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COMPACTNESS OF SUPPORT OF SOLUTIONS FOR SOME CLASSES OF NONLINEAR ELLIPTIC AND PARABOLIC SYSTEMS

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Abstract

In this paper, we obtain some existence Theorems of nonnegative solutions with compact support for homogeneous Dirichlet elliptic problems; we extend also these results to parabolic systems.

Supersolution and comparison principles are our main ingredients.

1. Introduction

This paper is concerned with the existence of nonnegative solutions with compact support in $X := W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ for the following systems:

$$(\mathcal{S})\begin{cases} -\Delta_p u + a|u|^{\alpha-1}u = f(x,u,v) & \text{in } \Omega \\ -\Delta_q v + b|v|^{\beta-1}v = g(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

and next

$$(\mathcal{P})\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u + a|u|^{\alpha-1}u = f(x,u,v) & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial v}{\partial t} - \Delta_q v + b|v|^{\beta-1}v = g(x,u,v) & \text{in } \Omega \times \mathbb{R}^+ \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(.,0) = u_0(.) & \text{in } \Omega \\ v(.,0) = v_0(.) & \text{in } \Omega, \end{cases}$$

where $p > 1$, $q > 1$, a, b, α and β are positive constants; the operator $\Delta_p u$, defined by $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$ is the well known "p-Laplacian"; f and g are nonnegative Caratheodory functions and u_0 and v_0 are some given functions.

During recent years, many papers are devoted to the study of reaction-diffusion systems which arise very often in applications such as, mathematical biology, chemical reactions and combustion theory. An excellent overview of the subject is the survey of [**1**].

Díaz and Herrero [**4**] and [**5**], study the case of a single equation of the form:

$$(\mathcal{E}_{a,f,g})\begin{cases} -\Delta_p u + a|u|^{\alpha-1}u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where a is a positive constant, $f \in L^\infty(\Omega)$, $g \in W^{1,p}(\Omega)$ and $g_{|\partial\Omega} \in L^\infty(\partial\Omega)$, both with compact support. Then, a necessary and sufficient condition for the existence of a solution $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ of $(\mathcal{E}_{a,f,g})$ with compact support is $0 < \alpha < p-1$. They obtain the same results for the associated parabolic problem.

Here we generalize the above results to some elliptic and parabolic systems and we used the iterative method based on the Comparison Principle for the problem $(\mathcal{P})$, taking in account the construction of sub-super solution introduced in [**11**].

Our paper is organized as follows:

1. Introduction; 2. Preliminaries; 3. Elliptic systems and, 4. Parabolic systems.

2. Preliminaries

We shall use the following notations:

For $p \in]1, +\infty[$, p^* is defined by $\frac{1}{p} + \frac{1}{p^*} = 1$.

For $a > 0$, $p > 1$ and $0 < \alpha < p-1$, set

$$K(a,p,\alpha) := \left[\frac{a(p-\alpha-1)^p}{p^{p-1}(p\alpha + N(p-\alpha-1))}\right]^{\frac{1}{(p-\alpha-1)}}. \tag{2.1}$$

For $T > 0$ and $R_0 > 0$ and $R_i > 0$ such that $R_0 < R_i$ for $(i = 1,2)$.

Consider the following sets:

$$\begin{aligned} D_0 &:= B(R_0), \\ D_i &:= B(R_i) \backslash B(R_0), \\ D_i' &:= \bar{B}(R_i)^c \text{ for } i = 1,2; \\ \tilde{\Omega} &:= \Omega \cap B(\max(R_1,R_2)+1), \\ Q_T &:= \Omega \times [0,T] \text{ and} \\ \Sigma_T &:= \partial\tilde{\Omega} \times [0,T] \end{aligned}$$

where $B(R) := \{x \in \bar{\Omega}/|x| < R\}$ and A^c is the complement any set of the A.

Definitions.

I) A pair $(\tilde{u}, \tilde{v}) - (\hat{u}, \hat{v})$ is said to be a weak sub-super solution for the Dirichlet problem $(\mathcal{S})$ if the following conditions are satisfied:

$$(\mathcal{C})\begin{cases} (\tilde{u}, \tilde{v});\ (\hat{u}, \hat{v}) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)\times & \\ \qquad\qquad \times W^{1,q}(\Omega) \cap L^\infty(\Omega) & \\ -\Delta_p \tilde{u} + a|\tilde{u}|^{\alpha-1}\tilde{u} - f(x, \tilde{u}, v) \le 0 \le & \\ \qquad \le -\Delta_p \hat{u} + a|\hat{u}|^{\alpha-1}\hat{u} - f(x, \hat{u}, v) & \text{in } \Omega\, \forall v \in [\tilde{v}, \hat{v}] \\ -\Delta_q \tilde{v} + b|\tilde{v}|^{\beta-1}\tilde{v} - g(x, u, \tilde{v}) \le 0 \le & \\ \qquad \le -\Delta_q \hat{v} + b|\hat{v}|^{\beta-1}\hat{v} - g(x, u, \hat{v}) & \text{in } \Omega\, \forall u \in [\tilde{u}, \hat{u}] \\ \tilde{u} \le \hat{u} & \text{in } \Omega \\ \tilde{v} \le \hat{v} & \text{in } \Omega \\ \tilde{u} \le 0 \le \hat{u} & \text{on } \partial\Omega \\ \tilde{u} \le 0 \le \hat{v} & \text{on } \partial\Omega. \end{cases}$$

Similar definitions can be found in Díaz-Hernández [**3**], Hernández [**7**].

