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NORMAL BASES FOR NON-ARCHIMEDEAN SPACES OF CONTINUOUS FUNCTIONS

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Abstract

K is a complete non-archimedean valued field and M is a compact, infinite, subset of K. $C(M \to K)$ is the Banach space of continuous functions from M to K, equipped with the supremum norm. Let $(p_n(x))$ be a sequence of polynomials, with $\deg p_n = n$. We give necessary and sufficient conditions for $(p_n(x))$ to be a normal basis for $C(M \to K)$. In the rest of the paper, K contains $\mathbb{Q}_p$, and V_q is the closure of the set $\{aq^n | n = 0, 1, 2 \dots\}$ where a and q are two units of $\mathbb{Z}_p$, q not a root of unity. We give necessary and sufficient conditions for a sequence of polynomials $(r_n(x))$ $(\deg r_n = n)$ to be a normal basis for $C(V_q \to K)$. Furthermore, if we define $\left\{ \begin{matrix} x \\ 0 \end{matrix} \right\} = 1$, $\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{(x/a-1)(x/(aq)-1)\dots(x/(aq^{n-1})-1)}{(q^n-1)\dots(q-1)}$ if $n \geq 1$, and if (j_n) is a sequence in $\mathbb{N}_0$, then we show that the sequence of polynomials $\left(\left\{ \begin{matrix} x \\ n \end{matrix} \right\}^{j_n} \right)$ forms a normal basis for $C(V_q \to K)$.

1. Introduction

The main aim of this paper is to find normal bases for spaces of continuous functions. Therefore we start by recalling some definitions and some previous results.

Let K be a non-archimedean valued field and suppose that K is complete for its valuation $|.|$. Take $M \subset K$ compact, infinite, and let $C(M \to K)$ be the Banach space of continuous functions from M to K, equipped with the supremum norm.

Let E be a non-archimedean Banach space over a non-archimedean valued field K. Let $e_1, e_2, \dots$ be a finite or infinite sequence of elements of E. We say that this sequence is orthogonal if $\|\alpha_1 e_1 + \dots + \alpha_k e_k\| = \max\{\|\alpha_i e_i\| : i = 1, \dots, k\}$ for all k in $\mathbb{N}$ (or for all k that do not exceed the length of the sequence) and for all $\alpha_1, \dots, \alpha_k$ in K. If the sequence

is infinite, it follows that $\left\|\sum_{i=1}^{\infty}\alpha_i e_i\right\| = \max\{\|\alpha_i e_i\| : i = 1, 2, \dots\}$ for all $\alpha_1, \alpha_2, \dots$ in K for which $\lim_{i\to\infty}\alpha_i e_i = 0$. An orthogoal sequence $e_1, e_2, \dots$ is called orthonormal if $\|e_i\| = 1$ for all i.

This leads us to the following definition:

If E is a non-archimedean Banach space over a non-archimedean valued field K, then a family (e_i) of elements of E is a (ortho)normal basis of E if the family (e_i) is orthonormal and also a basis.

An equivalent formulation is

If E is a non-archimedean Banach space over a non-archimedean valued field K, then a family (e_i) of elements of E is a (ortho)normal basis of E if each element x of E has a unique representation $x = \sum_i x_i e_i$ where $x_i \in K$ and $x_i \to 0$ if $i \to \infty$, and if the norm of x is the supremum of the norms of x_i.

In [6, chapter 5, 5.27 and 5.33] we find the following theorem which is due to Y. Amice:

Theorem 1.

Let K be a non-archimedean valued field, complete with respect to its norm $|.|$, and let M be a compact, infinite subset of K.

Let (u_n) be an injective sequence in M.

Define $p_0(x) = 1$, $p_n(x) = (x - u_{n-1})p_{n-1}(x)$ for $n \geq 1$, $q_n(x) = \frac{p_n(x)}{p_n(u_n)}$.

Then $(q_n(x))$ forms a normal basis for $C(M \to K)$ if and only if $\|q_n\| = 1$ $(\forall n)$.

If $(q_n(x))$ forms a normal basis for $C(M \to K)$ and f is an element of $C(M \to K)$, then

$$f(x) = \sum_{n=0}^{\infty} a_n q_n(x) \text{ where } a_n = p_n(u_n)\sum_{i=0}^{n}\frac{f(u_i)}{p'_{n+1}(u_i)}.$$

We remark that there always exist sequences $(q_n(x))$ such that $\|q_n\| = 1$ for all n.

We will call a sequence of polynomials $(p_n(x))$ a polynomial sequence if p_n is exactly of degree n for all n.

After all these definitions, we now give a survey of the results in this article.

In Section 2 of this paper, K is a non-archimedean complete field, and M is a compact, infinite subset of K. In Theorems 2 and 3 we will give necessary and sufficient conditions for a polynomial sequence $(p_n(x))$ to be a normal basis for $C(M \to K)$.

In Sections 3, 4, 5 and 6 we consider the following situation: $\mathbb{Z}_p$ is the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and K is a non-archimedean valued field, K containing $\mathbb{Q}_p$, and we supppose that K is complete for the valuation $|.|$, which extends the p-adic valuation. Let a and q be two units of $\mathbb{Z}_p$, q not a root of unity. We define V_q to be the closure of the set $\{aq^n | n = 0, 1, 2, \dots\}$. A description of the set V_q will be given in Section 3 (Lemmas 4 and 5). In Section 4, Theorem 4, we will give necessary and sufficient conditions for a polynomial sequence $(r_n(x))$ to be a normal basis for $C(V_q \to K)$.

If we put $\left\{ \begin{matrix} x \\ 0 \end{matrix} \right\} = 1$, $\left\{ \begin{matrix} x \\ n \end{matrix} \right\} = \frac{(x/a-1)(x/(aq)-1)\dots(x/(aq^{n-1})-1)}{(q^n-1)\dots(q-1)}$ if $n \geq 1$, and if (j_n) is a sequence in $\mathbb{N}_0$, then we show in Theorem 5 of Section 5 that $\left(\left\{ \begin{matrix} x \\ n \end{matrix} \right\}^{j_n} \right)$ forms a normal basis for $C(V_q \to \mathbb{Q}_p)$. The proof we give here is only valid when we work with a discrete valuation.

In Section 6, Theorem 6 we show that $\left(\left\{ \begin{matrix} x \\ n \end{matrix} \right\}^{j_n} \right)$ also forms a normal basis for $C(V_q \to K)$, where the valuation of K does not have to be discrete, as was the case in the previous section.

To prove this, we need the results of Section 5.

S. Caenepeel ([**3**]) proved the following: Let $\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}$ if $n \geq 1$, $\binom{x}{0} = 1$ (the binomial polynomials), then for each $s \in \mathbb{N}_0$, $\left(\binom{x}{n}^s \right)$ forms a normal basis for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$, and each function f in $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ can be written as a uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n^{(s)} \binom{x}{n}^s$$

where

$$a_n^{(s)} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^s f(k) \alpha_{n-k}^{(s)}$$

and

$$\alpha_0^{(s)} = 1$$

$$\alpha_m^{(s)} = \sum_{\substack{(j_1,\dots,j_r) \\ \sum j_i = m;\, 1 \leq j_i \leq m}} (-1)^{r+m} \binom{m}{j_1 \dots j_r}^s. \tag{1}$$

If (j_n) is a sequence in $\mathbb{N}_0$, then the sequence of polynomials $\left(\binom{x}{n}^{j_n}\right)$ also forms a normal basis of $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ ([4, p. 158]).

Now we can find an analogous result on the space $C(V_q \to K)$: each function f, element of $C(V_q \to K)$, can be written as a uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} b_n^{(s)} \left\{ \begin{matrix} x \\ n \end{matrix} \right\}^s$$

and we can give an expression for the coefficients $b_n^{(s)}$, which is analogous to the expression in (1). This result can be found in Proposition 1 of Section 6.

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2. Normal bases for $C(M \to K)$

In this section, K, is a non-archimedean valued field, complete with respect to his norm $|.|$, and M is a compact, infinite subset of K.

Before we generalize Amice's Theorem, we give a lemma.

Lemma 1.

Let (u_n) be an injective sequence in M, and let $q_0(x) = 1$, $q_n(x) = \frac{(x-u_0)\ldots(x-u_{n-1})}{(u_n-u_0)\ldots(u_n-u_{n-1})}$ for $n \geq 1$, where $\|q_n\| = 1$ for all n.

If $p(x)$ is a polynomial in $k[x]$ of degree n, then there exists an index i, $0 \leq i \leq n$, such that $\|p\| = |p(u_i)|$.

Proof:

There exist coefficients c_j such that $p(x) = \sum_{j=0}^{n} c_j q_j(x)$. Now suppose that $|p(u_i)| < \|p\|$ for all i, $0 \leq i \leq n$. This will lead to a contradiction.

