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# R-TREES AND THE BIERI-NEUMANN-STREBEL INVARIANT

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*Abstract*

Let $G$ be a finitely generated group. We give a new characterization of its Bieri-Neumann-Strebel invariant $\Sigma(G)$, in terms of geometric abelian actions on **R**-trees. We provide a proof of Brown's characterization of $\Sigma(G)$ by exceptional abelian actions of $G$, using geometric methods.

## Introduction.

In a 1987 paper at Inventiones [**BNS**], Bieri, Neumann and Strebel associated an invariant $\Sigma = \Sigma(G)$ to any finitely generated group $G$. This invariant may be viewed as a positively homogeneous open subset of $\mathrm{Hom}(G, \mathbf{R}) \setminus \{0\}$. It contains information about finitely generated normal subgroups of $G$ with abelian quotient.

In the same issue of Inventiones [**Br**], Brown introduced HNN-valuations and related $\Sigma$ to actions of $G$ on **R**-trees. In particular a nonzero homomorphism $\chi : G \to \mathbf{R}$ is in $\Sigma \cap -\Sigma$ if and only if **R** is the only **R**-tree admitting a minimal action of $G$ with length function $|\chi|$ (see Theorem 3.2 below).

A few months earlier, also in Inventiones [**Le 1**], this author studied singular closed differential one-forms on closed manifolds $M^n$ ($n \geq 3$). We defined *complete* forms by several equivalent geometric conditions; in the simplest case, a form $\omega$ is complete if and only if every path in $M$ is homotopic to a path $\gamma$ that is either transverse to $\omega$ or tangent to $\omega$ (i.e. $\omega(\gamma'(t))$ never vanishes or is identically 0).

We proved that any form cohomologous to a complete form is also complete, so that completeness defines a subset $U(M)$ in the De Rham cohomology space $H^1_{DR}(M, \mathbf{R}) \simeq \mathrm{Hom}(\pi_1 M, \mathbf{R})$. We also proved that $U(M)$ depends only on the group $G = \pi_1 M$, and in fact $U(M)$ is nothing but $\Sigma(\pi_1 M) \cap -\Sigma(\pi_1 M)$.

In this note we use (a generalization of) closed one-forms to give a new characterization of $\Sigma(G)$, this time in terms of *geometric* actions of $G$ on $\mathbf{R}$-trees (Theorem 3.1). Assuming for simplicity that $G$ is finitely presented, we say that an action of $G$ on an $\mathbf{R}$-tree is geometric if it comes from a measured foliation on a finite 2-complex $K$ with $\pi_1 K = G$ (see [**LP**] for a complete discussion). A consequence of Theorem 3.1 is:

**Corollary.** *Let $\chi : G \to \mathbf{R}$ be a nonzero homomorphism, with $G$ finitely generated.*

(1) *There exists a geometric action of $G$ on an $\mathbf{R}$-tree with length function $\ell = |\chi|$ if and only if $\chi \in \Sigma \cup -\Sigma$.*
(2) *The action of $G$ on $\mathbf{R}$ by translations associated to $\chi$ is geometric if and only if $\chi \in \Sigma \cap -\Sigma$.*

We also give a geometric proof of Brown's theorem, by associating a natural $\mathbf{R}$-tree $T^+(f)$ to any real-valued function $f$ defined on a path-connected space (there is a similar construction in terms of *romp-trees* in [**BS**, Chapter II]).

In Part 1 we define closed one-forms relative to a homomorphism $\chi : G \to \mathbf{R}$, and we reformulate the condition $\chi \in \Sigma$ in terms of closed one-forms. In Part 2 we recall known facts about abelian actions on $\mathbf{R}$-trees (see [**CuMo**], [**Sh**]). In Part 3 we prove both characterizations of $\Sigma$ mentioned above.

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## 1. Closed one-forms.

Let $\chi : G \to \mathbf{R}$ be a homomorphism. A *closed one-form* relative to $\chi$ consists of a path-connected space $X$ equipped with an action of $G$, together with a continuous function $f : X \to \mathbf{R}$ such that

$$f(gx) = f(x) + \chi(g)$$

for all $x \in X$ and $g \in G$.