II) In the case of parabolic problem we consider only sub-super solutions which do not depent on t. Such a pair $(\tilde{u}, \tilde{v}) - (\hat{u}, \hat{v})$ is sub-super solutions of $(\mathcal{P})$ if the following conditions are satisfied:

a) $(\mathcal{C})$,
b) $\tilde{u}(x) \le u_0(x) \le \hat{u}(x)$ in Ω,
c) $\tilde{v}(x) \le v_0(x) \le \hat{v}(x)$ in Ω.

In this paper we also use the following lemmas:

Lemma 1. [**2**, Lemma 1.6] *Assume that $p > 1$ and $0 < \alpha < p - 1$. Then the function $u(r) = Kr^{\frac{p}{p-\alpha-1}}$, where $K = K(a, p, \alpha)$ defined by (2.1), satisfies the following equation:* $\frac{-1}{r^{N-1}}\frac{d}{dr}(r^{N-1}|\frac{du}{dr}|^{p-2}\frac{du}{dr}) + a|u(r)|^{\alpha-1}u(r) = 0$.

Lemma 2. [**10**] *Suppose μ and ν are in $\mathbb{R}^N$. Then there exists $C > 0$ such that:*

$$\begin{cases} (|\mu|^{p-2}\mu - |\nu|^{p-2}\nu)(\mu - \nu) \ge C|\mu - \nu|^p & \text{if } p \ge 2 \\ (|\mu|^{p-2}\mu - |\nu|^{p-2}\nu)(\mu - \nu) \ge C\frac{(|\mu-\nu|)^2}{(|\mu|+|\nu|)^{2-p}} & \text{if } |\mu| + |\nu| \ne 0 \\ & \text{and } 1 < p \le 2. \end{cases}$$

Lemma 3. [8] *Let Y be a Banach space.*

If $u \in L^p(0,T;Y)$ and $\frac{\partial u}{\partial t} \in L^p(0,T;Y)(1 \le p \le +\infty)$. Then after an eventual modification on a set of measure zero of $(0,T)$, u is continuous from $[0,T]$ to Y.

Let $R_0 > 0$ be given. We seek a sub-super solution of $(\mathcal{S})$ and $(\mathcal{P})$ in the following way. Let $(\tilde{u},\tilde{v}) = (0,0)$ and $(\hat{u},\hat{v})$ be defined by: $\hat{u}(x) = G_1(|x|)$, $\hat{v}(x) = G_2(|x|)$ for $x \in \Omega$ where:

$$G_1(r) = \begin{cases} -A_1 r^{p^*} + B_1 & \text{if } r \le R_0 \\ K_1(R_1 - r)^{\frac{p}{p-\alpha-1}} & \text{if } R_0 \le r \le R_1 \\ 0 & \text{if } r \ge R_1, \end{cases} \tag{2.2}$$

$$G_2(r) = \begin{cases} -A_2 r^{q^*} + B_2 & \text{if } r \le R_0 \\ K_2(R_2 - r)^{\frac{q}{q-\beta-1}} & \text{if } R_0 \le r \le R_2 \\ 0 & \text{if } r \ge R_2; \end{cases}$$

with $K_1 := K(a,p,\alpha)$; $K_2 := K(b,q,\beta)$ (defined by (2.1)), A_i, B_i and $R_i (i = 1,2)$ are some positive constants.

First we need that $\hat{u} \in \mathcal{C}^1(\bar{\Omega})$ (resp. $\hat{v} \in \mathcal{C}^1(\bar{\Omega})$) which implies that the positive constants A_1, B_1 (resp. A_2, B_2) satisfy:

$$-A_1 R_0^{p^*} + B_1 = K_1 X_1^{\frac{p}{p-\alpha-1}}, \tag{2.3}$$

$$A_1 p^* R_0^{p^*-1} = K_1 \frac{p}{p-\alpha-1} X_1^{\frac{\alpha+1}{p-\alpha-1}}, \tag{2.4}$$

$$-A_2 R_0^{q^*} + B_2 = K_2 X_2^{\frac{q}{q-\beta-1}}, \tag{2.5}$$

$$A_2 q^* R_0^{q^*-1} = K_2 \frac{q}{q-\beta-1} X_2^{\frac{\beta+1}{q-\beta-1}}, \tag{2.6}$$

where $X_i = R_i - R_0$ for $i = 1,2$. These constants will be completely determined in each one of the following sections.

3. Elliptic systems

We study the elliptic system $(\mathcal{S})$, where

$(\mathcal{H}_0)$ Ω is a regular open set in $\mathbb{R}^N$ (not necessarly bounded); $a > 0$, $b > 0$, $p > 1$, $q > 1$, $0 < \alpha < p-1$, $0 < \beta < q-1$ are given numbers.

$(\mathcal{H}_1)$ $f(x,u,v)$ and $g(x,u,v)$ are Caratheory functions, $f(x,.,v)$ (resp. $g(x,u,.)$) is a nondecreasing function for fixed v (resp. fixed u).

