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ON A POINTWISE ERGODIC THEOREM FOR MULTIPARAMETER SEMIGROUPS

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Abstract

Let T_i $(i = 1, 2, \dots, d)$ be commuting null preserving transformations on a finite measure space $(X, \mathcal{F}, \mu)$ and let $1 \le p < \infty$. In this paper we prove that for every $f \in L_p(\mu)$ the averages

$$A_n f(x) = (n+1)^{-d} \sum_{0 \le n_i \le n} f(T_1^{n_1} T_2^{n_2} \dots T_d^{n_d} x)$$

converge a.e. on X if and only if there exists a finite invariant measure ν (under the transformations T_i) absolutely continuous with respect to μ and a sequence $\{X_N\}$ of invariant sets with $X_N \uparrow X$ such that $\nu B > 0$ for all nonnull invariant sets B and such that the Radon-Nikodym derivative $v = d\nu/d\mu$ satisfies $v \in L_q(x_N, \mu)$, $1/p + 1/q = 1$, for each $N \ge 1$.

1. Introduction

We refer to [**2**] for the basic notation in ergodic theory. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \to X$ $(i = 1, 2, \dots, d)$ be commuting null preserving transformations, where $d \ge 1$ is a fixed integer. Associated with these transformations T_i and for any measurable function f on X we have the averages

$$A_n f(x) = (n+1)^{-d} \sum_{0 \le n_i \le n} f(T_1^{n_1} T_2^{n_2} \dots T_d^{n_d} x) \quad (n \ge 0)$$

and the maximal operator

$$Mf = \sup_{n \ge 0} A_n |f|.$$

Further each T_i defines, by the Radon-Nikodym Theorem, a unique positive linear contraction operator T_i^* on $L_1(\mu)$ by the relation

$$\int_B T_i^* u \, d\mu = \int_{T_i^{-1}B} u \, d\mu \quad (u \in L_1(\mu), \, B \in \mathcal{F}).$$

Under the additional hypothesis that all the transformations T_i are invertible, Martín-Reyes [**3**] has recently proved the equivalence of the following conditions, for $1 \leq p < \infty$.

(a) The sequence $\{A_n f\}$ converges a.e. for all f in $L_p(\mu)$;
(b) There exists a positive measurable function U on X such that

$$\int_{\{|A_n f| > t\}} U \, d\mu \leq t^{-p} \int_X |f|^p \, d\mu \quad (t > 0, \, f \in L_p(\mu)).$$

In this paper, without assuming the invertibility hypothesis on the transformations T_i, we intend to characterize those finite measures μ for which (a) holds.

2. The result

Theorem. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \to X$ $(i = 1, 2, \ldots, d)$ be commuting null preserving transformations. If $1 \leq p < \infty$, then the following are equivalent.*

(a) *For any $f \in L_p(\mu)$ the sequence $\{A_n f\}$ converges to a finite limit a.e. on X.*
(b) *For any $u \in L_1(\mu)$ the averages*

$$A_n^* u = (n+1)^{-d} \sum_{0 \leq n_i \leq n} T_1^{*^{n_1}} T_2^{*^{n_2}} \ldots T_d^{*^{n_d}} u \quad (n \geq 0)$$

converge a.e. on X and also in the norm topology of $L_1(\mu)$; further to every $v \in L_1^+(\mu)$ with $T_i^* v = v$ for all $i = 1, 2, \ldots, d$ there corresponds a sequence $\{X_N\}$ of invariant sets with $X_N \uparrow X$ such that $v \in L_q(X_N, \mu)$, $1/p + 1/q = 1$, for all $N \geq 1$.
(c) *There exists $v \in L_1^+(\mu)$ with $T_i^* v = v$ for all $i = 1, 2, \ldots, d$ and a sequence $\{X_N\}$ of invariant sets with $X_N \uparrow X$ such that $\int_B v \, d\mu > 0$ for all nonnull invariant sets B and such that $v \in L_q(X_N, \mu)$, $1/p + 1/q = 1$, for all $N \geq 1$.*
(d) *$Mf < \infty$ for all $f \in L_p(\mu)$.*