Supposing that $|p(u_i)| < \|p\|$ for all i, $0 \leq i \leq n$, we will prove by induction that $|c_i| < \|p\|$ for $0 \leq i \leq n$.

Now $p(u_0) = \sum_{j=0}^{n} c_j q_j(u_0) = c_0$, so $|c_0| < \|p\|$.

Further, $p(u_1) = \sum_{j=0}^{n} c_j q_j(u_1) = c_0 + c_1$, so $|c_0 + c_1| < \|p\|$, and combining this with the previous we find $|c_1| < \|p\|$.

Suppose we already have that $|c_i| < \|p\|$ for $0 \leq i < k \leq n$.

Then $p(u_k) = \sum_{j=0}^{n} c_j q_j(u_k) = \sum_{j=0}^{k-1} c_j q_j(u_k) + c_k$, so $\left|\sum_{j=0}^{k-1} c_j q_j(u_k) + c_k\right| < \|p\|$. Since $|c_i| < \|p\|$ for $0 \leq i < k \leq n$ and since $\|q_j\| = 1$, we find that $|c_k| < \|p\|$.

So we may conclude that $|c_i| < \|p\|$ for $0 \leq i \leq n$.

But then we have, since $(q_n(x))$ forms a normal basis (Theorem 1), $\|p\| = \max_{0 \leq i \leq n} \{|c_i|\} < \|p\|$ which is clearly a contradiction.

Since $|p(u_i)| \leq \|p\|$, we may conclude that there exists an index i, $0 \leq i \leq n$, such that $\|p\| = |p(u_i)|$. ■

Proceeding from the theorem of Amice, we can marke more normal bases with the following theorem:

Theorem 2.

Let $(q_n(x))$ be a normal basis as found in Theorem 1.

Define $p_n(x) = \sum_{j=0}^{n} c_{n;j} q_j(x)$, $c_{n;j} \in K$, $c_{n;n} \neq 0$.

Then $(p_n(x))$ forms a normal basis for $C(M \to K)$ if and only if $\|p_n\| = 1$ and $|c_{n;n}| = 1$ for all n.

Proof:

Suppose that the sequence $(p_n(x))$ forms a normal basis for $C(M \to K)$.

It is clear that the norm of p_n must equal one. Since $(q_n(x))$ forms a normal basis, this implies that $|c_{n;n}| \leq 1$.

There exist coefficients $d_{n;j}$ such that $q_n = \sum_{j=0}^{n} d_{n;j} p_j(x)$ and so we have $1 = \|q_n\| = \max_{0 \leq j \leq n} \{|d_{n;j}|\}$ so $|d_{n;n}| \leq 1$.

Further, $q_n = \sum_{j=0}^{n} d_{n;j} p_j(x) = \sum_{j=0}^{n} d_{n;j} \sum_{i=0}^{j} c_{j;i} q_i = \sum_{i=0}^{n} q_i \sum_{j=i}^{n} d_{n;j} c_{j;i}$ and this implies $d_{n;n} c_{n;n} = 1$.

Combining this with the fact that $|d_{n;n}| \leq 1$ and $|c_{n;n}| \leq 1$, we conclude $|d_{n;n}| = 1$ and $|c_{n;n}| = 1$.

We now prove the other implication.

Let k be an arbitrary element of $\mathbb{N}$ and let $b_0, b_1, \ldots, b_k$ be arbitrary elements of K. For the orthonormality of the sequence $(p_n(x))$, we have to show

$$\|b_0 p_0 + \cdots + b_k p_k\| = \max_{0 \leq n \leq k} \{\|b_i p_i\|\} = \max_{0 \leq n \leq k} \{|b_i|\}.$$

If $\max_{0\le n\le k}\{|b_n|\}=0$ there is nothing to prove.

If $\max_{0\le n\le k}\{|b_n|\}>0$, then put $I=\{n|0\le n\le k|\,|b_n|=\max_{0\le j\le k}\{|b_j|\}\}$.

There exists an N such that $N=\max\{i\in I\}$.

We have $\left|\sum_{n=0}^{k} b_n p_n(x)\right| \le \max_{0\le n\le k}\{|b_n p_n(x)|\} \le |b_N|$, and so $\left\|\sum_{n=0}^{k} b_n p_n\right\| \le \max_{0\le n\le k}\{|b_n|\}$.

Put $\sum_{n=0}^{k} b_n p_n(x)=\sum_{n=0}^{N} b_n p_n(x)+\sum_{n=N+1}^{k} b_n p_n(x)=\tilde f(x)+\hat f(x)$, where we have $\|\tilde f\|\le \max_{0\le n\le k}\{|b_n|\}$, $\|\hat f\|<\max_{0\le n\le k}\{|b_n|\}$ (strict inequality).

$$\tilde f(x)=\sum_{n=0}^{N} b_n p_n(x)=\sum_{n=0}^{N} b_n\sum_{j=0}^{n} c_{n;j}q_j(x)=\sum_{j=0}^{N} q_j(x)\sum_{n=j}^{N} b_n c_{n;j}$$
$$=\sum_{j=0}^{N-1} q_j(x)\sum_{n=j}^{N} b_n c_{n;j}+q_N(x)b_N c_{N;N}.$$

We distinguish two cases:

a) $\left\|\sum_{j=0}^{N-1} q_j\sum_{n=j}^{N} b_n c_{n;j}\right\|<|b_N|$.

Since $|q_N(u_N)b_N c_{N;N}|=|b_N|$, it follows that $|\tilde f(u_N)|=|b_N|$, and so $\left\|\sum_{n=0}^{k} b_n p_n\right\|=|b_N|=\max_{0\le n\le k}\{|b_n|\}$.

b) $\left\|\sum_{j=0}^{N-1} q_j\sum_{n=j}^{N} b_n c_{n;j}\right\|=|b_N|$.

There exists an i, $0\le i\le N-1$, such that $\left|\sum_{j=0}^{N-1} q_j(u_i)\sum_{n=j}^{N} b_n c_{n;j}\right|=|b_N|$ (Lemma 1).

Then we have $|\tilde f(u_i)=\left|\sum_{j=0}^{N-1} q_j(u_i)\sum_{n=j}^{N} b_n c_{n;j}\right|=|b_N|$, and so $\left\|\sum_{n=0}^{k} b_n p_n\right\|=|b_N|=\max_{0\le n\le k}\{|b_n|\}$.

We conclude that the sequence $(p_n(x))$ is orthonormal.

By [**6**, p. 165, Lemma 5.1] and by Kaplansky's Theorem (see e.g. [**6**, p. 191, Theorem 5.28]) it follows that $(p_n(x))$ forms a basis of $C(M\to K)$, since the k linear span of the polynomials $p_n(x)$ contains $K[x]$. ■

Theorem 3.

Let $(p_n(x))$ be a polynomial sequence in $K[x]$, which forms a normal basis for $C(M \to K)$, and let $(r_n(x))$ be a polynomial sequence in $K[x]$ such that $r_n(x) = \sum_{j=0}^{n} e_{n;j} p_j(x)$, $e_{n;j} \in K$. Then the following are equivalent:

i) *$(r_n(x))$ forms a normal basis for $C(M \to K)$*
ii) $\|r_n\| = 1$, $|e_{n;n}| = 1$
iii) $|e_{n;j}| \leq 1$, $|e_{n;n}| = 1$.

Proof:

i) $\Leftrightarrow$ ii) follows from Theorem 2, using the expression $p_n(x) = \sum_{j=0}^{n} c_{n;j} q_j(x)$, and ii) $\Leftrightarrow$ iii) follows from the fact that $(p_n(x))$ forms a normal basis. ■

3. The set V_q

From now on, K is a non-archimedean valued field, K contains $\mathbb{Q}_p$, and K is complete for the valuation $|.|$, which extends the p-adic valuation.

The aim now is to find normal bases for the space $C(V_q \to K)$. Therefore, we start by giving a description of the set V_q (Lemmas 4 and 5 below).

Definition.

If b is an element of $\mathbb{Z}_p$, $b \equiv 1 \pmod{p}$, x an element of $\mathbb{Z}_p$, then we put $b^x = \lim_{n \to x} b^n$. The mapping: $\mathbb{Z}_p \to \mathbb{Z}_p : x \to b^x$ is continuous.

For more details, we refer the reader to [4, Section 32].

Lemma 2.