$(\mathcal{H}_2)$ The functions $f(x,u,v)$ and $g(x,u,v)$ satisfy:

$$0 \le f(x,u,v) \le cu^{\gamma}v^{\delta+1} + \varphi(x)v^{\eta} + f_1(x) \text{ and}$$
$$0 \le g(x,u,v) \le du^{\gamma+1}v^{\delta} + \psi(x)u^{\xi} + g_1(x) \text{ for } u,\ v \ge 0,$$

where

c, d, δ, η, γ and ξ are nonnegative constants;

φ, ψ, f_1 and g_1 are nonnegative, bounded measurable functions such that:

$$\operatorname{supp} f_1 \cup \operatorname{supp} g_1 \cup \operatorname{supp}\varphi \cup \operatorname{supp}\psi \subset B(R_0) \text{ for some } R_0 > 0.$$

We seek solutions $(u,v) \in X$ satisfying $(\mathcal{S})$ in the distributional sense.

First we give some conditions on α, β, f and g which insure that $(\hat{u},\hat{v})$ defined in (2.2) is a supersolution of $(\mathcal{S})$.

Proposition 1. *If $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are satisfied with $\gamma \ge \alpha$ and $\delta \ge \beta$, then, for sufficiently small nonnegative c, d, $\|\varphi\|_\infty$ and $\|\psi\|_\infty$, $(0,0) - (\hat{u},\hat{v})$ is a sub-super solution for $(\mathcal{S})$ in Ω.*

Proof: It is obvious that $(0,0)$ is a sub solution.

By the definition of a super solution, we have to prove that:

$$-\Delta_p \hat{u} + a(\hat{u})^\alpha - f(x,\hat{u},v) \ge 0 \quad \forall v \in [0,\hat{v}] \text{ in } \Omega. \tag{3.1}$$

In $B(R_0)$, we have:

$$-\Delta_p \hat{u} = N(A_1 p^*)^{p-1}.$$

Then, if we take:

$$N(A_1 p^*)^{p-1} \ge cB_1^{\gamma}B_2^{\delta+1} + \|\varphi\|_\infty B_2^{\eta} + \|f_1\|_\infty, \tag{3.2}$$

$\hat{u}$ satisfies (3.1) in $B(R_0)$.

In $B(R_1)\backslash B(R_0)$, $\hat{u}$ is a solution of $(\mathcal{E}_{\frac{a}{2},0,0})$ by Lemma 1 [**5**], thus (3.1) becomes:

$$\frac{a}{2}K_1^{\alpha}(R_1-r)^{\frac{p\alpha}{p-\alpha-1}} - cK_1^{\gamma}K_2^{\delta+1}(R_1-r)^{\frac{p\gamma}{p-\alpha-1}}(R_2-R_0)^{\frac{q(\delta+1)}{q-\beta-1}} \ge 0,$$

which is satisfied if we have:

$$\begin{cases} \gamma \ge \alpha \\ cX_1^{\frac{p(\gamma-\alpha)}{p-\alpha-1}}X_2^{\frac{q(\delta+1)}{q-\beta-1}} \le C_1 \end{cases} \tag{3.3}$$

and analogously for the other equation of $(\mathcal{S})$, we obtain:

$$\text{(3.4)}\qquad \begin{cases} \delta \geq \beta \\ dX_1^{\frac{p(\gamma+1)}{p-\alpha-1}} X_2^{\frac{q(\delta-\beta)}{q-\beta-1}} \leq C_1' \end{cases}$$

where:

$$C_1 := C_1(a,b;\, p,q;\, \gamma,\delta;\, \alpha,\beta) := \frac{1}{2} a^{\frac{p-\gamma-1}{p-\alpha-1}} b^{\frac{-(\delta+1)}{q-\beta-1}} \times$$

$$\left[\frac{(p-\alpha-1)^p}{2p^{p-1}(\alpha p + N(p-\alpha-1))}\right]^{\frac{\alpha-\gamma}{p-\alpha-1}} \left[\frac{(q-\beta-1)^q}{2q^{q-1}(\beta q + N(q-\beta-1))}\right]^{-(\delta+1)};$$

$C_1' := C_1(b,a;\, q,p;\, \delta,\gamma;\, \beta,\alpha)$,
$X_i := R_i - R_0$ for $i = 1,2$.

So $(\hat{u},\hat{v})$ is a supersolution of $(\mathcal{S})$ if (2.3) to (2.6), (3.2) to (3.4) are satisfied.

From (2.3) to (2.6), (3.2) is satisfied if:

(3.5)

$$C_3 X_1^{\frac{(p-1)(\alpha+1)}{p-\alpha-1}} \geq cC_4 X_1^{\frac{\gamma p}{p-\alpha-1}} X_2^{\frac{q(\delta+1)}{q-\beta-1}} (1 + C_5 X_1^{-1})^{\gamma} (1 + C_5' X_2^{-1})^{\delta+1} + \\ + \|\varphi\|_\infty K_2^{\eta} X_2^{\frac{q\eta}{q-\beta-1}} (1 + C_5' X_2^{-1})^{\eta} + \|f_1\|_\infty,$$

where:

$C_3 := C_3(p,\alpha,R_0) := N\left(K_1 \dfrac{p-1}{p-\alpha-1}\right)^{p-1} R_0^{-1}$,

$C_4 := C_4(a,b;\, p,q;\, \alpha,\beta,\gamma,\delta) := K_1^{\gamma} K_2^{\delta+1}$,

$C_5 := C_5(R_0; a; p; \alpha) = \dfrac{p-1}{p-\alpha-1} K_1 R_0$,

$C_5' := C_5(R_0; b; q; \beta)$,

$K_1 := K\left(\dfrac{a}{2}, p, \alpha\right)$ and $K_2 := K\left(\dfrac{b}{2}, q, \beta\right)$ (function K defined by (2.1)).

