PROJECTIONS ON HARDY SPACES
IN THE LIE BALL

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Abstract

On the Lie ball ω of $\mathbb{C}^n$, $n \geq 3$, we prove that for all $p \in [1, \infty)$, $p \neq 2$, the Hardy space $H^p(\omega)$ is an uncomplemented subspace of the Lebesgue space $L^p(\partial_0\omega, d\sigma)$, where $\partial_0\omega$ denotes the Shilov boundary of ω and $d\sigma$ is a normalized invariant measure on $\partial_0\omega$.

1. Introduction

Let $\omega = \omega_n$ denote the Lie ball of $\mathbb{C}^n$, $n \geq 3$, defined by

$$\omega = \left\{ z \in \mathbb{C}^n : 1 - 2|z|^2 + \left| \sum_{j=1}^{n} z_j^2 \right|^2 > 0 \text{ and } \left| \sum_{j=1}^{n} z_j^2 \right| < 1 \right\}.$$

The Shilov boundary $\partial_0\omega$ of ω is $\partial_0\omega = \{ e^{i\theta}x : \theta \in [0, 2\pi) \text{ and } x \in S_{n-1} \}$, where S_{n-1} denotes the unit sphere of $\mathbb{R}^n$. Denote by G the compact group consisting of the following transformations in $\mathbb{C}^n$: g belongs to G if there are $h \in SO(n)$ and $\theta \in [0, 2\pi)$ such that $g(z) = e^{i\theta}h(z)$ for all $z \in \mathbb{C}^n$. The group G is subgroup of the automorphism group of ω and G acts transitively on $\partial_0\omega$. Moreover, a normalized G-invariant measure on $\partial_0\omega$ is $d\sigma(e^{i\theta}x) = \frac{\Gamma(n/2)}{4\pi^{n/2+1}} \, d\theta \, d\mu(x)$, where $d\mu$ denotes the Lebesgue surface measure on S_{n-1}.

Recall that the Hardy space $H^p(\omega)$, $1 \leq p < \infty$, is defined as consisting of all functions f holomorphic in ω and such that

$$\sup_{0<r<1} \int_{\partial_0\omega} |f(rz)|^p \, d\sigma(z) < \infty.$$

Every $f \in H^p(\omega)$ has radial limits σ-a.e. on $\partial_0\omega$ and when we identify f with its boundary value function, the space $H^p(\omega)$ can be viewed as a closed subspace of $L^p(\partial_0\omega, d\sigma)$. For this fact, c.f. e.g. [**V**].

The main result of the present paper is the following:

Theorem 1.1. *For all $p \in [1, \infty)$, $p \neq 2$, $H^p(\omega)$ is an uncomplemented subspace of $L^p(\partial_0 \omega, d\sigma)$.*

In Chapter 12 of [**R1**], the same result is proved for $p = 1$ in the unit ball of $\mathbb{C}^n$. It turns out that the result for the Lie ball (Theorem 1.1) can be deduced exactly in the same way from the following theorem:

Theorem 1.2 [BeBo]. *In the Lie ball ω of $\mathbb{C}^n$, $n \geq 3$, the Szegö projection is L^p-unbounded for all $p \in (1, \infty)$ $p \neq 2$.*

We recall that the Szegö projection for ω is the orthogonal projection of $L^2(\partial_0 \omega, d\sigma)$ onto $H^2(\omega)$.

Let U denote the unit disc in the complex plane. Then the following question is open: are the Hardy spaces $H^p(\omega)$ and $H^p(U)$ isomorphic for $p \in (1, \infty)$, $p \neq 2$? P. Wojtaszczyk [**W**] answered this question affirmatively in the unit ball B_n of $\mathbb{C}^n$, $n \geq 2$. His proof uses the fact that $H^p(B_n)$ is a complemented subspace of $L^p(\partial B_n)$, by a theorem of [**KV**], and hence it cannot be applied to the Lie ball case. On the other hand, T. Wolniewicz [**Wo**] proved that for classical domains D of class I (in Hua's standard notation [**H**]), the Hardy space $H^1(D)$ is not isomorphic to $H^1(U)$ except when D is a unit ball. In particular, $H^1(D)$ is not isomorphic to $H^1(U)$ when D is the Lie ball ω_4 in C^4. It seems unlikely that $H^1(\omega_n)$ is isomorphic to $H^1(U)$ when $n = 3$ and $n \geq 5$ (i.e. for domains of class IV).