The closed one-form is *geometric* if $G$ acts as a group of covering transformations and the base $X/G$ is (homeomorphic to) a finite CW-complex. Note that this forces $G$ to be finitely generated.

**Example 1.** Let $G$ be the trivial group. Then any function on a path-connected space defines a closed one-form.

**Example 2.** Let $\omega$ be a closed differential one-form on a closed manifold $M$. Let $\chi : \pi_1 M \to \mathbf{R}$ be the homomorphism given by integrating $\omega$ along loops. Let $p : X \to M$ be the universal covering. Then any $f : X \to \mathbf{R}$ such that $df = p^*\omega$ defines a geometric closed one-form relative to $\chi$.

**Example 3.** Let $\Gamma$ be the Cayley graph of $G$ relative to some fixed generating system. Given a homomorphism $\chi : G \to \mathbf{R}$, view it as a function on the set of vertices of $\Gamma$, and extend it affinely and $G$-equivariantly to a function $f$ defined on the whole of $\Gamma$. This defines a closed one-form relative to $\chi$. It is geometric if and only if the generating system is finite.

**Example 4.** Any abelian action of $G$ on an $\mathbf{R}$-tree $T$ defines a closed one-form $f : T \to \mathbf{R}$ (see Part 2).

If $f$ is a closed one-form on $X$, we denote $X_{>c} = f^{-1}(c, +\infty)$ for $c \in \mathbf{R}$.

**Proposition 1.1.** *Let $f : X \to \mathbf{R}$ be a geometric closed one-form relative to a nonzero homomorphism $\chi : G \to \mathbf{R}$. For any $c \in \mathbf{R}$, the set $X_{>c}$ has at least one component on which $f$ is unbounded. This component is unique if and only if $\chi \in \Sigma(G)$.*

*Proof:* Since $f$ is geometric, the group $G$ acts on $X$ as a group of covering transformations. Let $X' = X/G'$, where $G'$ is the commutator subgroup of $G$. The function $f$ induces $f' : X' \to \mathbf{R}$. Let $X'_{>c} = f'^{-1}(c, +\infty)$. By [**BNS**, Part 5], there exists a unique component $A'$ of $X'_{>c}$ on which $f'$ is unbounded, and $\chi \in \Sigma(G)$ if and only if the natural map from $\pi_1 A'$ to $G'$ is onto (compare [**Le 1**, Parts IV and V] and [**Si**]). The proposition follows. ■

## 2. Abelian actions on R-trees.

Suppose a finitely generated group $G$ acts by isometries on an $\mathbf{R}$-tree $T$.

The *length function* $\ell : G \to \mathbf{R}^+$ is defined as $\ell(g) = \inf_{x \in T} d(x, gx)$. The action is *trivial* if there is a global fixed point (equivalently if $\ell \equiv 0$), *minimal* if there is no proper invariant subtree. The action (or the length function) is called *abelian* if $\ell$ is the absolute value of a nonzero homomorphism $\chi : G \to \mathbf{R}$. Two minimal actions of $G$ with the same length function $\ell$ are equivariantly isometric, except maybe if $\ell$ is abelian. Brown's theorem (Theorem 3.2 below) is concerned with this "maybe".

A nontrivial action is abelian if and only if there is a fixed end $e$. We can then define a closed one-form on $T$, as follows. Given $x \in T$, there is a unique isometric embedding $i_x : (-\infty, 0] \to T$ such that $i_x(-\infty) = e$

and $i_x(0) = x$. Fixing a basepoint $m \in T$, we define $f(x)$ as the only real number such that $i_x(t) = i_m(t + f(x))$ for $|t|$ large enough ("Busemann function"). Then $f$ is a closed one-form on $T$, relative to some nonzero homomorphism $\chi : G \to \mathbf{R}$ satisfying $\ell = |\chi|$. This homomorphism measures how much elements of $G$ push *away from* $e$.

An abelian action is called *exceptional* if there is only one fixed end $e$. We can then define $\chi$ unambiguously, and we say that the action is *associated* to $\chi$. If there are two fixed ends (i.e. if there is an invariant line), we say that the action is associated to both $\chi$ and $-\chi$.