We choose X_1 and $X_2 \gg 1$ such that: $1 + C_5 X_1^{-1} \leq 2$ and $1 + C_5' X_2^{-1} \leq 2$, so (3.5) is satisfied if:

$$\text{(3.6)}\qquad \begin{aligned} C_3 X_1^{\frac{(\alpha+1)(p-1)}{p-\alpha-1}} &\geq 3\|f_1\|_\infty, \\ C_3 X_1^{\frac{(\alpha+1)(p-1)}{p-\alpha-1}} &\geq 3 \times 2^{\delta+\gamma+1} cC_4 X_1^{\frac{\gamma p}{p-\alpha-1}} X_2^{\frac{q(\delta+1)}{q-\beta-1}}, \\ C_3 X_1^{\frac{(\alpha+1)(p-1)}{p-\alpha-1}} &\geq 3 \times 2^{\eta} \|\varphi\|_\infty K_1^{\eta} X_2^{\frac{q\eta}{q-\beta-1}}. \end{aligned}$$

Let $Z = X_1^{\frac{p(\gamma+1)}{p-\alpha-1}} = X_2^{\frac{q(\delta+1)}{q-\beta-1}}$ be large enough such that $Z \geq \left(\frac{3}{C_3}\|f_1\|_\infty\right)^{\frac{(\gamma+1)p^*}{\alpha+1}}$, c such that $c \leq \frac{C_3}{3\times 2^{\gamma+\delta+1}C_4} Z^{\frac{\alpha+1}{(\gamma+1)p^*}-\frac{\gamma}{\gamma+1}-1}$ and $\|\varphi\|_\infty \leq \frac{C_3}{3\times 2^\eta K_2^\eta} Z^{\frac{\alpha+1}{(\gamma+1)p^*}-\frac{\eta}{\delta+1}}$.

Similarly for the other equation of $(\mathcal{S})$, we take:

$Z \geq \left(\frac{3}{C_3'}\|g_1\|_\infty\right)^{\frac{(\delta+1)q^*}{(\beta+1)}}$, d such that $d \leq \frac{C_3'}{3\times 2^{\gamma+\delta+1}C_4'} Z^{\frac{\beta+1}{(\delta+1)q^*}-\frac{\delta}{\delta+1}-1}$ and $\|\psi\|_\infty \leq \frac{C_3'}{3\times 2^\xi K_1^\xi} Z^{\frac{\beta+1}{(\delta+1)q^*}-\frac{\xi}{\gamma+1}}$.

For (3.3) and (3.4), we take:

$$\begin{cases} c \leq C_1 Z^{\frac{\alpha-2\gamma-1}{\gamma+1}} \\ d \leq C_1' Z^{\frac{\beta-2\delta-1}{\delta+1}}. \end{cases} \tag{3.7}$$

So, consider Z large enough such that:

$$Z \geq \text{Max}\left(\left(\frac{2}{C_3}\|f_1\|_\infty\right)^{\frac{(\gamma+1)}{(\alpha+1)p^*}}, \left(\frac{2}{C_3'}\|g_1\|_\infty\right)^{\frac{(\delta+1)}{(\beta+1)q^*}}\right), \tag{3.8}$$

and choose c, d, $\|\varphi\|_\infty$ and $\|\psi\|_\infty$ small enough such that:

$$\begin{aligned} 0 \leq c &\leq \text{Min}\left(\frac{C_3}{3\times 2^{\delta+\gamma+1}C_4} Z^{\frac{\alpha+1}{(\gamma+1)p^*}-\frac{\gamma}{\gamma+1}-1}, C_1 Z^{\frac{\alpha-2\gamma-1}{\gamma+1}}\right), \\ 0 \leq d &\leq \text{Min}\left(\frac{C_3'}{3\times 2^{\delta+\gamma+1}C_4'} Z^{\frac{\beta+1}{(\delta+1)q^*}-\frac{\delta}{\delta+1}-1}, C_1' Z^{\frac{\beta-2\delta-1}{\delta+1}}\right), \\ \|\varphi\|_\infty &\leq \frac{C_3}{3\times 2^\eta K_2^\eta} Z^{\frac{\alpha+1}{(\gamma+1)p^*}-\frac{\eta}{\delta+1}}, \\ \|\psi\|_\infty &\leq \frac{C_3'}{3\times 2^\xi K_2^\xi} Z^{\frac{\beta+1}{(\delta+1)q^*}-\frac{\xi}{\gamma+1}}. \end{aligned} \tag{3.9}$$

Therefore, the existence of $(\hat{u}, \hat{v})$ in Ω is proved. ■

Theorem 1. *Suppose that the hypothesis of Proposition 1 are satisfied, that* c, d, $\|\varphi\|_\infty$ *and* $\|\psi\|_\infty$ *are nonnegative real numbers sufficiently small. Then, there exists at least one nonnegative solution with compact support* (u, v) *of problem* $(\mathcal{S})$.

Proof: We proceed in three steps.

i) Construction of an invariant set:

In view of applying Schauder's Fixed Point Theorem, let us introduce $E = L^p(\tilde{\Omega}) \times L^q(\tilde{\Omega})$, $K = [0, \hat{u}] \times [0, \hat{v}]$ and consider m_1 and m_2 such that $m_1 > a$ and $m_2 > b$.