Finally, it seems to be certain that Theorems 1.1 and 1.2 can be extended to all bounded symmetric domains of tube type. To do this, an appropriate reference would be [**K**].

In the second section, we recall a decomposition of $L^2(\partial_0 \omega, d\sigma)$ into a direct sum of G-invariant subspaces (c.f. e.g. [**BCK**]). We denote by N the polynomial on $\mathbb{C}^n$ defined by

$$(1) \qquad\qquad N(z) = \sum_{j=1}^{n} z_j^2$$

and by $\square$ the differential polynomial defined by

$$(2) \qquad\qquad \square_z = \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2}.$$

Denote by H_k, $k \in \mathbb{Z}_+$, the space consisting of restrictions to $\partial_0 \omega$ of homogeneous holomorphic polynomials p of degree k such that $\square p = 0$.

Then $L^2(\partial_0\omega, d\sigma) = \sum\limits_{(j,k)\in\mathbb{Z}\times\mathbb{Z}_+} \oplus N^j H_k$ and the spaces $N^j H_k$ are pairwise orthogonal with respect to the usual inner product in $L^2(\partial_0\omega, d\sigma)$.

In the third section, we study the orthogonal projection of $L^2(\partial_0\omega, d\sigma)$ into $N^j H_k$ and linear maps $T : N^{j_1} H_{k_1} \rightarrow N^{j_2} H_{k_2}$ that commute with G.

In the fourth section, we characterize the G-invariant closed subspaces of $L^2(\partial_0\omega, d\sigma)$, $1 \leq p < \infty$ and of $\mathcal{C}(\partial_0, \omega)$. This extends a theorem due to A. Nagel and W. Rudin (Theorem 12.3.6 in [**R1**]) for the unit ball B_n of $\mathbb{C}^n$. Denote by $H(p,q)$, $(p,q) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ the space of homogeneous harmonic polynomials on $\mathbb{C}^n$ that have total degree p in the variables $z_1, \ldots, z_n$, and total degree q in the variables $\bar{z}_1, \ldots, \bar{z}_n$. Then $L^2(\partial B_n)$ is the direct sum of the spaces $H(p,q)$ and the theorem of Nagel and Rudin in B_n is stated in terms of the $H(p,q)$ spaces. Their proof still works in the Lie ball case, when the spaces $H(p,q)$ are replaced by the spaces $N^j H_k$.

Finally, in the fifth section, we prove Theorem 1.1.

2. Decomposition of $L^2(\partial_0\omega, d\sigma)$

When we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we let $\mathcal{P}$ denote the space of all complex-valued polynomials on $\mathbb{C}^n$. By the Stone-Weierstrass Theorem, the space of restrictions to $\partial_0\omega$ of elements of $\mathcal{P}$ is dense in $\mathcal{C}(\partial_0\omega)$, and hence in $L^2(\partial_0\omega, d\sigma)$. Denote by $\mathcal{P}^{(k)}$ the space of restrictions to $\partial_0\omega$ of homogeneous complex-valued polynomials of degree k; then

$$(3) \qquad L^2(\partial_0\omega, d\sigma) = \sum_{k=0}^{+\infty} \mathcal{P}^{(k)}.$$

Let N be the polynomial on $\mathbb{C}^n$ defined by (1); then

$$\bar{z}_j|_{\partial_0\omega} = N(z)^{-1} z_j|_{\partial_0\omega}.$$

Furthermore, when P_k denote the space of restrictions to $\partial_0\omega$ of homogeneous holomorphic polynomials of degree k on $\mathbb{C}^n$, we obtain:

$$(4) \qquad \mathcal{P}^{(k)} = \sum_{j=0}^{k} \oplus N^{-j} P_k.$$

For $q \in P_k$, $q(z) = \sum\limits_{\alpha \in A} b_\alpha z^\alpha$, define $\tilde{q}$ by $\tilde{q}(z) = \sum\limits_{\alpha \in A} \tilde{b}_\alpha z^\alpha$ and equip P_k with the differential inner product

$$(p|q) = \frac{1}{k!} p(D)\tilde{q},$$

where $p(D) = \sum\limits_{\alpha \in A} a_\alpha \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}$ and $A = \{\alpha = (\alpha_1, \ldots, \alpha_n) : |\alpha| = k\}$.

Denote by H_k, $k \geq 2$, the orthogonal of NP_{k-2} in P_k relative to the inner product $(|)$ and set $H_1 = P_1$, $H_0 = P_0$.