Let α be an element of $\mathbb{Z}_p$, $\alpha \equiv 1 \pmod{p^r}$, $\alpha \not\equiv 1 \pmod{p^{r+1}}$ $r \geq 1$. If $(p, r) \neq (2, 1)$, $\beta \in \mathbb{Z}_p \backslash \{0\}$ then

$$\alpha^\beta \equiv (\mathit{mod}\ p^{r+\mathrm{ord}_p \beta})$$
$$\alpha^\beta \not\equiv 1 (\mathit{mod}\ p^{r+1+\mathrm{ord}_p \beta}).$$

Proof:

Let $\alpha = 1 + \gamma p^r$, and let $\gamma = \gamma_0 + \gamma_1 p + \ldots$, with $\gamma_0 \neq 0$, be the Henseldevelopment of the p-adic integer γ ([4, Section 3]).

Then we have

$$\alpha^p = (1+\gamma p^r)^p = \sum_{k=0}^{p} \binom{p}{k} (\gamma p^r)^k = 1 + p\gamma p^r + \ldots (\gamma p^r)^p$$
$$\equiv 1 + \gamma_0 p^{r+1} \ldots (\text{remark} : r+1 \neq rp),$$

and so $\alpha^p \equiv 1 \pmod{p^{r+1}}$, $\alpha^p \not\equiv 1 \pmod{p^{r+2}}$.

If we continue in this way, we find: $\alpha^{p^s} \equiv 1 \pmod{p^{r+s}}$, $\alpha^{p^s} \not\equiv 1 \pmod{p^{r+1+s}}$.

Now take k such that $2 \leq k \leq p-1$.

$$\alpha^k = (1+\gamma p^r)^k = \sum_{j=0}^{k} \binom{k}{j} (\gamma p^r)^j = 1 + k\gamma p^r + \ldots (\gamma p^r)^k.$$

$k\gamma$ cannot be a multiple of p, since neither k or γ is divisible by p.

So $\alpha^k \equiv 1 \pmod{p^r}$, $\alpha^k \not\equiv 1 \pmod{p^{r+1}}$.

Let n be an element of $\mathbb{N}_0$. If we combine the previous results then we find $\alpha^n \equiv 1 \pmod{p^{r+\mathrm{ord}_p n}}$, $\alpha^n \not\equiv 1 \pmod{p^{r+1+\mathrm{ord}_p n}}$.

The lemma follows by continuity. ■

Lemma 3.

Let α be an element of $\mathbb{Z}_2$, $\alpha \equiv 3$ (mod 4). Define a natural number n by $\alpha = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{n-1} = 1$, $\varepsilon_n = 0$.

If $\beta \in \mathbb{Z}_2 \backslash \{0\}$, $\mathrm{ord}_2 \beta = 0$ then

$$\alpha^\beta \equiv 1 (mod\ 2)$$
$$\alpha^\beta \not\equiv 1 (mod\ 4).$$

If $\beta \in \mathbb{Z}_2 \backslash \{0\}$, $\mathrm{ord}_2 \beta = k \geq 1$ then

$$\alpha^\beta \equiv 1 (mod\ 2^{n+2+\mathrm{ord}_2 \beta})$$
$$\alpha^\beta \not\equiv 1 (mod\ 2^{n+3+\mathrm{ord}_2 \beta}).$$

Proof:

$\alpha = 3 + 4\varepsilon$. Hence $\alpha^2 = 1 + 2^3(1+\varepsilon)(1+2\varepsilon)$.

Since $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{n-1} = 1$, $\varepsilon_n = 0$, $\mathrm{ord}_2(1+2\varepsilon) = 0$, we have $\alpha^2 \equiv 1 \pmod{2^{n+3}}$, $\alpha^2 \not\equiv 1 \pmod{2^{n+4}}$.

Then

$$\alpha^{2^{k+1}} = (\alpha^2)^{2^k} \equiv 1 (\mathrm{mod}\ 2^{n+3+k})$$
$$\not\equiv 1 (\mathrm{mod}\ 2^{n+4+k}) \text{ by Lemma 2 } (k \geq 1).$$

So $\alpha^{2^k} \equiv 1 \pmod{2^{n+2+k}}$, $\alpha^{2^k} \not\equiv 1 \pmod{2^{n+3+k}}$ $(k \geq 1)$.

In an analogous way as in the previous lemma, we show

If $s \in \mathbb{N}_0$, $\mathrm{ord}_2\, s = 0$, then $\alpha^s \equiv 1 \pmod 2$, $\alpha^s \not\equiv 1 \pmod 4$.

If $s \in \mathbb{N}_0$, $\mathrm{ord}_2\, s = k \geq 1$, then $\alpha^s \equiv 1 \pmod{2^{n+2+\mathrm{ord}_2\, s}}$, $\alpha^s \not\equiv 1 \pmod{2^{n+3+\mathrm{ord}_2\, s}}$.

The lemma follows by continuity. ■

In the following lemma, m is the smallest integer such that $q^m \equiv 1 \pmod p$. (Remark: $1 \leq m \leq p-1$).

Lemma 4.

Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$.

If $(p, k_0) \neq (2, 1)$, *then* $V_q = \bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$.

Proof:

We take the m balls $\{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$, $0 \leq r \leq m-1$.

Every element $aq^r(q^m)^n$ $(0 \leq r \leq m-1,\ n \in \mathbb{N})$ belongs to one of these balls: $|aq^r(q^m)^n - aq^r| = |aq^r|\,|(q^m)^n - 1| \leq p^{-k_0}$ (Lemma 2).

Since V_q is the closure of $\{aq^n | n = 0, 1, 2, \dots\} = \{aq^r(q^m)^n | 0 \leq r \leq m-1,\ n \in \mathbb{N}\}$, we have that $\bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\} \supset V_q$.

The m balls $\{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$ are pairwise disjoint: take r, $s \in \{0, 1, \dots, m-1\}$, $r \neq s$ e.g. $r > s$, then $|aq^r - aq^s| = |aq^s|\,|q^{r-s} - 1| = 1$.

We remark that it is impossible to take balls with a smaller radius: $|aq^r - aq^r q^m| = |aq^r|\,|1 - q^m| = p^{-k_0}$.

Let r be fixed: $\{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$.

We take the following p elements of V_q : $aq^r(q^m)^0$, $aq^r(q^m)^1, \dots, aq^r(q^m)^{p-1}$.

Each of these elements belongs to $\{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$: $|aq^r(q^m)^i - aq^r| = |aq^r|\,|(q^m)^i - 1| \leq p^{-k_0}$ by Lemma 2 $(0 \leq i \leq p-1)$.

Furthermore, if $i, j \in \{0, 1, \dots, p-1\}$, $i \neq j$, say $i > j$, then $|aq^r(q^m)^i - aq^r(q^m)^j| = |aq^r(q^m)^j|\,|(q^m)^{i-j} - 1| = p^{-k_0}$ by Lemma 2 since $0 < i - j \leq p-1$.

So these p elements define p disjoint balls with radius $p^{-(k_0+1)}$ which cover $\{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$.

We take $\{x \in \mathbb{Z}_p | \, |x - aq^r(q^m)^i| \leq p^{-(k_0+1)}\}$, $i \in \{0, 1, \ldots, p-1\}$, i fixed.

Take the p elements $aq^r(q^m)^{i+jp}$, $0 \leq j \leq p-1$.

These elements belong to $\{x \in \mathbb{Z}_p | \, |x - aq^r(q^m)^i| \leq p^{-(k_0+1)}\}$: $|aq^r(q^m)^{i+jp} - aq^r(q^m)^i| = |aq^r(q^m)^i| \, |(q^m)^{jp} - 1| \leq p^{-(k_0+1)}$ by Lemma 2 $(0 \leq j \leq p-1)$.

Furthermore, if $j, k \in \{0, 1, \ldots, p-1\}$, $k \neq j$, say $k > j$, then $|aq^r(q^m)^{i+kp} - aq^r(q^m)^{i+jp}| = |aq^r(q^m)^{i+jp}| \, |(q^m)^{(k-j)p} - 1| = p^{-(k_0+1)}$ by Lemma 2 since $0 < k - j \leq p-1$. So these p elements define p disjoint balls with radius $p^{-(k_0+2)}$ which cover $\{x \in \mathbb{Z}_p | \, |x - aq^r(q^m)^i| \leq p^{-(k_0+1)}\}$.

We can continue this way.

Suppose we have $\{x \in \mathbb{Z}_p | \, |x - aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n}| \leq p^{-(k_0+n+1)}\}$, $i_0, i_1, \ldots, i_n \in \{0, 1, \ldots, p-1\}$, $i_0, i_1, \ldots, i_n$ fixed.

We take the p elements $aq^r(q^m)^{i_0+i_1p+i_np^n+i_{n+1}p^{n+1}}$, $0 \leq i_{n+1} \leq p-1$.