Next we define the nonlinear operator $T : (u, v) \in K \to (w, z) \in E$ by:

$$
(3.10) \qquad \begin{cases} -\Delta_p w + m_1 |w|^{\alpha-1} w = (m_1 - a) u^\alpha + f(x, u, v) & \text{in } \tilde{\Omega} \\ -\Delta_p z + m_2 |z|^{\beta-1} z = (m_2 - b) v^\beta + g(x, u, v) & \text{in } \tilde{\Omega} \\ w = z = 0 & \text{on } \partial\tilde{\Omega}, \end{cases}
$$

Existence and uniqueness for solutions of (3.10) are well-known by [**8**] and [**4**], so that T is well defined. Moreover (w, z) is in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \times W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ and is nonnegative by [**3**]. Now we extend w and z by 0 out of $\tilde{\Omega}$.

ii) $T(K) \subset K$. Let $(u, v) \in K$, we have:

$$
\begin{aligned} (3.11) \quad & -(\Delta_p w - \Delta_p \hat{u}) + m_1 (|w|^{\alpha-1} w - |\hat{u}|^{\alpha-1} \hat{u}) \le \\ & \le (m_1 - a)(|u|^{\alpha-1} u - |\hat{u}|^{\alpha-1} \hat{u}) + f(x, u, v) - f(x, \hat{u}, v), \end{aligned}
$$

multiplying (3.11) by $(w - \hat{u})_+ := \max(w - \hat{u}, 0))$ and integrating over $\tilde{\Omega}$, we obtain:

$$
(3.12) \qquad \begin{aligned} & \int_{\tilde{\Omega}} (|\nabla_w|^{p-2} \nabla w - |\nabla \hat{u}|^{p-2} \nabla \hat{u}) \nabla (w - \hat{u})_+ \, dx + \\ & + m_1 \int_{\tilde{\Omega}} (|w|^{\alpha-1} w - |\hat{u}|^{\alpha-1} \hat{u})(w - \hat{u})_+ \, dx \le \\ & \le (m_1 - a) \int_{\tilde{\Omega}} (u^\alpha - (\hat{u})^\alpha)(w - \hat{u})_+ \, dx + \\ & + \int_{\tilde{\Omega}} (f(x, u, v) - f(x, \hat{u}, v))(w - \hat{u})_+ \, dx \le 0. \end{aligned}
$$

From $(\mathcal{H}_1)$ and Lemma 2, we have $(w - \hat{u})_+ = 0$, hence $0 \le w \le \hat{u}$.

The same is true for z, $0 \le z \le \hat{v}$, $T(K) \subset K$.

iii) T is completely continuous:

First we prove that T is compact, let (u_j, v_j) be a bounded sequence in K. By $(\mathcal{H}_2)$ $f(x, u_j, v_j)$ (resp. $g(x, u_j, v_j)$) is bounded in $L^{p^*}(\tilde{\Omega})$ (resp. $L^{q^*}(\tilde{\Omega})$).

Multiplying the first equation in (3.10) by w, we obtain for w_j:

$$\begin{aligned}
&\int_{\tilde\Omega} |\nabla w_j|^p \, dx + \\
&m_1 \int_{\tilde\Omega} |w_j|^{\alpha+1} \, dx = (m_1 - a) \int_{\tilde\Omega} |u_j|^{\alpha-1} u_j w_j \, dx + \\
&+ \int_{\tilde\Omega} f(x, u_j, v_j) w_j \, dx \leq C \left(m_1 \int_{\tilde\Omega} |w_j|^p \, dx \right)^{\frac{1}{p}},
\end{aligned} \tag{3.13}$$

Hence (w_j) is bounded in $W^{1,p}(\tilde\Omega)$ and it possesses a strongly convergent subsequence in $L^p(\tilde\Omega)$. The same is true for z_j in $L^q(\tilde\Omega)$.

Now we prove the continuity of T:

Suppose that $(u_j, v_j) \to (u, v)$ in K. By the Dominated Convergence Theorem, we have:

(3.14)

$$f(x, u_j, v_j) \to f(x, u, v) \text{ in } L^{p^*}(\tilde\Omega) \text{ and } |u_j|^{\alpha-1} u_j \to |u|^{\alpha-1} u \text{ in } L^{p^*}(\tilde\Omega).$$

Consider

(3.15)

$$\begin{cases} -(\Delta_p w_j - \Delta_p w) + m_1 (|w_j|^{\alpha-1} w_j - |w|^{\alpha-1} w) = \\ = (m_1 - a)(|u_j|^{\alpha-1} u_j - |u|^{\alpha-1} u) + f(x, u_j, v_j) - f(x, u, v) & \text{in } \tilde\Omega \\ w_j = w = 0 & \text{on } \partial\tilde\Omega. \end{cases}$$

Multiplying (3.15) by $(w_j - w)$ and integrating over $\tilde\Omega$ we obtain:

$$\begin{aligned}
&\int_{\tilde\Omega} (|\nabla w_j|^{p-2} \nabla w_j - |\nabla w|^{p-2} \nabla w) \nabla (w_j - w) \, dx + \\
&+ m_1 \int_{\tilde\Omega} (|w_j|^{\alpha-1} w_j - |w|^{\alpha-1} w)(w_j - w) \, dx = \\
&= (m_1 - a) \int_{\tilde\Omega} (|u_j|^{\alpha-1} u_j - |u|^{\alpha-1} u)(w_j - w) \, dx + \\
&+ \int_{\tilde\Omega} ((f(x, u_j, v_j) - f(x, u, v))(w_j - w) \, dx.
\end{aligned} \tag{3.16}$$

It follows from (3.14) that the right-hand side of (3.16) tend to zero as j tends to $+\infty$.