We have the following lemma:

Lemma 2.1. *The space H_k consists of all restrictions to $\partial_0\omega$ of homogeneous holomorphic polynomials of degree k that are annihilated by the differential polynomial $\square$ defined in (2).*

Moreover,

$$(6) \qquad P_k = \sum_{j=0}^{[k/2]} \oplus N^j H_{k-2j}.$$

Proof: This is just Lemma 3.2 in **[Co We]**. ∎

In the sequel, the letter Q will stand for $\mathbb{Z} \times \mathbb{Z}_+$. We prove next the following lemma:

Lemma 2.2. *For all $(j,k) \in Q$, the space $N^j H_k$ is G-invariant.*

Proof: We must prove that $(N^j p) \circ \phi \in N^j H_k$ for all $\phi \in G$ and $p \in H_k$. There is a constant $c_j(\phi)$ such that $(N_j p) \circ \phi = c_j(\phi) N^j \cdot (p \circ \phi)$, and hence it suffices to prove that $p \circ \phi \in H_k$. But $\square$ commutes with G, i.e. $\square(p \circ \phi) = (\square p) \circ \phi$ for all $\phi \in G$ and $p \in P_k$. Then, $\square p = 0$ implies $\square(p \circ \phi) = 0$ and the desired conclusion follows from the first assertion of Lemma 2.1. ∎

In the rest of the paper, $L^2(\partial_0\omega, d\sigma)$ will be equipped with the usual inner product:

$$(7) \qquad [f,g] = \int_{\partial_0\omega} f\bar{g}\, d\sigma.$$

We prove the following proposition:

Proposition 2.3. *The spaces $N^j H_k$, $(j,k) \in Q$, are pairwise orthogonal with respect to the inner product $[\ ,\ ]$.*

Proof: For all (j_1, k_1) and (j_2, k_2) in Q such that $(j_1, k_1) \neq (j_2, k_2)$ and for all $p \in H_{k_1}$ and $q \in H_{k_2}$, we have:

$$[N^{j_1}p, N^{j_2}q] = \left(\int_0^{2\pi} e^{2i\theta(j_1+k_1-j_2-k_2)}\, d\theta\right) \left(\int_{S_{n-1}} p\bar{q}\, d\mu\right).$$

Assume first that $k_1 \neq k_2$. On $\mathbb{R}^n$, p and q are homogeneous harmonic polynomials of respective degrees k_1 and k_2; it is known (cf. e.g. [**R1**, p. 254]) that the integral over S_{n-1} is then zero. Assume next that $k_1 = k_2$ and $j_1 \neq j_2$. In this case, the integral over $[0, 2\pi)$ is zero. This proves the proposition. ■

We deduce next the following corollary:

Corollary 2.4. *The space $L^2(\partial_0 \omega, d\sigma)$ is the direct sum of the pairwise orthogonal subspaces $N^j H_k$, $(j, k) \in Q$.*

Proof: In view of Proposition 2.3, it is enough to prove that $L^2(\partial_0 \omega, d\sigma)$ is contained in the sum of the spaces $N^j H_k$, $(j, k) \in Q$. Combining (4) and (6) yields:

$$\mathcal{P}^{(k)} = \sum_{\substack{j=0,1,\ldots,k \\ i=0,1,\ldots,\left[\frac{k}{2}\right]}} \oplus N^{-j+i} H_{k-2i}.$$

The conclusion then follows from (3). This completes the proof. ■

3. The spaces $N^j H_k$

Denote by S the representation of G in $\mathcal{P}$ defined by $S_\phi p(z) = p(\phi^{-1}(z))$ and denote by S_{jk} the restriction of S to $N^j H_k$. We have the following theorem:

Theorem 3.1. *The representation S_{jk} of G in $N^j H_k$ is irreducible, i.e. $N^j H_k$ has no proper G-invariant subspace.*

Proof: Since $SO(n) \subset G$, it is known (cf. Theorem 3.3 in [**Co We**]) that the representation S_{ok} in H_k is irreducible. On the other hand, $N^{-j} Y$ is a G-invariant subspace of H_k whenever Y is a G-invariant subspace of $N^j H_k$. The conclusion follows. ■

Fix $(j, k) \in Q$ and denote by π_{jk} the orthogonal projection of $L^2(\partial_0 \omega, d\sigma)$ onto its closed (since it is finite-dimensional) subspace $N^j H_k$. We prove first the following lemma:

Lemma 3.2. *The projection π_{jk} commutes with G, i.e. for all $\phi \in G$ and $f \in L^2(\partial_0 \omega, d\sigma)$, $\pi_{jk}(f \circ \phi) = (\pi_{jk} f) \circ \phi$.*

Proof: This is an immediate consequence of the G-invariance of $N^j H_k$ (Lemma 2.2) and $(N^j H_k)^\perp$. ■

We prove next the following theorem:

Theorem 3.3. *1. To every $z \in \partial_0 \omega$ corresponds a unique $K_z \in N^j H_k$ that satisfies*

$$(8) \qquad\qquad (\pi_{jk} f)(z) = [f, K_z] \qquad\qquad (f \in L^2(\partial_0 \omega, d\sigma)).$$

2. These kernel functions K_z have the following addtional properties:

$$(9) \qquad\qquad K_z(w) = \overline{K_w(z)} \qquad\qquad (z, w \in \partial_0 \omega),$$

$$(10) \qquad\qquad \pi_{jk} f = \int_{\partial_0 \omega} f(\zeta) K_\zeta \, d\sigma(\zeta) \qquad (f \in L^2(\partial_0 \omega, d\sigma)$$

$$(11) \qquad\qquad K_{\phi(z)} = K_z \circ \phi^{-1} \qquad\qquad (\phi \in G),$$

$$(12) \qquad\qquad K_z = K_z \circ \Psi$$

for all $\Psi \in G$ that fix z, and

$$(13) \qquad\qquad K_z(z) = K_w(w) > 0 \qquad\qquad (z, w \in \partial_0 \omega).$$

Proof: We write π for π_{jk}.

1. In the finite-dimensional space $N^j H_k$, the norms $\max\limits_{\partial_0 \omega} |N^j p|$ and $\|N^j p\|$ are equivalent.

The correspondence $f \rightarrow (\pi f)(z)$ is then a bounded linear functional on $L^2(\partial_0 \omega, d\sigma)$. Thus there is a unique $K_z \in L^2(\partial_0 \omega, d\sigma)$ that satisfies (8). Since $\pi f = 0$ for every $f \in (N^j H_k)^\perp$, it follows that $K_z \in N^j H_k$.

2. For $f = K_w$, (8) becomes

$$K_w(z) = [K_w, K_z] = \overline{[K_z, K_w]} = \overline{K_z(w)}.$$

This proves (9) and (9) shows that (10) is just another way of writing (8).

Since π commutes with G (Lemma 3.2), $(\pi f)(\phi(z)) = \pi(f \circ \phi)(z) = [f \circ \phi, K_z] = [f, K_z \circ \phi^{-1}]$ for every $f \in L^2(\partial_0 \omega, d\sigma)$. Combined with (8), this proves (11), hence also its special case (12).

Finally,

$$K_{\phi(z)}(\phi(z)) = (K_z \circ \phi^{-1})(\phi z)) = K_z(z)$$

is another consequence of (11). It proves (13), because

$$K_z(z) = [K_z, K_z] > 0. \qquad \blacksquare$$

Let us now prove the following proposition:

Proposition 3.4. *For each $z \in \partial_0 \omega$, the space $N^j H_k$ contains a unique f such that $f(z) = 1$ and $f = f \circ \Psi$ for every $\Psi \in G$ that fixes z.*

Proof: It follows from (12) and (13) that $f = \frac{K_z}{K_z(z)}$ satisfies the required properties. To prove the uniqueness, denote by e_1 the point $(1, 0, \ldots, 0)$ and assume $z = e_1$ without loss of generality. The invariance $f = f \circ \Psi$ for every $\Psi \in SO(n)$ that fixes e_1 shows then, for $f = N^j p$, $p \in H_k$ and for each w_1, that $(w_2, \ldots, w_n) \mapsto p(w_1, w_2, \ldots, w_n)$ is a polynomial in $w_2^2 + \cdots + w_n^2$. Since p is a homogeneous holomorphic polynomial of degree k on $\mathbb{C}^n$ that satisfies $p(e_1) = 1$, p therefore has the form

$$(14) \qquad p(w) = \sum_{i=1}^{r} c_i (w_2^2 + \cdots + w_n^2)^i w_1^{k-2i}$$

where $r = \left[\frac{k}{2}\right]$ and $c_0, \ldots, c_n$ are constants, $c_0 = 1$. Differentiation of (14) gives

$$(15) \qquad \Box p(w) = \sum_{i=0}^{r-1} b_i (w_2^2 + \cdots + w_n^2)^i w_1^{k-2i-2}$$

where

$$(16) \qquad b_i = (k - 2i)(k - 2i - 1)c_i + (2i + 1)(n + 2i)c_{i+1},$$

for all $i = 0, \ldots, r - 1$. Since $\Box p = 0$, (15) vanishes, so that $b_i = 0$ for all i and (16) successively determines $c_1, \ldots, c_r$. This proves the proposition. ∎