All these elements belong to $\{x \in \mathbb{Z}_p | \, |x - aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n}| \leq p^{-(k_0+n+1)}\}$:

$$|aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n+i_{n+1}p^{n+1}} - aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n}|$$
$$= |aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n}| \, |aq^r(q^m)^{i_{n+1}p^{n+1}} - 1| \leq p^{-(k_0+n+1)}.$$

Furthermore, if $j, k \in \{0, 1, \ldots, p-1\}$, $k \neq j$, say $k > j$, then

$$|aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n+kp^{n+1}} - aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n+jp^{n+1}}|$$
$$= |aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n+jp^{n+1}}| \, |(q^m)^{(k-j)p^{n+1}} - 1| = p^{-(k_0+n+1)}.$$

So these p elements define p disjoint balls with radius $p^{-(k_0+n+2)}$ which cover $\{x \in \mathbb{Z}_p | \, |x - aq^r(q^m)^{i_0+i_1p+\cdots+i_np^n}| \leq p^{-(k_0+n+1)}\}$.

Continuing this way, we find closed balls with radius tending to zero and whose centers are elements of $\{aq^n | n = 0, 1, 2, \ldots\}$, and these balls cover $\bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p | \, |x - aq^r| \leq p^{-k_0}\}$. So $\bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p | \, |x - aq^r| \leq p^{-k_0}\}$ is the closure of $\{aq^n | n = 0, 1, 2, \ldots\}$. But this means that $V_q = \bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p | \, |x - aq^r| \leq p^{-k_0}\}$. ■

Lemma 5.

Let $q \equiv 3$ (mod 4).

Then $V_q = \{x \in \mathbb{Z}_2 | \, |x - a| \leq 2^{-(N+3)}\} \cup \{x \in \mathbb{Z}_2 | \, |x - aq| \leq 2^{-(N+3)}\}$, where $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$.

Proof:

Every element aq^n belongs to $\{x \in \mathbb{Z}_2 \,|\, |x-a| \leq 2^{-(N+3)}\} \cup \{x \in \mathbb{Z}_2 \,|\, |x-aq| \leq 2^{-(N+3)}\}$: $|aq^{0+2k}-a| = |a|\,|q^{2k}-1| \leq 2^{-(N+3)}$ and $|aq^{1+2k}-aq| = |aq|\,|q^{2k}-1| \leq 2^{-(N+3)}$ by Lemma 3 ($k \in \mathbb{N}$). Since V_q is the closure of $\{aq^n | n = 0,1,2,\dots\}$, we have that $\{x \in \mathbb{Z}_2 \,|\, |x-a| \leq 2^{-(N+3)}\} \cup \{x \in \mathbb{Z}_2 \,|\, |x-aq| \leq 2^{-(N+3)}\} \supset V_q$.

The balls $\{x \in \mathbb{Z}_2 \,|\, |x-a| \leq 2^{-(N+3)}\}$ and $\{x \in \mathbb{Z}_2 \,|\, |x-aq| \leq 2^{-(N+3)}\}$ are disjoint: $|aq-a| = |a|\,|q-1| = 2^{-1}$.

We remark that it is impossible to take balls with a smaller radius: $|aq^2-a| = |a|\,|q^2-1| = 2^{-(N+3)}$ and $|aq^{1+2}-aq| = |aq|\,|q^2-1| = 2^{-(N+3)}$ (Lemma 3).

From now on, we can prove the lemma in an analogous way as Lemma 4. ■

We will need these lemmas in the sequel.

4. Normal bases for $C(V_q \to K)$

We want to give a theorem analogous to Theorem 3, but with $C(M \to K)$ replaced by $C(V_q \to K)$.

Therefore, we need some notations.

We introduce the following:

$[n]! = [n][n-1]\dots[1]$, $[0]! = 1$, where $[n] = \frac{q^n-1}{q-1}$ if $n \geq 1$.

$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ if $n \geq k$, $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $n < k$.

$(x-a)^{(n)} = (x-a)(x-aq)\dots(x-aq^{n-1})$ if $n \geq 1$, $(x-a)^{(0)} = 1$.

$\begin{Bmatrix} x \\ k \end{Bmatrix} = \frac{(x/a-1)(x/(aq)-1)\dots(x/(aq^{k-1})-1)}{(q^k-1)\dots(q-1)}$ if $k \geq 1$, $\begin{Bmatrix} x \\ 0 \end{Bmatrix} = 1$.

Lemma 6.

i) $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}$.

ii) $\begin{bmatrix} n \\ k \end{bmatrix}$ *is a polynomial in* q.

iii) $\left|\begin{bmatrix} n \\ k \end{bmatrix}\right| \leq 1$.

Proof:

i) follows immediately from the definition, ii) and iii) follow from i). ■

The polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$ are the Gauss-polynomials.

We will need the following properties of these symbols: $\left\| \begin{Bmatrix} x \\ k \end{Bmatrix} \right\| = 1$, since $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} x \\ k \end{Bmatrix}$ if $x = aq^n$, $\left| \begin{bmatrix} n \\ k \end{bmatrix} \right| \leq 1$ for all n, k in $\mathbb{N}$, $\begin{Bmatrix} aq^k \\ k \end{Bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = 1$ and since $\begin{Bmatrix} x \\ k \end{Bmatrix}$ is continuous. $\frac{(x-a)^{(n)}}{[n]!} = \begin{Bmatrix} x \\ n \end{Bmatrix} (q-1)^n q^{n(n-1)/2} a^n$, so $\left\| \frac{(x-a)^{(n)}}{[n]!} \right\| = |(q-1)^n|$.

Definition.

If $f : V_q \to K$ then we define the operator D_q as follows:

$$(D_q f)(x) = \frac{f(qx) - f(x)}{x(q-1)}.$$

The following properties are easily verified:

$$\begin{array}{ll} D_q^j x^k = [k][k-1]\dots[k-j+1]x^{k-j} & \text{if } k \geq j \geq 1, \\ D_q^j x^k = 0 & \text{if } k < j \\ D_q^j (x-y)^{(k)} = [k][k-1]\dots[k-j+1](x-y)^{(k-j)} & \text{if } k \geq j \geq 1, \\ D_q^j (x-y)^{(k)} = 0 & \text{if } j > k. \end{array}$$

Lemma 7.

$$x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} (x-a)^{(k)}.$$

Proof:

We know we can write x^n as $x^n = \sum_{k=0}^{n} a_k (x-a)^{(k)}$. Since $D_q x^n = [n] x^{n-1}$ and if we apply the operator D_q k times and put $x = a$, we find $[n][n-1]\dots[n-k+1]a^{n-k} = a_k [k]!$. ■

Lemma 7 and it's proof can also be found in [5, p. 121].

Lemma 8.

Take an injective sequence (u_n) in V_q and define

$$q_n(x) = \frac{(x-u_0)\dots(x-u_{n-1})}{(u_n-u_0)\dots(u_n-u_{n-1})} \text{ for } n \geq 1, \qquad q_0(x) = 1.$$

Then (q_n) forms a normal basis for $C(V_q \to K)$ if and only if $\|q_n\| = 1$ for all n.

Proof:

Put $M = V_q$ in Theorem 1. ■

Corollary.
$\left(\left\{ \begin{matrix} x \\ n \end{matrix} \right\}\right)$ *forms a normal basis for* $C(V_q \to K)$.

Proof:
Put $u_n = aq^n$. ■

Theorem 4.

Let $(p_n(x))$ *be a polynomial sequence in* $K[x]$ *which forms a normal basis for* $C(V_q \to K)$, *and let* $(r_n(x))$ *be a polynomial sequence such that* $r_n(x) = \sum_{j=0}^{n} c_{n;j} p_j(x) = \sum_{j=0}^{n} b_{n;j} x^j$, $c_{n;j}$, $b_{n;j} \in K$, $c_{n;n} \neq 0$, $b_{n;n} \neq 0$.

Then the following are equivalent:

i) $(r_n(x))$ *forms a normal basis for* $C(V_q \to K)$.
ii) $\|r_n\| = 1$, $|c_{n;n}| = 1$.
iii) $|c_{n;j}| \leq 1$, $|c_{n;n}| = 1$.
iv) $\|r_n\| = 1$, $|b_{n;n}| = \frac{1}{|[n]!(q-1)^n|}$.
v) $\left| \sum_{j=k}^{n} b_{n;j} \left[\begin{matrix} j \\ k \end{matrix} \right] a^j \right| \leq \frac{1}{|[k]!(q-1)^k|}$, $|b_{n;n}| = \frac{1}{|[n]!(q-1)^n|}$.

Proof:
i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) follows from Theorem 3, by putting $M = V_q$, ii) $\Leftrightarrow$ iv) and iii) $\Leftrightarrow$ v) follow from Lemma 7, by putting $p_n(x) = \left\{ \begin{matrix} x \\ n \end{matrix} \right\}$. ■

Some examples.

