(\varrho)|$ the maps $g^c_\varrho \circ p$ and q^c are τ_c-close.

(33) Each $R \in \varrho$ admits a $T_R \in \tau_c$ such that $g^c_\varrho(*^R_\varrho) \subset T_R$ and $q^c(R) \subset T_R$.

Without loss of generality, we can assume that the function φ^* also satisfies the condition $\varphi^*(c) > \{\mu_c\}$ for every $c \in \check{Y}$. By assumption, there is a map $g^c_b : Y_b \to Y_c$ such that

$$q^c \overset{\tau_c}{=} g^c_b \circ q^b, \tag{34}$$

and each $R \in \beta$ admits a $T_R \in \tau_c$ such that $g^c_b(*^R_\beta) \subset T_R$ and $q^c(R) \subset T_R$. But, the function B_β satisfies $B_\beta(*^R_\beta) \subset R$ for every $R \in \beta$ so that both $g^c_b(*^R_\beta)$ and $q^c \circ B_\beta(*^R_\beta)$ are contained in T_R for every $R \in \beta$. It follows that g^c_b and $q^c \circ B_\beta$ are $(*_\beta, \tau_c)$-close. This means that by selecting the function φ carefully, we can achieve that the map B_ε is small enough so that

$$g^c_b \circ f^b \circ p^d \circ B_\varepsilon \overset{\tau_c}{=} q^c \circ B_\beta \circ f^b \circ p^d \circ B_\varepsilon. \tag{35}$$

Now we can apply Lemma 5 again, this time in the case $X = X$, $M = X_d$, $\sigma = \kappa_c$, where $\kappa_c = (g^c_b \circ f^b)^{-1}(\tau_c)$, and $h = p^d$ to get a cover ν_c of X such that for every refinement ϱ of ν_c there exist a map $h^d_\varrho : |N(\varrho)| \to X_d$ with the following properties:

(36) For any canonical map $p : X \to |N(\varrho)|$ the maps $h^d_\varrho \circ p$ and p^d are κ_c-close.

(37) Each $R \in \varrho$ admits a $K_R \in \kappa_c$ such that $h^d_\varrho(*^R_\varrho) \subset K_R$ and $p^d(R) \subset K_R$.

Without loss of generality, we can assume that the function φ also satisfies the condition $\varphi(c) > \{\nu_c\}$ for every $c \in \tilde{Y}$. Recall that $e = \varphi(c)$. By assumption, there is a map $h_e^d : X_e \to X_d$ such that

$$p^d \overset{\kappa_c}{\cong} h_e^d \circ p^e, \tag{38}$$

and each $E \in \varepsilon$ admits a $K_E \in \kappa_c$ such that $h_e^d(*_\varepsilon^E) \subset K_E$ and $p^d(E) \subset K_E$. But, the function B_ε satisfies $B_\varepsilon(*_\varepsilon^E) \subset E$ for every $E \in \varepsilon$ so that both $h_e^d(*_\varepsilon^E)$ and $p^d \circ B_\varepsilon(*_\varepsilon^E)$ are subsets of K_E for every $E \in \varepsilon$. It follows that h_e^d and $p^d \circ B_\varepsilon$ are κ_c-close. Hence,

$$g_b^c \circ f^b \circ p^d \circ B_\varepsilon \overset{\tau_c}{\cong} g_b^c \circ f^b \circ h_e^d. \tag{39}$$

Since $\tau_c \in \sigma_c^*$, the relations (35) and (39) give that the map $g_b^c \circ f^b \circ h_e^d$ is σ_c-close to the composition $q^c \circ B_\beta \circ f^b \circ p^d \circ B_\varepsilon$. Therefore, we can take $g_b^c \circ f^b \circ h_e^d$ as the map φ^c.