## 3. Characterizations of $\Sigma$.

Let $f : X \to \mathbf{R}$ be continuous, with $X$ path-connected. Assume $f$ has bounded variation in the following sense: given $x, y \in X$, there exists a path $\gamma : [0,1] \to X$ from $x$ to $y$ such that $f \circ \gamma$ has bounded variation. The infimum of the total variation of $f \circ \gamma$ over all paths $\gamma$ from $x$ to $y$ then defines a pseudometric $d(x,y)$ on $X$.

We let $T(f)$ be the associated metric space: points of $T(f)$ are equivalence classes for the relation $d(x,y) = 0$. Denote $\pi : X \to T(f)$ the natural projection and $\lambda : T(f) \to \mathbf{R}$ the map such that $\lambda \circ \pi = f$.

If $f$ is a closed one-form relative to some $\chi : G \to \mathbf{R}$, there is an induced isometric action of $G$ on $T(f)$ with $\lambda(gx) = \lambda(x) + \chi(g)$. When $T(f)$ is an $\mathbf{R}$-tree, the length function $\ell$ of this action satisfies $\ell \geq |\chi|$ (since $\lambda$ does not increase distances).

**Definition.** Consider an abelian action of a finitely generated group $G$ on an $\mathbf{R}$-tree $T$, associated to $\chi : G \to \mathbf{R}$. The action is *geometric* if and only if there exists a geometric closed one-form $f : X \to \mathbf{R}$ relative to $\chi$ such that $T(f)$ is $G$-equivariantly isometric to $T$.

**Theorem 3.1.** *Let $\chi : G \to \mathbf{R}$ be a nontrivial homomorphism, with $G$ a finitely generated group. There exists a geometric abelian action of $G$ on an $\mathbf{R}$-tree associated to $\chi$ if and only if $\chi \in -\Sigma$.*

*Proof:*

Let $f : X \to \mathbf{R}$ be a geometric closed one-form relative to $\chi$. Assume that $T(f)$ is an $\mathbf{R}$-tree and the action of $G$ on $T(f)$ is abelian, associated to $\chi$. We show $\chi \in -\Sigma$.

Fix $g \in G$ with $\chi(g) < 0$, and fix $x \in X$ with, say, $f(x) = 0$. For $A$ large enough, the path component $U$ of $f^{-1}(-A, A)$ containing $x$ meets every orbit for the action of $G$ on $X$: this is because $X/G$ is a finite complex. We may also assume that $A$ has been chosen so that $gx \in U$.

We then claim that any $y \in X$ with $f(y) \leq -A$ belongs to the same component of $f^{-1}(-\infty, A)$ as $x$. This will imply $\chi \in -\Sigma$ by Proposition 1.1.

Choose an infinite path $\gamma : [0, +\infty) \to X$ such that $\gamma_{|[0,1]}$ is a path from $x$ to $gx$ in $U$ and $\gamma(t+n) = g^n\gamma(t)$ for $n \in \mathbf{N}$ and $t \in [0,1]$. Since $\chi(g) < 0$ this path is contained in $f^{-1}(-\infty, A)$.

Given $y \in X$ with $f(y) \leq -A$, fix $h \in G$ such that $hy \in U$, and choose a path $\delta$ from $hy$ to $x$ in $U$. Consider the infinite path $\rho$ obtained by applying $h^{-1}$ to $\delta\gamma$: it starts at $y$ and passes through $h^{-1}x$, $h^{-1}gx$, $h^{-1}g^2x, \ldots$. It is contained in $f^{-1}(-\infty, A)$ since $f(y) \leq -A$.

The image of $\gamma$ in $T(f)$ contains all points $g^n\pi(x)$ ($n \in \mathbf{N}$), while the image of $\rho$ contains all points $h^{-1}g^n\pi(x) = (h^{-1}gh)^n\pi(h^{-1}x)$. Now the translation axes of $g$ and $h^{-1}gh$ intersect in a half-line containing the fixed end $e$ (unless they are equal). Furthermore $g$ and $h^{-1}gh$ both push towards $e$ since $\chi(g) < 0$. It follows that $\gamma$ and $\rho$ are contained in the same component of $f^{-1}(-\infty, A)$, so that $\chi \in \Sigma$.