1) Put $p_n(x) = \left\{ \begin{matrix} qx \\ n \end{matrix} \right\}$.

   Then the sequence $(p_n(x))$ forms a normal basis of $C(V_q \to K)$: apply iv).

2) If the polynomial sequence $(p_n(x))$ forms a normal basis of $C(V_q \to K)$, then so does $(p_n(qx))$: If $p_n(x) = \sum_{j=0}^{n} c_{n;j} \left\{ \begin{matrix} x \\ j \end{matrix} \right\}$, then $p_n(qx) = \sum_{j=0}^{n} c_{n;j} \left\{ \begin{matrix} qx \\ j \end{matrix} \right\}$.

   Use example i) and apply iii).

3) If the sequence $(p_n(x))$ forms a normal basis of $C(V_q \to K)$, then so does $(p_n(q^k x))$ where k is a fixed natural number: use Example 2).

4) If the sequence $(p_n(x))$ forms a normal basis of $C(V_q \to K)$, then so does $(p_n(q^{k_n}x))$, where (k_n) is a sequence in $\mathbb{N}$: use Example 3).
5) If the sequence $(p_n(x))$ forms a normal basis of $C(V_q \to K)$, then so does $(r_n(x))$, where $r_n(x) = p_0(x) + p_1(x) + \cdots + p_n(x)$: apply iii).
6) If the polynomial sequence $(p_n(x))$ forms a normal basis of $C(V_q \to K)$, then so does $((q-1)^j D_q^j p_n(x))_{n \geq j}$, $j \in \mathbb{N}$, j fixed: apply iii).

To end this chapter, we give the valuation of $|b_{n;n}| = \frac{1}{|[n]!(q-1)^n|}$.

If n is different from zero, then $\frac{1}{|[n]!(q-1)^n|} = \frac{1}{|(q^n-1)(q^{n-1}-1)\ldots(q-1)|}$ and this leads us to the following lemma:

Lemma 9.

Take $m \geq 1$, m the smallest integer such that $q^m \equiv 1$ (mod p).

i) *If $q^m \equiv 1$ (mod p^r), $q^m \not\equiv 1$ (mod p^{r+1}) $(r \geq 1)$, and $(p,r) \neq (2,1)$ then*

$$|(q^k-1)(q^{k-1}-1)\ldots(q-1)| = p^{-[k/m]r}|[k/m]!|$$

where $[x] = \max\{k \in \mathbb{Z} | k \leq x\}$.

ii) *If $q \equiv 3$ (mod 4), where*

$$q = 1 + 2 + 2^2\varepsilon,$$
$$\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots, \varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1, \varepsilon_N = 0$$

then

$$\begin{aligned} &|(q^k-1)(q^{k-1}-1)\ldots(q-1)| \\ &= 2^{-2k}2^{-Nk/2}|(k/2)!| \qquad \text{if } k \text{ is even,} \\ &= 2^{(-kN-4k+N+2)/2}2^{-Nk/2}|((k-1)/2)!| \quad \text{if } k \text{ is odd.} \end{aligned}$$

We remark that (see [4, Section 25.5]) $|j!| = p^{-\lambda(j)}$ with $\lambda(j) = \frac{j-s_j}{p-1}$,

$$j = \sum_{i=0}^{t} \gamma_i p^i, \qquad s_j = \sum_{i=0}^{t} \gamma_i.$$

Proof:

i) Suppose $q^m \equiv 1 \pmod{p^r}$, $q^m \not\equiv 1 \pmod{p^{r+1}}$, $r \geq 1$, $(p,r) \neq (2,1)$. First, take $p \neq 2$. $q^k - 1 = q^{mj+s} - 1$ with $0 \leq s < m$.

Then $(q^m)^j \equiv 1 \pmod{p^{r+\mathrm{ord}_p j}}$, $(q^m)^j \not\equiv 1 \pmod{p^{r+1+\mathrm{ord}_p j}}$ (Lemma 2), so $(q^m)^j = 1 + \alpha p^{r+\mathrm{ord}_p j}$ $\mathrm{ord}_p \alpha = 0$.

If s is different from zero, then $q^s = \beta$ with $\beta = \beta_0 + \beta_1 p + \beta_2 p^2 + \dots$ with $\beta_0 \neq 0$, $\beta_0 \neq 1$, so $q^{mj+s} - 1 = (1 + \alpha p^{r+\mathrm{ord}_p j})\beta - 1 = \beta + \alpha\beta p^{r+\mathrm{ord}_p j} - 1$ and thus $q^{mj+s} - 1$ is a unit if s is different from zero.

Then

$$\begin{aligned}|(q^k-1)(q^{k-1}-1)\dots(q-1)| &= |((q^m)^j-1)\dots(q^m-1)| \\ &= p^{-(r+\mathrm{ord}_p j)}\dots p^{-(r+\mathrm{ord}_p 1)} \\ &= p^{-rj}|j!| = p^{-\lfloor k/m\rfloor r}|\lfloor k/m\rfloor !|.\end{aligned}$$

If p is equal to 2 then m equals one and thus

$$\begin{aligned}|(q^k-1)(q^{k-1}-1)\dots(q-1)| &= 2^{-(r+\mathrm{ord}_p k)}\dots 2^{-(r+\mathrm{ord}_p 1)} \\ &= 2^{-rk}|k!| = 2^{-\lfloor k/m\rfloor r}|\lfloor k/m\rfloor !|.\end{aligned}$$

ii) Suppose $q \equiv 3 \pmod 4$. We use Lemma 3.

If k is even then $|(q^k-1)(q^{k-1}-1)\dots(q-1)|$

$$\begin{aligned}&= 2^{-k/2}2^{-(N+2+\mathrm{ord}_2 k)}2^{-(N+2+\mathrm{ord}_2(k-2))}\dots 2^{-(N+2+\mathrm{ord}_2 2)} \\ &= 2^{-k/2}2^{-(N+2)k/2}|k|\,|k-2|\dots|2| \\ &= 2^{-2k}2^{-Nk/2}|(k/2)!|\end{aligned}$$

and if k is odd $|(q^k-1)(q^{k-1}-1)\dots(q-1)|$

$$\begin{aligned}&= 2^{-(k+1)/2}2^{-(N+2+\mathrm{ord}_2(k-1))}2^{-(N+2+\mathrm{ord}_2(k-3))}\dots 2^{-(N+2+\mathrm{ord}_2 2)} \\ &= 2^{-(k+1)/2}2^{-(N+2)(k-1)/2}2^{-(k-1)/2}|((k-1)/2)!| \\ &= 2^{(-Nk-4k+N+2)/2}|((k-1)/2)!| \text{ which proves the lemma. } \blacksquare\end{aligned}$$

5. More bases for $C(V_q \to \mathbb{Q}_p)$

We want to make new normal bases, using the basis $\left(\left\{ \begin{matrix} x \\ n \end{matrix} \right\}\right)$.

Now, if E is a non-archimedean Banach space over a non-archimedean valued field L, and E has a normal basis, then the norm of E satisfies the following condition: for each element x of E there exists ν in L such that the norm of x is equal to $|\nu|$. Y. Amice ([**2**, p. 82]) calls this condition (N).

So, if we want to make more normal bases for $C(V_q \to \mathbb{Q}_p)$ we can use the following result ([2, p. 82, Prop. 3.1.5]):

Let E be a Banach space over a non-archimedean valued field L. If L has a discrete valuation and if E satisfies condition (N), then for a family (e_n) of E for which $\|e_n\| \leq 1$ for all n the following are equivalent:

i) (e_n) is a normal basis of E,

ii) $(p(e_n))$ is a basis of the vector space $\bar{E}$.

where $E_0 = \{x \in E | \, \|e\| \leq 1\}$, $E_0' = \{x \in E | \, \|e\| < 1\}$, $\bar{E} = E_0/E_0'$ and p is the canonical projection of E_0 on $\bar{E}$.

Since the valuation of L has to be discrete, we use this result to find normal bases for $C(V_q \to \mathbb{Q}_p)$. We start with some lemmas.

Lemma 10.

$$\begin{bmatrix} i+j \\ n \end{bmatrix} = \sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ n-k \end{bmatrix} q^{-k(-j+n-k)}.$$

Proof:

If n is zero or $i+j$ is strictly smaller then n, then the lemma surely holds.

From now on we suppose $i+j$ greater than n.

If $i+j$ is equal to n then $\begin{bmatrix} i+j \\ n \end{bmatrix}$ is equal to one and $\sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ n-k \end{bmatrix} q^{-k(-j+n-k)} = \sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} n-i \\ n-k \end{bmatrix} q^{-k(i-k)} = 1$ since the only term different from zero is the term where k equals i.