(e) *There exists a positive measurable function U on X such that*

$$\int_{\{Mf>t\}} U\,d\mu \le t^{-p}\int_X |f|^p\,d\mu \quad (t>0,\ f\in L_p(\mu)).$$

(f) *There exists a positive measurable function U on X, a constant $r>0$, and a subsequence $\{n(k)\}$ of $\{n\}$ such that*

$$\int_{\{|A_{n(k)}f|>t\}} U\,d\mu \le t^{-r}\left(\int_X |f|^p\,d\mu\right)^{r/p} \quad (t>0,\ f\in L_p(\mu)).$$

We begin by proving the following lemma, which deals with the case $p=\infty$.

Lemma. *Let $(X,\mathcal{F},\mu)$ be a finite measure space and let $T_i : X\to X$ $(i=1,2,\dots,d)$ be commuting null preserving transformations. Then the following are equivalent.*

(a) *The sequence $\{A_n f\}$ converges a.e. on X for all $f\in L_\infty(\mu)$.*
(b) *The sequence $\{A_n^* u\}$ converges in the norm topology of $L_1(\mu)$ for all $u\in L_1(\mu)$.*
(c) *For any $u\in L_1^+(\mu)$ with $\|u\|_1>0$ the pointwise limit $u_0^*(x)=\lim_n A_n^*(x)$ exists a.e. on X and satisfies $\|u_0^*\|_1>0$.*
(d) *For any $u\in L_1^+(\mu)$ with $\|u\|_1>0$ we have*

$$0<\|\liminf_n A_n^* u\|_1<\infty.$$

Proof:

(a) ⇒ (b) follows from a mean ergodic theorem (see e.g. [**2**, Theorem 2.1.5]).

(b) ⇒ (a) and (c). Let $v_0=\text{strong-}\lim_n A_n^* 1 (\in L_1^+(\mu))$. Since $T_i^* v_0 = v_0$ for all $i=1,2,\dots,d$, we have

$$Y\subset T_i^{-1}Y \text{ for all } i=1,2,\dots,d, \text{ where } Y=\{v_0>0\}.$$

Since the measure $\nu=v_0\,d\mu$ is invariant under the T_i's, we may regard the transformations T_i as commuting measure preserving transformations on a finite measure space $(Y, v_0\,d\mu)$. Then, by the classical multiparameter pointwise ergodic theorem, for any $f\in L_\infty(\mu)$ the sequence

$\{A_n f\}$ converges a.e. on Y. To prove the a.e. convergence of $\{A_n f\}$ on $X \backslash Y$, it is sufficient to show that

$$(T_1 T_2 \dots T_d)^{-n} Y \uparrow X.$$

To do this, let $B = \lim_n (T_1 T_2 \dots T_d)^{-n} Y$. We see easily that $T_i^{-1} B = B$ for all $i = 1, 2, \dots, d$, i.e., B is an invariant set. Hence

$$\mu(X \backslash B) = \int_{X \backslash B} A_n^* 1 \, d\mu \to \int_{X \backslash B} v_0 \, d\mu = \int_{\{v_0 = 0\}} v_0 \, d\mu = 0.$$

To prove (c), let $u \in L_1^+(\mu)$ and $\|u\|_1 > 0$. Since $\|A_n^* u\|_1 = \|u\|_1 > 0$ and $\{A_n^* u\}$ converges in the norm topology of $L_1(\mu)$, it is sufficient to prove the a.e. convergence of $\{A_n^* u\}$. Since the transformations T_i preserve the measure $\nu = v_0 \, d\mu$, the classical pointwise ergodic theorem for multiparameter semigroups of Dunford-Schwartz operators and an approximation argument imply that $\{A_n^* u\}$ converges a.e. on Y.