Our next result is the following:

Theorem 3.5. *The irreducible representations $S_{j_1 k_1}$ and $S_{j_2 k_2}$ of G in $N^{j_1} H_{k_1}$ and $N^{j_2} H_{k_2}$ respectively are inequivalent when $(j_1, k_1) \neq (j_2, k_2)$. More precisely, suppose $T : N^{j_1} H_{k_1} \to N^{j_2} H_{k_2}$ is linear and commutes with G. Then*

1. $T = 0$ *when $(j_1, k_1) \neq (j_2, k_2)$,*
2. *when $(j_1, k_1) = (j_2, k_2)$, there is a constant c such that $T = cI$ where I is the identity operator.*

Proof: The assertion 2 follows from Schur's Lemma, as well as the following assertion: either $T = 0$ or T is invertible when $(j_1, k_1) \neq (j_2, k_2)$. Let us prove that $T = 0$ in the latter case.

Let $K_z^1 \in N^{j_1} H_{k_1}$ and $K_z^2 \in N^{j_2} H_{k_2}$ be as in Theorem 3.3. If $\Psi \in G$ fixes z, then

$$(17) \qquad (TK_z^1) \circ \Psi = T(K_z^1 \circ \Psi) = TK_z^1$$

by (12), since T commutes with G. By Proposition 3.4, (17) shows that to every $z \in \partial_0 \omega$ corresponds a constant $c(z)$ such that $TK_z^1 = c(z)K_z^2$. Hence $TK_z^1(z) = c(z)K_z^2(z)$. By (13), $K_z^2(z)$ is independent of z. If $w = \phi(z)$, $\phi \in G$, then by (11), $(TK_w^1)(w) = T(K_z^1 \circ \phi^{-1})(\phi(z)) = TK_z^1(z)$.

We conclude from this that $c(z) = c$, the same for all $z \in \partial_0 \omega$.

If $f \in N^{j_1} H_{k_1}$, (10) shows that

$$(18) \qquad f = \int_{\partial_0 \omega} f(\zeta) K_\zeta^1 \, d\sigma(\zeta).$$

Apply T to (18) and use $TK_\zeta^1 = cK_\zeta^2$:

$Tf = c \int_{\partial_0 \omega} f(\zeta) K_\zeta^2 \, d\sigma(\zeta) = c\pi_{j_2 k_2} f$, for every $f \in N^{j_1} H_{k_1}$. If $(j_1, k_1) \neq (j_2, k_2)$, $\pi_{j_1 k_1} f = 0$ and hence $Tf = 0$. This proves the theorem. ∎

4. G-invariant spaces on $\partial_0 \omega$

For $\Omega \subset Q$ the algebraic sum of all $N^j H_k$ with $(j, k) \in \Omega$ will be denoted by E_Ω. We adopt the convention that $E_\Omega = \{0\}$ when Ω is empty.

The letter X will stand for any of the Banach spaces $\mathcal{C}(\partial_0 \omega)$ or $L^p(\partial_0 \omega, d\sigma)$, $1 \leq p < \infty$. The X-closure of E_Ω will be denoted by X_Ω. Trivially, every X_Ω is a G-invariant closed subspace in X. The proof of the converse is the main topic of the section. In the sequel, we write $\mathcal{C}$ for $\mathcal{C}(\partial_0 \omega)$ and L^p for $L^p(\partial_0 \omega, d\sigma)$.

We begin with the case $X = L^2$:

Theorem 4.1. *If Y is a G-invariant closed subspace of $L^2(\partial_0 \omega, d\sigma)$ and if Ω is the set of all $(j, k) \in Q$ such that $\pi_{jk} Y \neq \{0\}$, then $Y = (L^2)_\Omega$.*

Proof: Pick $(j, k) \in \Omega$. Since Y is G-invariant and π_{jk} commutes with G, $\pi_{jk} Y$ is a nontrivial invariant subspace of $N^j H_k$. By Theorem 3.1, we have therefore $\pi_{jk} Y = N^j H_k$.