From now on we proceed by (double) induction.

$$\begin{bmatrix} i+j \\ n \end{bmatrix} = \begin{bmatrix} i+j-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} i+j-1 \\ n \end{bmatrix} \text{ (by Lemma 6)}$$

$$= \sum_{k=0}^{n-1} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j-1 \\ n-k-1 \end{bmatrix} q^{-k(-j+n-k)}$$

$$+ q^n \sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j-1 \\ n-k \end{bmatrix} q^{-k(-j+1+n-k)}$$

(by the induction hypothesis)

$$= \begin{bmatrix} i \\ n \end{bmatrix} q^{nj} + \sum_{k=0}^{n-1} \begin{bmatrix} i \\ k \end{bmatrix} q^{-k(-j+n-k)} \left(\begin{bmatrix} j-1 \\ n-k-1 \end{bmatrix} + \begin{bmatrix} j-1 \\ n-k \end{bmatrix} q^{n-k} \right)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \begin{bmatrix} j \\ n-k \end{bmatrix} q^{-k(-j+n-k)} \text{ (by Lemma 6). } \blacksquare$$

Lemma 11.

Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ *with* $(p, k_0) \neq (2, 1)$.

If $x, y \in V_q$, $|x - y| \leq p^{-(k_0+t)}$ *then* $\left|\left\{ \begin{matrix} x \\ n \end{matrix} \right\}^s - \left\{ \begin{matrix} y \\ n \end{matrix} \right\}^s\right| \leq 1/p$, *where* $s \in \mathbb{N}$, $0 \leq n < mp^t$.

Proof:

The lemma holds if s is equal to zero.

If $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ with $(p, k_0) \neq (2, 1)$, we then have $V_q = \bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p \,|\, |x - aq^r| \leq p^{-k_0}\}$ (Lemma 4).

So V_q is the union of m disjoint balls with radius p^{-k_0}.

By the proof of Lemma 4, we have that V_q is the union of mp^t disjoint balls with radius $p^{-(k_0+t)}$ and with centers $aq^r(q^m)^k$, $0 \leq r \leq m-1$, $0 \leq k < p^t$.

Take $x, y \in \{aq^j | j = 0, 1, 2, \ldots\}$ with $|x - y| \leq p^{-(k_0+t)}$. Then, by Lemmas 2 and 4, there exist natural numbers n_x and n_y such that $x = aq^r(q^m)^{n_x}$ and $y = aq^r(q^m)^{n_y}$ with $|n_x - n_y| \leq p^{-t}$ $(n_x, n_y \in \mathbb{N})$.

Then

$$\left|\left\{ \begin{matrix} x \\ n \end{matrix} \right\} - \left\{ \begin{matrix} y \\ n \end{matrix} \right\}\right| = \left|\left[\begin{matrix} r + mn_x \\ n \end{matrix} \right] - \left[\begin{matrix} r + mn_y \\ n \end{matrix} \right]\right|.$$

Further,

$$\begin{aligned}
\left[\begin{matrix} r + mn_x \\ n \end{matrix} \right] &= \left[\begin{matrix} m(n_x - n_y) + r + mn_y \\ n \end{matrix} \right] \\
&= \sum_{k=0}^{n} \left[\begin{matrix} m(n_x - n_y) \\ k \end{matrix} \right] \left[\begin{matrix} r + mn_y \\ n - k \end{matrix} \right] q^{-k(-(r+mn_y)+n-k)} \\
&\qquad \text{(Lemma 10 if } n_x \geq n_y) \\
&= \left[\begin{matrix} r + mn_y \\ n \end{matrix} \right] \\
&\quad + \sum_{k=1}^{n} \left[\begin{matrix} m(n_x - n_y) \\ k \end{matrix} \right] \left[\begin{matrix} r + mn_y \\ n - k \end{matrix} \right] q^{-k(-(r+mn_y)+n-k)} \\
&\qquad (n \geq 1).
\end{aligned}$$

Since $\left[\begin{matrix} i \\ j \end{matrix} \right] = \frac{[i]}{[j]} \left[\begin{matrix} i-1 \\ j-1 \end{matrix} \right]$ $(i \geq j \geq 1)$ we have $\left|\left[\begin{matrix} i \\ j \end{matrix} \right]\right| \leq \left|\frac{[i]}{[j]}\right|$ $(i \geq 0,\ j \geq 1)$, so $\left|\left[\begin{matrix} m(n_x - n_y) \\ k \end{matrix} \right]\right| \leq \left|\frac{[m(n_x - n_y)]}{[k]}\right| = \left|\frac{(q^m)^{n_x - n_y} - 1}{q^k - 1}\right| \leq 1/p$ by Lemma 2 since $1 \leq k \leq n < mp^t$ and $|n_x - n_y| \leq p^{-t}$.

Then

$$\left|\begin{bmatrix} r+mn_x \\ n \end{bmatrix} - \begin{bmatrix} r+mn_y \\ n \end{bmatrix}\right|$$
$$\leq \max_{1\leq k\leq n}\left\{\left|\begin{bmatrix} m(n_x-n_y) \\ k \end{bmatrix}\begin{bmatrix} r+mn_y \\ n-k \end{bmatrix} q^{-k(-(r+m_y)+n-k)}\right|\right\} \leq 1/p.$$

So $\left|\begin{Bmatrix} x \\ n \end{Bmatrix} - \begin{Bmatrix} y \\ n \end{Bmatrix}\right| \leq 1/p$ and this also holds if n is zero.

Finally, if s is greater then one,

$$\left|\begin{Bmatrix} x \\ n \end{Bmatrix}^s - \begin{Bmatrix} y \\ n \end{Bmatrix}^s\right| = \left|\begin{Bmatrix} x \\ n \end{Bmatrix} - \begin{Bmatrix} y \\ n \end{Bmatrix}\right| \left|\sum_{i=0}^{s-1} \begin{Bmatrix} x \\ n \end{Bmatrix}^i \begin{Bmatrix} y \\ n \end{Bmatrix}^{s-1-i}\right| \leq 1/p. \quad \blacksquare$$

The lemma follows by continuity.

Lemma 12.
Let $q \equiv 3$ (mod 4),

$$q = 1 + 2 + 2^2\varepsilon$$
$$\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots,\ \varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1,\ \varepsilon_N = 0.$$

If $x, y \in V_q$, $|x-y| \leq p^{-(N+2+t)}$ then $\left|\begin{Bmatrix} x \\ n \end{Bmatrix}^s - \begin{Bmatrix} y \\ n \end{Bmatrix}^s\right| \leq 1/2$, where $s \in \mathbb{N}$, $0 \leq n < 2^t$ $(t \geq 1)$.

Proof:

The lemma holds if s is equal to zero.

$V_q = \{x \in \mathbb{Z}_2 | \, |x-a| \leq 2^{-(N+3)}\} \cup \{x \in \mathbb{Z}_2 | \, |x-aq| \leq 2^{-(N+3)}\}$, by Lemma 5.

By the proof of Lemma 5, we have that V_q is the union of 2^t disjoint balls with radius $2^{-(N+2+t)}$ and with centers aq^n, $0 \leq n < 2^t$ $(t \geq 1)$.

Take $x, y \in \{aq^j | j = 0, 1, 2, \ldots\}$ with $|x-y| \leq 2^{-(N+2+t)}$. Then, by Lemmas 3 and 5 we must have that $x = aq^{n_x}$ and $y = aq^{n_y}$ with $|n_x - n_y| \leq 2^{-t}$ $(n_x, n_y \in \mathbb{N})$. Then

$$\left|\begin{Bmatrix} x \\ n \end{Bmatrix} - \begin{Bmatrix} y \\ n \end{Bmatrix}\right| = \left|\begin{bmatrix} n_x \\ n \end{bmatrix} - \begin{bmatrix} n_y \\ n \end{bmatrix}\right|.$$

Further,

$$\begin{bmatrix} n_x \\ n \end{bmatrix} = \begin{bmatrix} (n_x - n_y) + n_y \\ n \end{bmatrix}$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n_x - n_y \\ k \end{bmatrix} \begin{bmatrix} n_y \\ n-k \end{bmatrix} q^{-k(-n_y+n-k)} \text{ (Lemma 10 if } n_x \geq n_y)$$
$$= \begin{bmatrix} n_y \\ n \end{bmatrix} + \sum_{k=1}^{n} \begin{bmatrix} n_x - n_y \\ k \end{bmatrix} \begin{bmatrix} n_y \\ n-k \end{bmatrix} q^{-k(-n_y+n-k)} \, (n \geq 1)$$

Since $\begin{bmatrix} i \\ j \end{bmatrix} = \frac{[i]}{[j]} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}$ $(i \geq j \geq 1)$ we have $\left|\begin{bmatrix} i \\ j \end{bmatrix}\right| \leq \left|\frac{[i]}{[j]}\right|$ $(i \geq 0, j \geq 1)$, so $\left|\begin{bmatrix} n_x - n_y \\ k \end{bmatrix}\right| \leq \left|\frac{[n_x - n_y]}{[k]}\right| = \left|\frac{q^{n_x - n_y} - 1}{q^k - 1}\right| \leq 1/2$ by Lemma 3 since $1 \leq k \leq n < 2^t$ and $|n_x - n_y| \leq 2^{-t}$.