To prove that $\lim_n A_n^* u(x) = 0$ a.e. on $X \backslash Y$, we use Brunel's Theorem (see e.g. [**2**, Theorem 6.3.4]) concerning an ergodic inequality for commuting linear contraction operators on $L_1(\mu)$: there exists a constant $K_d > 0$ and a positive linear operator Q on $L_\infty(\mu)$ of the form

$$Qf(x) = \sum_{n_i \ge 0} a(n_1, n_2, \dots, n_d) f(T_1^{n_1} T_2^{n_2} \dots T_d^{n_d} x),$$

where $a(n_1, n_2, \dots, n_d) > 0$ and $\sum_{n_i \ge 0} a(n_1, n_2, \dots, n_d) = 1$, such that if Q^* denotes the positive linear operator on $L_1(\mu)$ associated with Q, then

$$\limsup_n A_n^* u \le K_d \cdot \limsup_n (n+1)^{-1} \sum_{i=0}^{n} Q^{*i} u \quad (u \in L_1^+(\mu)).$$

Let $C = \{x : \sum_{i=0}^{\infty} Q^{*i} u = \infty\} \backslash Y$. Since $\|Q^*\|_1 = 1$, it follows that $Q 1_C \ge 1_C$, 1_C being the indicator function of C. Thus we have $C \subset T_i^{-1} C$ for all $i = 1, 2, \dots, d$, and hence

$$\mu C \le \int_X A_n 1_C \, d\mu = \int_C A_n^* 1 \, d\mu \to \int_C v_0 \, d\mu = \int_{\{v_0 = 0\}} v_0 \, d\mu = 0.$$

This proves that $\lim_n A_n^* u(x) = 0$ a.e. on $X \backslash Y$.

(c) $\Rightarrow$ (d). Obvious.

(d) $\Rightarrow$ (b). There exists $v_0 \in L_1^+(\mu)$ with $T_i^* v_0 = v_0$ for all $i = 1, 2, \ldots, d$ such that if $v \in L_1^+(\mu)$ satisfies $T_i^* v = v$ for all $i = 1, 2, \ldots, d$ then $\{v > 0\} \subset \{v_0 > 0\}$. Let $Y = \{v_0 > 0\}$ and

$$B = \lim_n (T_1 T_2 \ldots T_d)^{-n} Y.$$

Since B and $X \backslash B$ are invariant sets, it follows that if $u \in L_1^+(X \backslash B, \mu)$ and $\|u\|_1 > 0$ then the function

$$\tilde{u}_0 = \liminf_n A_n^* u \quad \text{satisfies} \quad \{\tilde{u}_0 > 0\} \cap \{v_0 > 0\} = \phi.$$

But this is impossible, since $T_i^* \tilde{u}_0 = \tilde{u}_0 \in L_1^+(\mu)$ for all $i = 1, 2, \ldots, d$ and (d) implies that $\mu\{\tilde{u}_0 > 0\} > 0$. We conclude that

$$(T_1 T_2 \ldots T_d)^{-n} Y \uparrow X.$$

Hence by an approximation argument we see that $\{A_n^* u\}$ converges in the norm topology of $L_1(\mu)$ for all $u \in L_1(\mu)$, completing the proof. ■

Proof of the Theorem: (a) $\Rightarrow$ (b). The first part of (b) follows from the lemma. To prove the second part, let $v \in L_1^+(\mu)$ be such that $T_i^* v = v$ for all $i = 1, 2, \ldots, d$. Putting $Y = \{v > 0\}$, we see that the transformations T_i can be regarded as commuting null preserving transformations on the measure space (Y, μ). Since the measure $\nu = v\,d\mu$ is invariant under the transformations T_i, it follows that these T_i are conservative on (Y, μ).

By this and the fact that for each f in $L_p(Y, \mu)$ the sequence $\{A_n f\}$ converges to a finite limit a.e. on Y, we can apply Theorem 3.1 in [**4**] to infer that there exists a sequence $\{Y_N\}$ of sets in $\mathcal{I}_Y$, where

$$\mathcal{I}_Y = \{B \in \mathcal{F} : B \subset Y,\, B = Y \cap T_i^{-1} B \text{ for all } i = 1, 2, \ldots, d\},$$

such that $Y_N \uparrow Y$ and $v \in L_q(Y_N, \mu)$ for all $N \geq 1$. Then, letting

$$X_N = \left[\lim_n (T_1 T_2 \ldots T_d)^{-n} Y_N\right] \cup \left[X \backslash \lim_n (T_1 T_2 \ldots T_d)^{-n} Y\right],$$

we have $v \in L_q(X_N, \mu)$ for all $N \geq 1$, $X_N \uparrow X$, and $X_N \in \mathcal{I}$ where

$$\mathcal{I} = \{B \in \mathcal{F} : B = T_i^{-1} B \text{ for all } i = 1, 2, \ldots, d\}.$$