Conversely, suppose $\chi \in -\Sigma$. First assume that $G$ is finitely presented. Let $M$ be a closed manifold with $\pi_1 M = G$. Consider a geometric closed one-form $f : X \to \mathbf{R}$ as in Example 2 of Part 1. To fix ideas we may assume that $f$ is a Morse function.

Since $X$ is simply connected (it is the universal covering of $M$), it is known [**GS**] that $T(f)$ is an $\mathbf{R}$-tree (see [**Le 2**, Corollary III.5]). Since $\chi \in -\Sigma$ the function $\lambda : T(f) \to \mathbf{R}$ is bounded on every component of $\lambda^{-1}(-\infty, c)$ but one. It follows that the action of $G$ on $T(f)$ is abelian, associated to $\chi$: letting $\lambda$ go to $-\infty$ defines an end $e$ of $T(f)$ which is invariant under the action.

Now let $G$ be any finitely generated group. Using (i) $\Leftrightarrow$ (iii) in [**BNS**, Proposition 2.1] we can find an epimorphism $q : H \to G$, with $H$ finitely presented, such that $\chi' = \chi \circ q$ belongs to $-\Sigma(H)$. Apply the previous construction to $H$ and $\chi'$. Let $Y$ be the normal covering of $M$ with group $G$ and $g : Y \to \mathbf{R}$ the map induced by $f$.

The length function of the action of $H$ on the $\mathbf{R}$-tree $T(f)$ is $\ell = |\chi'|$. It vanishes on the kernel $K$ of $q$. It follows from [**Le 2**, corollary of Theorem 2] that $T(g) = \widehat{T(f)/K}$ is an $\mathbf{R}$-tree. The action of $G$ on this $\mathbf{R}$-tree is abelian, associated to $\chi$. ■

**Theorem 3.2 (Brown).** *Let $\chi : G \to \mathbf{R}$ be a nontrivial homomorphism, with $G$ finitely generated. Then $\chi \in \Sigma$ if and only if there exists no exceptional abelian action associated to $\chi$.*

We start the proof with a general construction. Let $f : X \to \mathbf{R}$ be continuous, with $X$ path connected. We construct an $\mathbf{R}$-tree $T^+(f)$ as follows. Given $x, y \in X$, define

$$\delta(x,y) = f(x) + f(y) - 2 \sup_{\gamma} \min_{t \in [0,1]} f(\gamma(t)),$$

the supremum being over all paths $\gamma : [0,1] \to X$ from $x$ to $y$. This is a pseudodistance on $X$ and we let $T^+(f)$ be the associated metric space.

**Proposition 3.3.** *The space $T^+(f)$ is an $\mathbf{R}$-tree. If $\mu : T^+(f) \to \mathbf{R}$ is the map induced by $f$, all sets $\mu^{-1}(-\infty, c)$ are path-connected, so that $T^+(f)$ has a preferred end $e = \mu^{-1}(-\infty)$.*

*Proof:*

We first prove that $T^+(f)$ is an $\mathbf{R}$-tree. By [**AB**, Theorem 3.17] it suffices to show that $\delta$ satisfies the 0-hyperbolicity inequality

$$\delta(x,y) + \delta(z,t) \leq \max\{\delta(x,z) + \delta(y,t), \delta(x,t) + \delta(y,z)\}.$$

By linearity we need only worry about the terms $\delta' = \sup \min f \circ \gamma$. They satisfy inequalities such as

$$\delta'(x,y) \leq \min\{\max(\delta'(x,z), \delta'(z,y)), \max(\delta'(x,t), \delta'(t,y))\}$$

and we conclude by applying the inequality

$$\min\{\max(a,c), \max(b,d)\} + \min\{\max(a,b), \max(c,d)\} \leq \max(a+d, b+c),$$

valid for any four real numbers $a, b, c, d$.

