EXISTENCE DOMAINS FOR HOLOMORPHIC $L^p$ FUNCTIONS

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Abstract

If $\Omega$ is a domain of holomorphy in $\mathbb{C}^n$, having a compact topological closure into another domain of holomorphy $U \subset \mathbb{C}^n$ such that $(\Omega, U)$ is a Runge pair, we construct a function $F$ holomorphic in $\Omega$ which is singular at every boundary point of $\Omega$ and such that $F$ is in $L^p(\Omega)$, for any $p \in (0, +\infty)$.

1. Statement of the problem

The following notation and terminology will be used without further explanation. The open polydisc in $\mathbb{C}^n$ with center $\alpha$ and radius $r$ is denoted by $\Delta^n(\alpha; r)$; if $n = 1$, then we use the notation $\Delta(\alpha; r)$. For every open set $D$ in $\mathbb{C}^n$, $\theta(D)$ denotes the space of all holomorphic functions in $D$. If $K$ is a compact subset of $D$, we define the $\theta(D)$-hull $K_D$ of $K$ by $K_D := \{z \in D; |f(z)| \leq \sup_{w \in K} |f(w)|$, for all $f \in \theta(D)\}$. For $p \in (0, +\infty]$, we set $\theta L^p(D) := \theta(D) \cap L^p(D)$. Obviously, $\theta L^\infty(D)$ equals the algebra $H^\infty(D)$ of bounded holomorphic functions in $D$. If $D$ carries a function $F \in \theta L^p(D)$, which cannot be holomorphically extended across the boundary of $D$, then $D$ is said to be an existence domain for $\theta L^p$ or of type $\theta L^p$.

Asking for the conditions under which a bounded domain of holomorphy $\Omega$ is of type $\theta L^p$, we recall the following result: If $\Omega \subset \subset \mathbb{C}^n$ is a domain of holomorphy with $C^\infty$ boundary and $(\alpha_\nu \in \Omega; \nu \in \mathbb{N})$ is a sequence such that $\lim_{\nu \to \infty} \alpha_\nu \in \partial \Omega$, then there exists a function $F \in \theta L^2(\Omega)$ satisfying $\lim_{\nu \to \infty} |F(\alpha_\nu)| = +\infty$ ([4]). The question we are interested is the following: Is any bounded domain of holomorphy in $\mathbb{C}^n$ existence domain for $\theta L^p$, for every $p \in (0, +\infty)$? In [1] Catlin showed that any smoothly bounded domain of holomorphy in $\mathbb{C}^n$ is of type $\theta L^\infty$, and consequently of type $\theta L^p$, for every $p \in (0, +\infty)$). However in [6], Sibony showed that there is a bounded Runge complete Hartogs domain of holomorphy $\Omega_S \subset \Delta^2(0; 1) (\Omega_S \neq \Delta^2(0; 1))$ such that all bounded
holomorphic functions in $\Omega_S$ extend holomorphically to the open unit bidisc, that is $\Omega_S$ is not of type $\theta L^\infty$.

The concern of this note is to give an answer to the above question. Our approach illustrates a partial extension of Catlin's improvement. More precisely, we shall prove that any domain of holomorphy $\Omega \subset \subset \mathbb{C}^n$, having a compact topological closure into another domain of holomorphy $U$ such that $(\Omega, U)$ is a Runge pair, is of type $\theta L^p$ for any $p \in (0, +\infty)$.

2. Unbounded holomorphic functions in Runge domains

Let $\Omega \subset \subset U$ be domains of holomorphy in $\mathbb{C}^n$. Assume that $\Omega$ is a bounded Runge domain relative to $U$.

Let $(z_m; m \in \mathbb{N})$ be a dense sequence in $\Omega$, such that every point of the sequence is counted infinitely many times. Let $r_m$ be the largest number with $\Delta^n(z_m; r_m) \subset \Omega$. $\Omega$ can be exhausted by compact sets $E_j$, so that $E_j \subset E_{j+1}$. Letting $K_1 := E_1$, we find a point $w_1 \in \Delta^n(z_1; r_1) - \hat{K}_{1,U}$. Obviously, there exists a $j_1 > 1$, with $w_1 \in E_{j_1}$. Put $K_2 := E_{j_1}$. Now, there is a point $w_2 \in \Delta^n(z_2; r_2) - \hat{K}_{2,U}$. If we set $K_3 := E_{j_2}$ ($j_2 > j_1$), then $w_2 \in E_{j_2}$. Continuing like this, we find an exhaustive sequence $(K_m; m \in \mathbb{N})$ of compact subsets of $\Omega$ and a sequence $(w_m; m \in \mathbb{N})$ of points of $\Omega$, with the following properties:

- $w_m \in K_{m+1} - \hat{K}_{m,U}$ ($m \in \mathbb{N}$),
- whenever $w \in \partial \Omega \cap \Delta^n(\xi; \rho)$ for a polydisc $\Delta^n(\xi; \rho)$ and $V$ is a connected component of $\Omega \cap \Delta^n(\xi; \rho)$ clustering at $w$, there exists a subsequence of $(w_m; m \in \mathbb{N})$ converging to $w$ in $V$.

To each $w_m$ there corresponds a holomorphic function $f_m \in \theta(U)$, such that $|f_m(w_m)| > \sup_{z \in K_m} |f_m(z)| = 1$. If we let $0 < \varepsilon_m < |f_m(w_m)| - 1$, then $|f_m(z)| < |f_m(w_m) - \varepsilon_m|$, whenever $z \in K_m$. Hence, for suitably chosen numbers $v_m > 0$, the series

$$F(z) = \sum_{m=1}^{\infty} \left( [f_m(z)]^{v_m} / [f_m(w_m) - \varepsilon_m]^{v_m} \right)$$

converges absolutely and compactly on $\Omega$ and $|F(w_m)| > m$, for any $m$. It follows that whenever $w \in \partial \Omega$, $\Delta^n(\xi; \rho)$ is a polydisc containing $w$ and $V$ is a connected component of $\Omega \cap \Delta^n(\xi; \rho)$ clustering at $w$, $F$ is unbounded in $V$. So, $F$ is a function holomorphic on $\Omega$, which is singular (unbounded) at every boundary point of $\Omega$ ([3]).
3. Runge domains of type $\theta L^p$

Let the notations and assumptions be as in Section 2. The principal purpose of this paragraph is to announce the following:

**Theorem 1.** Let $\Omega \subset U$ be domains of holomorphy in $\mathbb{C}^n$ such that $(\Omega, U)$ is a Runge pair. Then, $F \in \theta L^p(\Omega)$, for any $p \in (0, +\infty)$.

**Proof:** The evaluation of more useful choice of $\nu_m$ is our first aim. Let $\delta > 2$. For each $m \in \mathbb{N}$, choose $\nu_m$ so that $|f_m(w_m) - \epsilon_m|^{\nu_m} \geq \delta^m$. It is easily seen that the power series

$$h(\zeta) = \sum_{m=1}^{\infty} [f_m(w_m) - \epsilon_m]^{-\nu_m} \cdot \zeta^m$$

converges into the disc $\Delta(0; \delta)$. Define a linear functional

$$\Lambda_h : \mathbb{P}(\mathbb{C}) \to \mathbb{C}; \quad x^m \to \Lambda_h(x^m) := [f_m(w_m) - \epsilon_m]^{-\nu_m},$$

where $\mathbb{P}(\mathbb{C})$ is the vector space of complex polynomials in $\mathbb{C}$. In order to prove the theorem two lemmas play crucial role:

**Lemma 1.** ([2]) The functional $\Lambda_h$ is continuous and there is a continuous extension of $\Lambda_h$ into $\theta(\Delta(0; \delta^{-1}))$. Further, for each $\zeta \in \Delta(0; \delta)$ there holds $\Lambda_h((1 - x \zeta)^{-1}) = h(\zeta)$ ($x \in \Delta(0; \delta^{-1})$).

**Proof of Lemma 1:** Let $r < \delta$. If $p(x)$ is a polynomial in $x \in \mathbb{C}$, then by Cauchy's integral formula we have

$$|\Lambda_h(p)| \leq M(r) \cdot \sup_{|x| \leq r} |h(x)| \cdot \sup_{|x| \leq r^{-1}} |p(x)|,$$

where the constant $M(r)$ depends only on $r$. Hence, by density, there is a continuous extension of $\Lambda_h$ on $\theta(\Delta(0; \delta^{-1}))$. If now $\zeta \in \Delta(0; \delta)$ and if $\zeta$ is fixed, then the number $\Lambda_h((1 - x \zeta)^{-1})$ is well defined ($\Lambda_h$ acts on the variable $x \in \Delta(0; \delta^{-1})$ and $\zeta$ is regarded as a parameter). By the continuity of $\Lambda_h$, we obtain $\Lambda_h((1 - x \zeta)^{-1}) = h(\zeta)$. $\blacksquare$

The next lemma is a consequence of Lemma 1, but is much more useful since the choice of the functional $\Lambda_h$ is eliminated.
Lemma 2. If $z \in \Omega$, then there holds

$$|F(z)| \leq f \left( \frac{1}{\tau} \right) \cdot \sum_{m=1}^{\infty} \frac{|[f_m(z)]^{m_m}|}{\tau^m},$$

for any $\tau \in (2, \delta)$ and where the constant $\mathcal{L} \left( \frac{1}{\tau} \right)$ depends only on $\tau$ but is independent of $z$.