(b) $\Rightarrow$ (c). It is enough to put $v = \text{strong-}\lim_n A_n^* 1$.

(c) $\Rightarrow$ (a). Put $Y = \{v > 0\}$. It follows (cf. the proof of the lemma) that

$$(T_1 T_2 \ldots T_d)^{-n} Y \uparrow X.$$

Hence it is sufficient to prove that for each f in $L_p(Y,\mu)$ the sequence $\{A_n f\}$ converges to a finite limit a.e. on Y; this follows from the equivalence of (a) and (f) of Theorem 3.1 in [**4**], since the transformations T_i may be regarded as commuting conservative null preserving transformations on the measure space (Y,μ).

(a) ⇒ (d). Obvious.

(d) ⇒ (e). This follows from Nikishin's Theorem (see e.g. [**1**, p. 536]).

(e) ⇒ (f). Obvious.

(f) ⇒ (a). We may suppose that $0 < U \le 1$ on X. Using an approximation argument we see (cf. the proof of (d) ⇒ (a) of Theorem 3.1 in [**4**]) that

$$\lim_{\mu B\to 0} \sup_{k\ge 1} \int_B A^*_{n(k)} 1\, d\mu = 0.$$

Hence by a mean ergodic theorem we see that the sequence $\{A^*_n 1\}$ converges in the norm topology of $L_1(\mu)$. Write $v = \text{strong-}\lim_n A^*_n 1$ and $Y = \{v > 0\}$. Since $T^*_i v = v$ for all $i = 1, 2, \ldots, d$ and $(T_1 T_2 \ldots T_d)^{-n} Y \uparrow X$, it follows from the classical multiparameter pointwise ergodic theorem that for any $f \in L^+_p(\mu)$ the limit

$$f^*(x) = \lim_n A_n f(x)$$

exists a.e. on X (but may be equal to infinity on some subset of X).

To prove that $f^* < \infty$ a.e. on X, we observe that $\{f^* = \infty\} \subset \liminf_k \{A_{n(k)} f > t\}$ for all $t > 0$; hence by Fatou's Lemma and (f)

$$\int_{\{f^*=\infty\}} U\, d\mu \le \liminf_k \int_{\{A_{n(k)} f > t\}} U\, d\mu \le t^{-r} \left(\int_X f^p\, d\mu \right)^{r/p}.$$

Letting $t \uparrow \infty$, we have $\int_{\{f^*=\infty\}} U\, d\mu = 0$ and $\mu\{f^* = \infty\} = 0$. The proof is complete. ■

Since the above proofs of the implications (a) ⇒ (d) ⇒ (e) ⇒ (f) ⇒ (a) hold for $0 < p < \infty$, we have the

Corollary. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and let $T_i : X \to X$ $(i = 1, 2, \ldots, d)$ be commuting null preserving transformations. Let $0 < p$, $r < \infty$. If there exists a subsequence $\{n(k)\}$ of $\{n\}$ such that the operators $A_{n(k)}$ are equicontinuous mappings from $L_p(\mu)$ to $L_r(\mu)$ then* *for any $f \in L_p(\mu)$ the sequence $\{A_n f\}$ converges to a finite limit a.e. on X.*

Proof: This follows from the equivalence of (a) and (f) of the theorem. ■

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Added in proof: An obvious argument shows that condition (c) of the theorem may be sharpened as follows. (c') *There exists $v \in L_q^+(\mu)$ with $T_i^* v = v$ for all $1 \leq i \leq d$ such that $\int_B v\, d\mu > 0$ for all nonnull invariant sets B.*