It remains to see that the morphisms $\underline{f}$ and $(\varphi, \{\varphi^c\}_{c\in\tilde{Y}})$ are equivalent, i. e., that for every $c \in \tilde{Y}$ there is an $a > e$, x such that

$$f^c \circ p_a^x \simeq g_b^c \circ f^b \circ h_e^d \circ p_a^e. \tag{40}$$

The relations (38) and (34) imply

$$g_b^c \circ f^b \circ h_e^d \circ p^e \simeq g_b^c \circ f^b \circ p^d, \tag{41}$$

and

$$q^c \simeq g_b^c \circ q^b. \tag{42}$$

From the relation (42) and the property (E2) for the expansion $\mathbf{q}$, we get the existence of an index $i > b$, c with

$$q_i^c \simeq g_b^c \circ q_i^b. \tag{43}$$

On the other hand, since $\underline{f}$ is a morphism, we have

$$f^b \circ p^d \simeq q_i^b \circ f^i \circ p^y, \tag{44}$$

where $y = f(i)$. Thus,

$$g_b^c \circ f^b \circ p^d \simeq g_b^c \circ q_i^b \circ f^i \circ p^y \simeq q_i^c \circ f^i \circ p^y \simeq f^c \circ p^x. \tag{45}$$

It follows from (41) and (45) that $f^c \circ p^x \simeq g_b^c \circ f^b \circ h_e^d \circ p^e$. Now, we can use the property (E2) for the expansion $\mathbf{p}$ to get an index a such that (40) holds. ∎

Claim 9. *For every multi-net* $\varphi = \{F_s\}_{s\in\tilde{Y}} : X \to Y$ *we have* $\zeta \circ \theta([\varphi]) = [\varphi]$.

Proof: Let $\theta([\varphi]) = [(\varphi, \{\varphi^c\}_{c\in\tilde{Y}})]$, where $\varphi^c : X_{\varphi(c)} \to Y_c$ is a map such that

$$\varphi^c \overset{\sigma_c}{=} q^c \circ F_{\varphi^*(c)} \circ B_{\overline{\varphi(c)}} \qquad \text{for every } c \in \tilde{Y}. \tag{46}$$

Then $\zeta \circ \theta([\varphi]) = [\overline{\varphi}]$, where $\overline{\varphi} = \{\varphi_c\}_{c\in\tilde{Y}}$ is a multi-net from X into Y and $\varphi_c = B_{\overline{c}} \circ \varphi^c \circ p^{\varphi(c)}$ for every $c \in \tilde{Y}$.

We must show that multi-nets φ and $\overline{\varphi}$ are homotopic, i. e., that for every $\sigma \in \hat{Y}$ there is a $c \in \tilde{Y}$ such that

$$F_d \overset{\sigma}{\simeq} \varphi_d \qquad \text{for every } d > c. \tag{47}$$

Let a $\sigma \in \hat{Y}$ be given. Let $\tau \in \sigma^{*3}$. Since φ is a multi-net there is a $c > \{\tau\}$ such that

$$F_e \overset{\tau}{\simeq} F_d \qquad \text{for all } e,\, d > c. \tag{48}$$

Let $d > c$. Let $\delta = \overline{d}$. Since B_δ is a $(*_\delta, \delta)$-function and σ_d refines $*_\delta$, from (46) we get

$$\varphi_d \overset{\tau}{=} B_\delta \circ q^d \circ F_m \circ B_\nu \circ p^n, \tag{49}$$

where $m = \varphi^*(d)$, $n = \varphi(d)$, and $\nu = \overline{n}$. Also, φ_d is a τ-small function while the right hand side of (49) is a τ-small function provided we make sure that the functions φ^* and φ increase sufficiently fast (so that $m > c$ and ν is such that F_m is a (ν, τ)-function). It follows from Lemma 6 that

$$\varphi_d \overset{st(\tau)}{\simeq} B_\delta \circ q^d \circ F_m \circ B_\nu \circ p^n. \tag{50}$$

Since the composition $B_\delta \circ q^d$ is $(\delta, st(\delta))$-homotopic to the id_X, we see again that by a careful selection of functions φ^* and φ, we can achieve that

$$B_\delta \circ q^d \circ F_m \circ B_\nu \circ p^n \overset{st(\tau)}{\simeq} F_m \circ B_\nu \circ p^n. \tag{51}$$

Since $m > c$, there is a normal cover λ of X such that F_m is a (λ, τ)-function. Hence, if we require in addition that $st(\nu)$ refines λ, then

$$F_m \circ B_\nu \circ p^n \overset{\tau}{\simeq} F_m, \tag{52}$$

because $B_\nu \circ p^n$ is $st(\nu)$-homotopic to the id_X.

Finally, since $m,\, d > c$ from (48) we get

$$F_m \overset{\tau}{\simeq} F_d. \tag{53}$$

The relations (50)–(53) together imply the relation (47). ∎

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Rebut el 12 de Gener de 1993