Then

$$\left|\begin{bmatrix} n_x \\ n \end{bmatrix} - \begin{bmatrix} n_y \\ n \end{bmatrix}\right| \leq \max_{1 \leq k \leq n} \left\{ \left| \begin{bmatrix} n_x - n_y \\ k \end{bmatrix} \begin{bmatrix} n_y \\ n-k \end{bmatrix} q^{-k(-n_y+n-k)} \right| \right\} \leq 1/2.$$

So $\left|\begin{Bmatrix} x \\ n \end{Bmatrix} - \begin{Bmatrix} y \\ n \end{Bmatrix}\right| \leq 1/2$ and this also holds if n is zero.

Finally, if s is greater then one,

$$\left|\begin{Bmatrix} x \\ n \end{Bmatrix}^s - \begin{Bmatrix} y \\ n \end{Bmatrix}^s\right| = \left|\begin{Bmatrix} x \\ n \end{Bmatrix} - \begin{Bmatrix} y \\ n \end{Bmatrix}\right| \left| \sum_{i=0}^{s-1} \begin{Bmatrix} x \\ n \end{Bmatrix}^i \begin{Bmatrix} y \\ n \end{Bmatrix}^{s-1-i} \right| \leq 1/2.$$

The lemma follows by continuity. ■

Since $C(V_q \to \mathbb{Q}_p)$ has a normal basis, its norm satisfies condition (N), and so we can use [**2**, p. 82, Prop. 3.1.5] to prove the following:

Theorem 5.

Let (j_n) be a sequence in $\mathbb{N}_0$. Then the sequence of polynomials $\left(\begin{Bmatrix} x \\ n \end{Bmatrix}^{j_n}\right)$ forms a normal basis for $C(V_q \to \mathbb{Q}_p)$.

Proof:

This proof is analogous to the proof of Theorem 1.1 in [**3**].

By [**2**, Proposition 3.1.5, p. 82] it suffices to prove that $\left(\overline{\begin{Bmatrix} x \\ n \end{Bmatrix}^{j_n}}\right)$ forms a vectorial basis of $C(V_q \to \mathbb{F}_p)$.

We distinguish two cases.

If $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ with $(p,k_0) \neq (2,1)$, define C_t the space of the functions from V_q to $\mathbb{F}_p$ constant on balls of the type $\{x \in \mathbb{Z}_p \,|\, |x-\alpha| \leq p^{-(k_0+t)}\}$, $\alpha \in V_q$. Since $C(V_q \to \mathbb{F}_p) = \cup_{t\geq 0} C_t$ it suffices to prove that $\left(\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} | n < mp^t\right)$ forms a basis of C_t. By the proof of Lemma 4, we can write V_q as the union of mp^t disjoint balls with radius $p^{-(k_0+t)}$ and with centers $aq^r(q^m)^n$, $0 \leq r \leq m-1$, $0 \leq n < p^t$. Let χ_i be the characteristic function of the ball with center aq^i. Using Lemma 11, we have

$$\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} = \sum_{i=0}^{mp^t-1} \chi_i(x) \overline{\left\{ \begin{matrix} aq^i \\ n \end{matrix} \right\}}^{j_n} = \sum_{i=n}^{mp^t-1} \chi_i(x) \overline{\left\{ \begin{matrix} aq^i \\ n \end{matrix} \right\}}^{j_n},$$

hence the transition matrix from $(\chi_n | n < mp^t)$ to $\left(\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} | n < mp^t\right)$ is triangular, so $\left(\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} | n < mp^t\right)$ forms a basis of C_t.

If $q \equiv 3 \pmod 4$, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$, define C_t the space of the functions from V_q to $\mathbb{F}_2$ constant on balls of the type $\{x \in \mathbb{Z}_2 \,|\, |x-\alpha| \leq 2^{-(N+2+t)}\}$, $\alpha \in V_q$.

Since $C(V_q \to \mathbb{F}_2) = \cup_{t\geq 1} C_t$ it suffices to prove that $\left(\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} | n < 2^t\right)$ forms a basis of C_t. By the proof of Lemma 5, we can write V_q as the union of 2^t disjoint balls with radius $2^{-(N+2+t)}$ and with centers aq^n, $0 \leq n < 2^t$. Let χ_i be the characteristic function of the ball with center aq^i. Using Lemma 12 we have

$$\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} = \sum_{i=0}^{2^t-1} \chi_i(x) \overline{\left\{ \begin{matrix} aq^i \\ n \end{matrix} \right\}}^{j_n} = \sum_{i=n}^{2^t-1} \chi_i(x) \overline{\left\{ \begin{matrix} aq^i \\ n \end{matrix} \right\}}^{j_n},$$

hence the transition matrix from $(\chi_n | n < 2^t)$ to $\left(\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} | n < 2^t\right)$ is triangular, so $\left(\overline{\left\{ \begin{matrix} x \\ n \end{matrix} \right\}}^{j_n} | n < 2^t\right)$ forms a basis of C_t. This proves the theorem. ■

6. Extension to $C(V_q \to K)$

Let K be as in Chapters 3 and 4. We want to show that $\left(\left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right)$ forms a normal basis for $C(V_q \to K)$. To prove this, we need the results from Section 5. We remark that the valuation of K does not have to be discrete, as was the case in Section 5.

Theorem 6.

Let (j_n) be a sequence in $\mathbb{N}_0$. Then $\left(\left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right)$ forms a normal basis for $C(V_q \to K)$.

Proof:

It is clear that $\left\|\left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right\| = 1$.

We now prove the orthogonality of the sequence. Let n be in $\mathbb{N}$, $\alpha_0, \ldots, \alpha_n$ in K. We prove $\left\|\alpha_0 \left\{\begin{matrix} x \\ 0 \end{matrix}\right\}^{j_0} + \cdots + \alpha_n \left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right\| = \max\limits_{0 \le i \le n} \{|\alpha_i|\}$.

It is clear that $\left\|\alpha_0 \left\{\begin{matrix} x \\ 0 \end{matrix}\right\}^{j_0} + \cdots + \alpha_n \left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right\| \le \max\limits_{0 \le i \le n} \{|\alpha_i|\}$.

Put $M = \max\limits_{0 \le i \le n} \{|\alpha_i|\}$, $N = \min\{i | 0 \le i \le n \text{ and } |\alpha_i| = M\}$. Then

$$\left|\alpha_0 \left\{\begin{matrix} aq^N \\ 0 \end{matrix}\right\}^{j_0} + \cdots + \alpha_n \left\{\begin{matrix} aq^N \\ n \end{matrix}\right\}^{j_n}\right|$$

$$= \max\left\{\left|\alpha_0 \left\{\begin{matrix} aq^N \\ 0 \end{matrix}\right\}^{j_0} + \cdots + \alpha_{N-1} \left\{\begin{matrix} aq^N \\ N-1 \end{matrix}\right\}^{j_{N-1}}\right|, \left|\alpha_N \left\{\begin{matrix} aq^N \\ N \end{matrix}\right\}^{j_N}\right|\right\}$$

$$= \left|\alpha_N \left\{\begin{matrix} aq^N \\ N \end{matrix}\right\}^{j_N}\right| = M$$

since $\left|\alpha_0 \left\{\begin{matrix} aq^N \\ 0 \end{matrix}\right\}^{j_0} + \cdots + \alpha_{N-1} \left\{\begin{matrix} aq^N \\ N-1 \end{matrix}\right\}^{j_{N-1}}\right| < M$, $\left|\alpha_N \left\{\begin{matrix} aq^N \\ N \end{matrix}\right\}^{j_N}\right| = M$. So $\left\|\alpha_0 \left\{\begin{matrix} x \\ 0 \end{matrix}\right\}^{j_0} + \cdots + \alpha_n \left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right\| = \max\limits_{0 \le i \le n} \{|\alpha_i|\}$.

Finally, we prove that the sequence forms a basis.