From Lemma 2 and Holder's Inequality applied to the left-hand side of (3.16), we obtain:

$$\int_{\tilde\Omega} |\nabla (w_j - w)|^p \, dx + m_1 \int_{\tilde\Omega} |(w_j - w)|^p \, dx \to 0 \text{ as } j \to +\infty.$$

A similar argument can be used for z. Since K is a convex, bounded and closed subset of E, we can apply Schauder's Fixed Point Theorem and obtain the existence of a fixed point for T, which gives the existence of at least one solution (u,v) of $(\mathcal{S})$ such that $0 \le u \le \hat{u}$ and $0 \le v \le \hat{v}$. ■

4. Parabolic systems

In this section, we consider $p > 2$, $q > 2$ and $R_0 > 0$ such that:

$$\operatorname{supp}\varphi \cup \operatorname{supp}\psi \cup \operatorname{supp} f_1 \cup \operatorname{supp} g_1 \cup \operatorname{supp} u_0 \cup \operatorname{supp} v_0 \subset B(R_0);$$

where f_1, g_1, φ, ψ defined in $(\mathcal{H}_2)$.

We add the following hypothesis:

$(\mathcal{H}_3)$ $\forall M > 0$, $\forall N > 0$, $\exists K^i_{M,N} > 0$ $i = 1,2$ such that:

$$f(x,u_1,v_1) - f(x,u_2,v_2) \le K^1_{M,N}((u_1 - u_2) + (v_1 - v_2));$$

$$g(x,u_1,v_1) - g(x,u_2,v_2) \le K^2_{M,N}((u_1 - u_2) + (v_1 - v_2))$$

for $0 \le u_2 \le u_1 \le M$ and $0 \le v_2 \le v_1 \le N$.

Proposition 2. *Assume that the hypothesis $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$ are satisfied and the numbers α, β, γ and δ are such that: $1 \le \alpha < p - 1$, $1 \le \beta < q - 1$, $\gamma \ge \alpha$, $\delta \ge \beta$. Then, for sufficiently small numbers c, d, $\|\varphi\|_\infty$ and $\|\psi\|_\infty$, $(0,0) - (\hat{u},\hat{v})$ is a sub-super solution of $(\mathcal{P})$.*

Proof: From the definition of $\hat{u}$, for (II-b) it is sufficient to have:

$$\|u_0\|_\infty \le K_1 X_1^{\frac{p}{p-\alpha-1}}. \tag{4.1}$$

Similarly for $\hat{v}$,

$$\|v_0\|_\infty \le K_2 X_2^{\frac{q}{q-\beta-1}}, \tag{4.2}$$

where $K_1 := K(a,p,\alpha)$, $K_2 := K(b,q,\beta)$ (defined by (2.1)) and $X_i := R_i - R_0$ for $i = 1,2$.

From the elliptic case (Proposition 1), (4.1) and (4.2), we choose the real numbers Z, c, d, $\|\varphi\|_\infty$ and $\|\psi\|_\infty$ such that:

$$Z \ge \operatorname{Max}\left(\left(\frac{2}{C_3}\|f_1\|_\infty\right)^{\frac{\gamma+1}{\alpha+1}\times p^*}, \left(\frac{2}{C_3'}\|g_1\|_\infty\right)^{\frac{\delta+1}{\beta+1}\times q^*}, \left(\frac{\|u_0\|_\infty}{K_1}\right)^{\gamma+1}, \left(\frac{\|v_0\|_\infty}{K_2}\right)^{\delta+1} \right),$$

$$0 \le c \le \operatorname{Min}\left(\frac{C_3}{3\times 2^{\delta+\gamma+1} C_4} Z^{\frac{\alpha+1}{(\gamma+1)p^*} - \frac{\gamma}{\gamma+1} - 1}, C_1 Z^{\frac{\alpha-2\gamma-1}{\gamma+1}} \right),$$

$$0 \le d \le \operatorname{Min}\left(\frac{C_3'}{3\times 2^{\delta+\gamma+1} C_4'} Z^{\frac{\beta+1}{(\delta+1)q^*} - \frac{\delta}{\delta+1} - 1}, C_1' Z^{\frac{\beta-2\delta-1}{\delta+1}} \right),$$

Then, the existence of $(\hat{u}, \hat{v})$ as super solution of $(\mathcal{P})$ in Q_T is proved. ■

Theorem 2. *Assume that $(\mathcal{H}_3)$ and the hypothesis of Proposition 2 are satisfied. Then the problem $(\mathcal{P})$ admits a nonnegative unique solution in $\mathcal{C}(\mathbb{R}^+, L^2(\Omega)) \cap L^\infty(0,T; W_0^{1,p}(\Omega)) \times \mathcal{C}(\mathbb{R}^+, L^2(\Omega)) \cap L^\infty(0,T; W_0^{1,q}(\Omega))$ such that: $0 \le u(x,t) \le \hat{u}(x)$ and $0 \le v(x,t) \le \hat{v}(x)$ in $\Omega \times \mathbb{R}^+$.*

Proof: Using an iterative method, we proceed in five steps.