Let $\pi^+$ be the projection from $X$ to $T^+(f)$. Given $x, y \in X$ in $f^{-1}(-\infty, c)$ with, say, $f(y) \leq f(x)$, choose a path $\gamma : [0,1] \to X$ from $x$ to $y$. If $(p,q)$ is a maximal interval in $(f \circ \gamma)^{-1}(f(x), +\infty)$, we have $(\pi^+ \circ \gamma)(p) = (\pi^+ \circ \gamma)(q)$ and we can change $\pi^+ \circ \gamma$ on $(p,q)$ so that it becomes constant on $[p,q]$. Doing this for all intervals $(p,q)$ yields a path from $\pi^+(x)$ to $\pi^+(y)$ in $\mu^{-1}(-\infty, c)$. ∎

If $f$ is a closed one-form relative to $\chi : G \to \mathbf{R}$, the natural action of $G$ on $T^+(f)$ fixes $e$. It is abelian, associated to $\chi$ (note that this action is nongeometric whenever $\chi \notin -\Sigma$, by Theorem 3.1).

To prove Theorem 3.2, we fix a finite generating system $S$ for $G$ with $\chi(s) > 0$ for every $s \in S$ and we consider the corresponding Cayley graph $\Gamma$.

First assume $\chi \notin \Sigma$. Let $f : \Gamma \to \mathbf{R}$ be as in Example 3 of Part 1. We claim that the abelian action of $G$ on $T^+(f)$ is exceptional.

Let $u_1$ and $u_2$ be vertices of $\Gamma$ belonging to distinct components $U_1, U_2$ of some $f^{-1}(c, +\infty)$. Fix $s \in S$. The whole ray $u_i, u_i s, u_i s^2, \dots, u_i s^n, \dots$ is contained in $U_i$. Writing $u_i s^n = (u_i s u_i^{-1})^n u_i$ we see that $u_1 s u_1^{-1}$ and $u_2 s u_2^{-1}$ do not have the same translation axis in $T^+(f)$. This means that the action is exceptional.

Now assume $\chi \in \Sigma$. Let $T$ be an $\mathbf{R}$-tree with a minimal abelian action associated to $\chi$. We show that the action is not exceptional.

Choose $x \in T$ belonging to the translation axis of every $s \in S$. Consider a $G$-equivariant map $\varphi : \Gamma \to T$, affine on each edge, sending 1 to $x$. It is surjective because the action is minimal.

Define $f : T \to \mathbf{R}$ as in Part 2. The choice of $x$ implies that $g = f \circ \varphi$ is monotonous on each edge of $\Gamma$. It follows that $g$ is unbounded on every component of $g^{-1}(c, +\infty)$, so that $g^{-1}(c, +\infty)$ is connected for every $c \in \mathbf{R}$ by Proposition 1.1. Projecting to $T$ we see that every $f^{-1}(c, +\infty)$ is connected: the action is not exceptional. ■

Combining Theorems 3.1 and 3.2 we get:

**Corollary.** *Let $\chi : G \to \mathbf{R}$ be a nonzero homomorphism, with $G$ finitely generated.*

(1) *If $\chi \in \Sigma \cap -\Sigma$, the action of $G$ on $\mathbf{R}$ by translations is the only minimal action with length function $\ell = |\chi|$. It is geometric.*
(2) *If $\chi \in \Sigma$ but $\chi \notin -\Sigma$, there exist geometric exceptional abelian actions associated to $-\chi$. The only minimal action associated to $\chi$ is the action on $\mathbf{R}$, it is not geometric.*
(3) *If $\chi \notin \Sigma \cup -\Sigma$, there exist both exceptional abelian actions associated to $\chi$ and exceptional actions associated to $-\chi$. No action with length function $|\chi|$ is geometric.*

Combining with Theorem B.1 from [**BNS**] we obtain:

**Corollary.** *Let $G$ be finitely generated. The following conditions are equivalent:*

(1) *Every nontrivial action of $G$ on $\mathbf{R}$ by translations is geometric.*
(2) *The commutator subgroup $G'$ is finitely generated.*

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