By [6, p. 165, Lemma 5.1] and by Kaplansky's Theorem (see e.g. [6, p. 191, Theorem 5.28]), it suffices to prove that the k linear span of the polynomials $\left(\left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right)$ contains $K[x]$. Since each $\left\{\begin{matrix} x \\ k \end{matrix}\right\}$ is an element of $\mathbb{Q}_p[x]$ and $\left(\left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right)$ forms a normal basis of $C(V_q \to \mathbb{Q}_p)$ (Theorem 5), we can write $\left\{\begin{matrix} x \\ k \end{matrix}\right\}$ as a uniformly convergent expansion $\left\{\begin{matrix} x \\ k \end{matrix}\right\} = \sum_{n=0}^{\infty} a_n q_n(x)^{j_n}$. So if $\alpha_0, \alpha_1, \ldots, \alpha_n$ are elements of K then there exists coefficients $d_n^{(j_n)}$ in K such that $\sum_{i=0}^{n} \alpha_i \left\{\begin{matrix} x \\ i \end{matrix}\right\} = \sum_{n=0}^{\infty} d_n^{(j_n)} \left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}$ where the right-hand-side is uniformly convergent.

Let p be an element of $K[x]$. By the previous remark there exist coefficients $c_n^{(j_n)}$ such that $p(x) = \sum_{i=0}^{N} \beta_i \left\{\begin{matrix} x \\ n \end{matrix}\right\} = \sum_{n=0}^{\infty} c_n^{(j_n)} \left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}$. So the k linear span of the polynomials $\left(\left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}\right)$ contains $K[x]$. This finishes the proof. ■

If f is an element of $C(V_q \to K)$, there exist coefficients $(b_n^{(j_n)})$ such that $f(x) = \sum_{n=0}^{\infty} b_n^{(j_n)} \left\{\begin{matrix} x \\ n \end{matrix}\right\}^{j_n}$ where the series on the right-hand-side is uniformly convergent. In some cases it is possible to give an expression for the coefficients:

Proposition 1.

Let s be in $\mathbb{N}_0$. Then each continuous function $f : V_q \to K$ can be written as a uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} b_n^{(s)} \left\{\begin{matrix} x \\ n \end{matrix}\right\} \quad \textit{with } \|f\| = \max_{n \geq 0}\{|b_n^{(s)}|\}$$

where

$$b_n^{(s)} = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}^s f(aq^k) \beta_{n-k}^{(s)} \tag{2}$$

and $\beta_0^{(s)} = 1$, $\beta_m^{(s)} = \sum_{\substack{(j_1,\ldots,j_r) \\ \sum j_i = m;\ 1 \leq j_i \leq m}} (-1)^{r+m} \begin{bmatrix} m \\ j_1 \ldots j_r \end{bmatrix}^s$, $\begin{bmatrix} m \\ j_1 \ldots j_r \end{bmatrix} = \frac{[m]!}{[j_1]! \ldots [j_r]!}$.

Proof:

The proof is equal to the proof of Corollary 1.2 in [**3**].

$\|f\| = \max_{n\geq 0}\{|b_n^{(s)}|\}$ follows from the fact that $\left(\left\{ \begin{matrix} x \\ n \end{matrix} \right\}^s\right)$ forms a normal basis. If $f(x) = \sum_{n=0}^{\infty} b_n^{(s)} \left\{ \begin{matrix} x \\ n \end{matrix} \right\}^s$ then $f(aq^k) = \sum_{n=0}^{k} b_n^{(s)} \left[\begin{matrix} k \\ n \end{matrix} \right]^s$, and so $b_0^{(s)} = f(a)$, $b_k^{(s)} = f(aq^k) - \sum_{n=0}^{k-1} b_n^{(s)} \left[\begin{matrix} k \\ n \end{matrix} \right]^s$ if $k \geq 1$.

If k is equal to zero, the formulas certainly hold.

We proceed by induction. Suppose the formulas hold for $0 \leq j \leq N$.

$$
\begin{aligned}
b_{N+1}^{(s)} &= f(aq^{N+1}) - \sum_{n=0}^{N} b_n^{(s)} \left[\begin{matrix} N+1 \\ n \end{matrix} \right]^s \\
&= f(aq^{N+1}) - \sum_{n=0}^{N} \left[\begin{matrix} N+1 \\ n \end{matrix} \right]^s \sum_{k=0}^{n} \left[\begin{matrix} n \\ k \end{matrix} \right]^s (-1)^{n-k} f(aq^k) \beta_{n-k}^{(s)} \\
&= f(aq^{N+1}) - \sum_{k=0}^{N} \sum_{n=k}^{N} (-1)^{n-k} f(aq^k) \\
&\qquad \sum_{\sum_{i=1}^{r} j_i = n-k} (-1)^{r+n-k} \left[\begin{matrix} n-k \\ j_1 \dots j_r \end{matrix} \right]^s \left[\begin{matrix} n \\ k \end{matrix} \right]^s \left[\begin{matrix} N+1 \\ n \end{matrix} \right]^s \\
&= f(aq^{N+1}) + \sum_{k=0}^{N} \sum_{n=k}^{N} f(aq^k) \\
&\qquad \sum_{\sum_{i=1}^{r} j_i = n-k} (-1)^{r+1} \left(\frac{[N+1]!}{[j_1]! \dots [j_r]! [k]! [N+1-n]!} \right)^s
\end{aligned}
$$

put $j_{r+1} = N + 1 - n$

$$
\begin{aligned}
&= f(aq^{N+1}) + \sum_{k=0}^{N} \left[\begin{matrix} N+1 \\ k \end{matrix} \right]^s f(aq^k) \\
&\qquad \sum_{\sum_{i=1}^{r+1} j_i = N+1-k} (-1)^{r+1} \left[\begin{matrix} N+1-k \\ j_1 \dots j_{r+1} \end{matrix} \right]^s \\
&= f(aq^{N+1}) + \sum_{k=0}^{N} \left[\begin{matrix} N+1 \\ k \end{matrix} \right]^s f(aq^k) (-1)^{N+1-k} \beta_{N+1-k}^{(s)} \\
&= \sum_{k=0}^{N+1} \left[\begin{matrix} N+1 \\ k \end{matrix} \right]^s f(aq^k) (-1)^{N+1-k} \beta_{N+1-k}^{(s)}
\end{aligned}
$$

which proves the proposition. ■

Proposition 2.

For each $s \in \mathbb{N}_0$, the sequence of polynomials $\left(\left(\frac{(x-a)^{(n)}}{[n]!}\right)^s\right)$ form a basis of $C(V_q \to K)$. Each continuous function $f : V_q \to K$ can be written as a uniformly convergent series

$$f(x) = \sum_{n=0}^{\infty} c_n^{(s)} \left(\frac{(x-a)^{(n)}}{[n]!}\right)^s \quad \text{with } \|f\| = \max_{n\geq 0}\{|c_n^{(s)}(q-1)^{ns}|\}$$

where $c_n^{(s)} = b_n^{(s)}/((q-1)^n q^{n(n-1)/2} a^n)^s$.

Proof:

This follows from the fact that $\frac{(x-a)^{(n)}}{[n]!} = \left\{ \begin{matrix} x \\ n \end{matrix} \right\} (q-1)^n q^{n(n-1)/2} a^n$.

If we put s equal to one in Proposition 2, we find Jackson's interpolation formula for continuous functions from V_q to K ([5]). ■

An example.

We have

$$\begin{aligned}
\beta_0^{(s)} &= \beta_1^{(s)} = 1 \\
\beta_2^{(s)} &= [2]^s - 1 = (q+1)^s - 1 \\
\beta_3^{(s)} &= [3]^s[2]^s - 2[3]^s + 1 = (q^2+q+1)^s(q+1)^s - 2(q^2+q+1)^s + 1 \\
\beta_4^{(s)} &= [4]^s[3]^s[2]^s - 3[4]^s[3]^s + \frac{[4]^s[3]^s}{[2]^s} + 2[4]^s - 1 \\
&= (q^3+q^2+q+1)^s(q^2+q+1)^s(q+1)^s \\
&\quad - 3(q^3+q^2+q+1)^s(q^2+q+1)^s + (q^2+1)^s(q^2+q+1)^s \\
&\quad + 2(q^3+q^2+q+1)^s - 1
\end{aligned}$$

...

and after some calculations we find

$$\left\{ \begin{matrix} x \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} x \\ 1 \end{matrix} \right\}^2 - q(q+1)\left\{ \begin{matrix} x \\ 2 \end{matrix} \right\}^2 + (q^2+q+1)(q+1)^2q^2\left\{ \begin{matrix} x \\ 3 \end{matrix} \right\}^2$$
$$- q^3(q^3+q^2+q+1)(q^2+q+1)(q^4+3q^3+3q^2+3q+1)\left\{ \begin{matrix} x \\ 4 \end{matrix} \right\}^2 + \ldots$$

which gives us a uniformly convergent series.

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