Construction of sequence $(\underline{u}_n)$ (resp. $(\underline{v}_n)$).

i) Determination of $\underline{u}_0$ (resp. $\underline{v}_0$).
From [**6**, Theorem 4] $\underline{u}_0$ (resp. $\underline{v}_0$) exists as solution of:

$$\begin{cases} \frac{\partial \underline{u}_0}{\partial t} - \Delta_p \underline{u}_0 + a|\underline{u}_0|^{\alpha-1}\underline{u}_0 = f(x,0,0) & \text{in } \tilde{Q}_T := \tilde{\Omega} \times [0,T] \\ \underline{u}_0(x,t) = 0 & \text{on } \tilde{\Sigma}_T := \partial\tilde{\Omega} \times [0,T] \\ \underline{u}_0(.,0) = u_0(.) & \text{in } \tilde{\Omega}, \end{cases}$$

resp.

$$\begin{cases} \frac{\partial \underline{v}_0}{\partial t} - \Delta_q \underline{v}_0 + b|\underline{v}_0|^{\beta-1}\underline{v}_0 = g(x,0,0) & \text{in } \tilde{Q}_T \\ \underline{v}_0(x,t) = 0 & \text{on } \tilde{\Sigma}_T \\ \underline{v}_0(.,0) = v_0(.) & \text{in } \tilde{\Omega}, \end{cases}$$

such that: $0 \le \underline{u}_0 \le \hat{u}$ (resp. $0 \le \underline{v}_0 \le \hat{v}$).

ii) Suppose $(\underline{u}_n)$ (resp. $(\underline{v}_n)$) is defined as nonnegative function on $\bar{\Omega} \times [0,T]$ $(T > 0)$ with initial value u_0 (resp. v_0) in Ω, zero on $\tilde{\Sigma}_T$ and $0 \le \underline{u}_n \le \hat{u}$ (resp. $0 \le \underline{v}_n \le \hat{v}$) also we define $\underline{u}_{n+1}$ (resp. $\underline{v}_{n+1}$) as solution of the problem:

$$\begin{cases} \frac{\partial \underline{u}_{n+1}}{\partial t} - \Delta_p \underline{u}_{n+1} + a|u_{n+1}|^{\alpha-1}u_{n+1} = f(x,\underline{u}_n,\underline{v}_n) & \text{in } \tilde{Q}_T \\ \underline{u}_{n+1}(x,t) = 0 & \text{on } \tilde{\Sigma}_T \\ \underline{u}_{n+1}(.,0) = u_0(.) & \text{in } \tilde{\Omega}, \end{cases}$$

resp.

$$\begin{cases} \frac{\partial \underline{v}_{n+1}}{\partial t} - \Delta_q \underline{v}_{n+1} + b|\underline{v}_{n+1}|^{\beta-1}\underline{v}_{n+1} = g(x,\underline{u}_n,\underline{v}_n) & \text{in } \tilde{Q}_T \\ \underline{v}_{n+1}(x,t) = 0 & \text{on } \tilde{\Sigma}_T \\ \underline{v}_{n+1}(.,0) = v_0(.) & \text{in } \tilde{\Omega}. \end{cases}$$

$\underline{u}_{n+1}$ (resp. $\underline{v}_{n+1}$) exists by [**6**, Theorem 4] such that: $0 \le \underline{u}_{n+1} \le \hat{u}$ (rsp. $0 \le \underline{v}_{n+1} \le \hat{v}$).

iii) Estimations of $\underline{u}_{n+1}$, $\frac{\partial \underline{u}_{n+1}}{\partial t}$ (resp. $\underline{v}_{n+1}$, $\frac{\partial \underline{v}_{n+1}}{\partial t}$).
Consider:

$$\begin{cases} \frac{\partial \underline{u}_{n+1}(x,t)}{\partial t} - \Delta_p \underline{u}_{n+1} + a|\underline{u}_{n+1}|^{\alpha-1}\underline{u}_{n+1} = f(x, \underline{u}_n, \underline{v}_n) & \text{in } \tilde{Q}_T \\ \underline{u}_{n+1}(x,t) = 0 & \text{on } \tilde{\Sigma}_T \\ \underline{u}_{n+1}(.,0) = u_0(.) & \text{in } \tilde{\Omega}. \end{cases}$$

Multiplying by $\frac{\partial \underline{u}_{n+1}}{\partial t}$ and integrating over $\tilde{Q}_T$, we get:

$$\int_{\tilde{Q}_T} \left(\frac{\partial \underline{u}_{n+1}(x,t)}{\partial t}\right)^2 - \int_{\tilde{Q}_T} \Delta_p \underline{u}_{n+1} \frac{\partial \underline{u}_{n+1}(x,t}{\partial t}\, dx\, dt +$$
$$+a \int_{\tilde{Q}_T} |\underline{u}_{n+1}|^{\alpha-1} \underline{u}_{n+1} \frac{\partial \underline{u}_{n+1}(x,t)}{\partial t}\, dx\, dt =$$
$$= \int_{\tilde{Q}_T} f(x, \underline{u}_n, \underline{v}_n) \frac{\partial \underline{u}_{n+1}(x,t)}{\partial t}\, dx\, dt,$$