Proof of Lemma 2: Assuming that $z \in \Omega$, $x \in \Delta(0; \delta^{-1})$ and $\tau \in (2, \delta)$, we have by Cauchy’s integral formula and by Lemma 1:

$$|F(z)| = \left| \sum_{m=1}^{\infty} \Lambda_h(x^m) \cdot [f_m(z)]^{m_m} \right| =$$

$$= \left| \frac{1}{2\pi i} \cdot \int_{|\zeta| = \frac{1}{\tau}} \Lambda_h(1/(\zeta - x)) \cdot \left( \sum_{m=1}^{\infty} [f_m(z)]^{m_m} \cdot \zeta^m \right) \ d\zeta \right| \leq$$

$$\leq \mathcal{L} \left( \frac{1}{\tau} \right) \cdot \left( \sup_{|\zeta| = \frac{1}{\tau}} |\Lambda_h(1/(\zeta - x))| \right) \left( \sup_{|\zeta| = \tau} \left\{ \sum_{m=1}^{\infty} \frac{|[f_m(z)]^{m_m}|}{\zeta^m} \right\} \right),$$

that is

$$|F(z)| \leq \mathcal{L} \left( \frac{1}{\tau} \right) \cdot \sum_{m=1}^{\infty} \frac{|[f_m(z)]^{m_m}|}{\tau^m}. \quad \blacksquare$$

End of Proof of Theorem 1: Let $0 < p < +\infty$. By Lemma 2 and by Fatou’s Theorem, it is enough to show that

$$\sup \left\{ \int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^{m_m}|}{\tau^m} \right)^p \ d\lambda(z); \ v \in \mathbb{N} \right\} < +\infty,$$

for some $\tau \in (2, \delta)$. ($\lambda(\cdot)$ is the Lebesgue measure in $\mathbb{C}^n$).

Suppose $\tau \in (2, \delta)$. For any $v \in \mathbb{N}$, choose a positive number $\frac{2k_v - 1}{k_v}$ ($, k_v \in \mathbb{N}$), such that

$$\int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^{m_m}|}{\tau^m} \right)^p \ d\lambda(z) \leq \left( \frac{2k_v - 1}{k_v} \right)^p.$$

This choice permits us to obtain the following inequalities

$$\int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^{m_m}|}{2} \right)^p \ d\lambda(z) \leq \left( \frac{2k_v - 1}{2k_v} \right)^p \leq 1,$$
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for any $v \in \mathbb{N}$. Therefore,
\[ \int_{\Omega} \left( \sum_{m=1}^{v} \frac{|f_m(z)|^{v_m}}{\tau} \right)^p d\lambda(z) \leq 1, \]
for any $v \in \mathbb{N}$ and consequently,
\[ \int_{\Omega} \left( \sum_{m=1}^{v} \frac{|[f_m(z)]^{v_m}|}{\tau^m} \right)^p d\lambda(z) \leq 1, \]
for any $v \in \mathbb{N}$, which completes the proof. ■

We are now in position to formulate the main result of this note, which is an immediate consequence of Theorem 1:

**Theorem 2.** Let $\Omega \subset U$ be domains of holomorphy in $\mathbb{C}^n$. Assume that $\Omega$ is a bounded Runge domain relative to $U$. Then, $\Omega$ is an existence domain for $\theta L^p$, for any $p \in (0, +\infty)$. In particular, any bounded Runge domain of holomorphy is of type $\theta L^p$, for any $p \in (0, +\infty)$.

We finally turn to the question whether Sibony's example $\Omega_S$ in [6] is an existence domain of $L^p$ holomorphic functions. The answer is a direct consequence of Theorem 2: Since Sibony's example is a bounded Runge domain of holomorphy, it is an existence domain for $\theta L^p$, for any $p \in (0, +\infty)$.

References


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