Let $Y_0 = \{f \in Y : \pi_{jk} f = 0\}$, and let Y_1 be the orthogonal complement of Y_0 in Y. Then Y_0 is G-invariant, hence so is Y_1. Moreover, $\pi_{jk} : Y_1 \to N^j H_k$ is an isomorphism whose inverse we denote by Λ.

Choose $(r, s) \in Q$, $(r, s) \neq (j, k)$, and consider the linear map $T = \pi_{rs} \circ \Lambda : N^j H_k \to N^r H_s$. It is clear that T commutes with G; hence, by Theorem 3.5, $T = 0$ and $\pi_{rs} Y_1 = \{0\}$ for every $(r, s) \neq (j, k)$. By Corollary 2.4, Y_1 is a nontrivial subspace of $N^j H_k$, isomorphic to $N^j H_k$; since Y_1 is G-invariant, then $Y_1 = N^j H_k$ by Theorem 3.1. Furthermore, $N^j H_k \subset Y$, for every $(j, k) \in \Omega$; in other words, $(L^2)_\Omega \subset Y$. Since $\pi_{rs} Y = \{0\}$ for every $(r, s) \neq (j, k)$, a second look at Corollary 2.4 completes the proof of the theorem. ∎

We shall need the following three lemmas:

Lemma 4.2. *If $f \in X$, then $\phi \to f \circ \phi$ is a continuous map of G into X.*

Proof: If $\varepsilon > 0$, then $\|f - g\| < \varepsilon$ for some $g \in C$. There is a neighbourhood N of the identity in G such that $\|g - g \circ \phi\|_\infty < \varepsilon$ for every $\phi \in N$. Since $|f - f \circ \phi| \leq |f - g| + |g - g \circ \phi| + |(g - f) \circ \phi|$, we obtain $\|f - f \circ \phi\| < 3\varepsilon$ for every $\phi \in N$. ∎

Lemma 4.3. *If Y is a G-invariant closed subspace of X, then $Y \cap C$ is dense in Y.*

Proof: Pick $f \in Y$, choose N as in the proof of Lemma 4.2, let $\Lambda : G \to [0, \infty)$ be continuous, with support in N and $\int_G \Lambda \, d\phi = 1$. Define

$$g = \int_G \Lambda(\phi) f \circ \phi \, d\sigma.$$

The integrand is a continuous Y-valued function. Hence $g \in Y$. If $\Psi \in G$ is such that $\Psi(e_1) = \zeta$, where $e_1 = (1, 0, \ldots, 0)$, the invariance of the Haar measure shows that

$$g(\zeta) = \int_G \Lambda(\phi \psi^{-1}) f(\phi(e_1)) \, d\sigma.$$

Thus, $g \in Y \cap C$. Finally, the relation

$$f - g = \int_N \Lambda(\phi)(f - f \circ \phi) \, d\phi$$

proves that $\|f - g\| < 3\varepsilon$, since $\|f - f \circ \phi\| < 3\varepsilon$ if $\phi \in N$. ∎

Lemma 4.4. *If $Y \subset C$, Y is a G-invariant space, and some $g \in C$ is not in the uniform closure of Y, then g is not in the L^2-closure of Y.*

Proof: There is a regular complex Borel measure m on $\partial_0 \omega$ such that $\int f \, dm = 0$ for all $f \in Y$, but $\int g \, dm = 1$. There is a neighbourhood

N of the identity in G such that $Re \int g \circ \phi \, dm > \frac{1}{2}$ for every $\phi \in N$. Associate Λ to N as in the proof of Lemma 4.3, and define $L \in C^*$ by

$$(19) \qquad L(h) = \int_{\partial_0 \omega} dm(\zeta) \int_G \Lambda(\phi) h(\phi(\zeta)) \, d\phi.$$

The Schwarz inequality shows that the square of the absolute value of the inner integral in (19) is at most

$$\left(\int_G |\Lambda(\phi)|^2 \, d\phi \right) \left(\int_G |h(\phi(\zeta))|^2 \, d\phi \right) = c\|\Lambda\|_2^2 \int_{\partial_0 \omega} |h|^2 \, d\sigma,$$

so that

$$|L(h)| \leq c\|m\| \, \|\Lambda\|_2 \, \|h\|_2.$$

Thus L extends to a bounded linear functional on L^2. If we interchange the two integrals in (19), we see that $L(h) = 0$ for every $h \in Y$, whereas $Re L(g) \geq \frac{1}{2}$. This completes the proof. ∎