then

$$\int_{\tilde{Q}_T} \left(\frac{\partial \underline{u}_{n+1}(x,t)}{\partial t}\right)^2 dx\, dt + \frac{1}{p}\int_{\tilde{\Omega}} |\nabla \underline{u}_{n+1}(x,T)|^p\, dx +$$
$$+\frac{a}{\alpha+1}\int_{\tilde{\Omega}} |\underline{u}_{n+1}(x,T)|^{\alpha+1}\, dx \leq c \int_{\tilde{Q}_T} \underline{u}_n^{\gamma} \underline{v}_n^{\delta+1}\, dx\, dt +$$
$$+ \int_{\tilde{Q}_T} \varphi(x) \underline{v}_n^{\eta} \frac{\partial \underline{u}_{n+1}(x,t)}{\partial t}\, dx\, dt + \int_{\tilde{Q}_T} f_1(x) \frac{\partial \underline{u}_{n+1}(x,t)}{\partial t}\, dx\, dt +$$
$$+\frac{1}{p}\int_{\tilde{\Omega}} |\nabla u_0(x)|^p\, dx + \frac{a}{\alpha+1}\int_{\tilde{\Omega}} |u_0(x)|^{\alpha+1}\, dx.$$

Using Young's Inequality, we have:

$$\left\| \frac{\partial \underline{u}_{n+1}(x,t)}{\partial t} \right\|_{L^2(\tilde{Q}_T)} \leq C(T) \tag{4.3}$$

$$\|\underline{u}_{n+1}\|_{L^\infty(0,T;W_0^{1,p}(\tilde{\Omega}))} \leq C(T) \tag{4.4}$$

$$\|\underline{u}_{n+1}\|_{L^\infty(\tilde{Q}_T)} \leq C(T), \tag{4.5}$$

where $C(T)$ is a positive constant.

From (4.3), (4.4) and (4.5), there exists a subsequence which converges to u in the sense of weak $*$ topology in $L^\infty(0,T;W_0^{1,p}(\tilde\Omega))$, which converges also to u weakly in $L^p(0,T;W_0^{1,p}(\tilde\Omega))$ and $\frac{\partial \underline{u}_{n+1}}{\partial t}$ converges weakly to $\frac{\partial u}{\partial t}$ in $L^2(\tilde Q_T)$.

By the argument of monotonicity [8], $\Delta_p \underline{u}_n$ converges weakly to $\Delta_p u$ in $L^{p^*}(0,T;W_0^{-1,p^*}(\tilde\Omega))$. Since $\underline{u}_n \to u$ a.e. and $\underline{v}_n \to v$ a.e., then we have by Convergence Dominated Theorem: $\underline{u}_{n+1}^\gamma \underline{v}_n^{\delta+1}$ converges to $u^\gamma v^{\delta+1}$.

Hence (u,v) is solution of $(\mathcal{P})$ in $\tilde Q_T$ such that:

$$0 \le u(x,t) \le \hat u(x) \text{ and } 0 \le v(x,t) \le \hat v(x).$$

We extend u by 0 and v by 0 out of $\tilde\Omega \times [0,T]$ $(\forall T>0)$.

v) Uniqueness: Suppose that there exists (u_1,u_2) and (v_1,v_2) two solutions of problem $(\mathcal{P})$, we have:

$$\frac{\partial(u_1-u_2)}{\partial t} - (\Delta_p u_1 - \Delta_p u_2) + a(u_1^\alpha - u_2^\alpha) = f(x,u_1,v_1) - f(x,u_2,v_2),$$

multiplying by (u_1-u_2), we obtain by using the monotonicity of operator $-\Delta_p$ and $(\mathcal{H}_3)$:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\tilde\Omega}(u_1-u_2)^2(x,t)\,dx\,dt \le K^1_{B_1,B_2}\int_{\tilde\Omega}(u_1-u_2)^2(x,t)\,dx\,dt + \\ + K^1_{B_1,B_2}\int_{\tilde\Omega}(u_1-u_2)(v_1-v_2)(x,t)\,dx\,dt.$$

Similarly:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\tilde\Omega}(v_1-v_2)^2(x,t)\,dx\,dt \le K^2_{B_1,B_2}\int_{\tilde\Omega}(v_1-v_2)^2(x,t)\,dx\,dt + \\ + K^2_{B_1,B_2}\int_{\tilde\Omega}(u_1-u_2)(v_1-v_2)(x,t)\,dx\,dt,$$

where the constants $K^1_{B_1,B_2}$ and $K^2_{B_1,B_2}$ are positive constant; where B_1 and B_2 are defined in (2.2).

By Holder's Inequality, we get:

$$\frac{1}{2}\frac{\partial}{\partial t}[\|(u_1-u_2)(x,t)\|^2_{L^2(\tilde\Omega)} + \|(v_1-v_2)(x,t)\|^2_{L^2(\tilde\Omega)}] \le \\ \le C[\|(u_1-u_2)(x,t)\|^2_{L^2(\tilde\Omega)} + \|(v_1-v_2)(x,t)\|^2_{L^2(\tilde\Omega)}],$$

where C is a positive constant.

Then from Gronwall's Lemma, we obtain: $u_1 = u_2$ and $v_1 = v_2$.

If $p > \frac{2N}{N+2}$ or $p = N$, $W^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$ and as $u \in L^\infty(0,T;W_0^{1,p}(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(Q_T)$ then from Lemma 3 we have $u \in \mathcal{C}(\mathbb{R}^+, L^2(\Omega))$ and similarly for $v \in \mathcal{C}(\mathbb{R}^+, L^2(\Omega))$. ■

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