We can now prove the following theorem, announced at the beginning of the section:

Theorem 4.5. *If Y is a G-invariant closed subspace of X, and if Ω is the set of all $(j, k) \in Q$ for which $\pi_{jk} Y \neq \{0\}$, then $Y = X_\Omega$.*

.Proof: Observe first that the domain of π_{jk} can be extended to L^1, by (10). Define $\tilde{Y}$ to be the L^2-closure of $Y \cap C$. Since Y is X-closed, $Y \cap C$ is uniformly closed, so that Lemma 4.4 gives

$$(20) \qquad \tilde{Y} \cap C = Y \cap C.$$

Observe next that $Y \cap C$ is L^2-dense in $\tilde{Y}$, by definition, and is X-dense in Y, by Lemma 4.3. Since $\int_{\partial_0 \omega} |K_\zeta(z)|^2 \, d\sigma(\zeta) = K_z(z) = c$ is independent of z by (13), we get $|K_\zeta(z)| = \left| \int_{\partial_0 \omega} K_\zeta(w) \overline{K_z(w)} \, d\sigma(w) \right| \leq c$ for all ζ, z in $\partial_0 \omega$, so that by (10), we obtain

$$|\pi_{jk} f(z)| \leq c\|f\|_{L^p},$$

for all $p \in [1, \infty]$ and $z \in \partial_0 \omega$. Hence each π_{jk} is X-continuous as well as L^2-continuous. It follows that $\pi_{jk} Y = \{0\}$ if and only if $\pi_{jk} \tilde{Y} = \{0\}$. Theorem 4.1 shows therefore that $\tilde{Y}$ is the L^2-closure of E_Ω. Since $E_\Omega \subset C$, another application of Lemma 4.4 gives

$$(21) \qquad \tilde{Y} \cap C = \text{uniform closure of } E_\Omega.$$

Since $Y \cap C$ is X-dense in Y (Lemma 4.3), (20) and (21) imply that Y is the X-closure of E_Ω. This is the assertion of the theorem. ∎

We deduce the following corollary:

Corollary 4.6. *Suppose* $T : X_1 \to X_2$ *is a continuous linear map that commutes with* G. *Then there exist* $c(j, k) \in \mathbb{C}$, *for every* $(j, k) \in Q$, *such that*

$$(22) \qquad\qquad Tf = c(j, k)f \text{ for all } f \in N^j H_k.$$

The numbers $c(j, k)$ *determine* T.

Proof: The map $\pi_{rs}T : N^j H_k \to N^r H_s$ commutes with G, hence by Theorem 3.5, $\pi_{rs}Tf = 0$ if $f \in N^j H_k$ and $(r, s) \neq (j, k)$. Consequently, $T(N^j H_k)$ is a (finite dimensional, hence closed) G-invariant subspace of X_2 that is annihilated by π_{rs} wherever $(r, s) \neq (j, k)$. By Theorem 4.5, $T(N^j H_k) = N^j H_k$ or $\{0\}$. Now (22) follows from Theorem 3.5. ∎

5. Proof of Theorem 1.1

Assume, to reach a contradiction, that for some $p \in [1, \infty)$, $p \neq 2$, there is a continuous linear projection P of L^p onto $H^p = H^p(\omega)$. Define $T : L^p \to H^p$ by

$$Tf = \int_G [P(f \circ \phi^{-1})] \circ \phi \, d\phi.$$

It is proved in [**R2**] that T is then a continuous linear projection of L^p onto H^p that commutes with G. Thus T is as in (22), (the case where $X_1 = X_2 = L^p$ in Corollary 4.6), with $c(j, k) = 1$ for all $(j, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ and $c(j, k) = 0$ whenever $j < 0$ and $k \geq 0$. Since it is well known (cf. e.g. [**H**]) that

$$H^2 = \sum_{(j, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+} \oplus N^j H_k,$$

this implies that Tf is the boundary value function of the Szegö projection of f. We conclude that the Szegö projection of ω is L^p-bounded for some $p \in [1, \infty)$, $p \neq 2$. This contradicts Theorem 1.2. ∎

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