NORMAL BASES FOR THE SPACE OF CONTINUOUS FUNCTIONS DEFINED ON A SUBSET OF $\mathbb{Z}_p$

ANN VERDOODT

Abstract

Let $K$ be a non-archimedean valued field which contains $\mathbb{Q}_p$ and suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. $V_q$ is the closure of the set $\{aq^n|n = 0, 1, 2, \ldots\}$ where $a$ and $q$ are two units of $\mathbb{Z}_p$, $q$ not a root of unity. $C(V_q \rightarrow K)$ is the Banach space of continuous functions from $V_q$ to $K$, equipped with the supremum norm. Our aim is to find normal bases $(r_n(x))$ for $C(V_q \rightarrow K)$, where $r_n(x)$ does not have to be a polynomial.

1. Introduction

The main aim of this paper is to find normal bases $(r_n(x))$ for the space of continuous functions on $V_q$, where $r_n(x)$ does not have to be a polynomial.

Therefore we start by recalling some definitions and some previous results.

Let $E$ be a non-archimedean Banach space over a non-archimedean valued field $L$.

Let $f_1, f_2, \ldots$ be a finite or infinite sequence of elements of $E$. We say that this sequence is orthogonal if $\|\alpha_1f_1 + \cdots + \alpha_kf_k\| = \max\{\|\alpha_i f_i\| : i = 1, \ldots, k\}$ for all $k$ in $\mathbb{N}$ (or for all $k$ that do not exceed the length of the sequence) and for all $\alpha_1, \ldots, \alpha_k$ in $L$. If the sequence is infinite, it follows that $\left\|\sum_{i=1}^{\infty} \alpha_i f_i\right\| = \max\{\|\alpha_i f_i\| : i = 1, 2, \ldots\}$ for all $\alpha_1, \alpha_2, \ldots$ in $L$ for which $\lim_{i \to \infty} \alpha_i f_i = 0$. An orthogonal sequence $f_1, f_2, \ldots$ is called orthonormal if $\|f_i\| = 1$ for all $i$.

This leads us to the following definition:
If $E$ is a non-archimedean Banach space over a non-archimedean valued field $L$, then a family $(f_i)$ of elements of $E$ is a (ortho)normal basis of $E$ if the family $(f_i)$ is orthonormal and also a basis.

An equivalent formulation is (see [1, Propositions 50.4 and 50.6])

If $E$ is a non-archimedean Banach space over a non-archimedean valued field $L$, then a family $(f_i)$ of elements of $E$ is a (ortho)normal basis of $E$ if each element $x$ of $E$ has a unique representation $x = \sum x_i f_i$ where $x_i \in L$ and $x_i \to 0$ if $i \to \infty$, and if the norm of $x$ is the supremum of the norms of $x_i$.

Let $\mathbb{Z}_p$ be the ring of $p$-adic integers, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $K$ is a non-archimedean valued field, $K$ containing $\mathbb{Q}_p$, and we suppose that $K$ is complete for the valuation $|\cdot|$, which extends the $p$-adic valuation. Let $a$ and $q$ be two units of $\mathbb{Z}_p$, $q$ not a root of unity. We define $V_q$ to be the closure of the set $\{aq^n | n = 0, 1, 2, \ldots \}$. The set $V_q$ has been described in [3]. Let $C(V_q \to K)$ (resp. $C(\mathbb{Z}_p \to K)$) be the Banach space of continuous functions from $V_q$ to $K$ (resp. $\mathbb{Z}_p$ to $K$) equipped with the supremum norm. $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0$ is the set of natural numbers without zero.

We introduce the following:

If $x$ is an element of $\mathbb{Q}_p$, $x$ can be written in the following way:

$$x = \sum_{j=-\infty}^{+\infty} a_j p^j \quad \text{where } a_{-i} \text{ is zero for } i \text{ sufficiently large } (i \in \mathbb{N}) \quad \text{(see [1, section 3 and section 4]).}$$

This is called the Hensel development of the $p$-adic integer $x$. We then define the $p$-adic entire part $[x]_p$ of $x$ by $[x]_p = \sum_{j=-\infty}^{-1} a_j p^j$ and we put $x_n = p^n [p^{-n} x]_p = \sum_{j=-\infty}^{-n-1} a_j p^j \quad (n \in \mathbb{N})$.

We write $m < x$, if $m$ is one of the numbers $x_0, x_1, \ldots$. We then say that "$m$ is an initial part of $x$" or "$x$ starts with $m$" (see [1, section 62]).

If $n$ belongs to $\mathbb{N}_0$, $n = \sum_{j=0}^{s-1} a_j p^j \text{ where } a_s \neq 0$, then we put $n_- = \sum_{j=0}^{s-1} a_j p^j$. We remark that $n_- < n$.

In [1, Theorem 62.2], we find the following result which is due to van der Put:

**Theorem.**

The functions $g_0, g_1, \ldots$ defined by

$$g_n(x) = \begin{cases} 1 & \text{if } n < x, \\ 0 & \text{otherwise,} \end{cases}$$
form a normal basis for $C(\mathbb{Z}_p \to K)$. If $f$ is an element of $C(\mathbb{Z}_p \to K)$, then $f$ can be written as a uniformly convergent series $f(x) = \sum_{k=0}^{\infty} \gamma_k g_k(x)$ where $\gamma_0 = f(0)$ and $\gamma_n = f(n) - f(n-1)$ if $n \in \mathbb{N}_0$.

We now survey the content of this paper: In Theorem 1 of section 2, our aim is to find a basis $(e_n(x))$ analogous to van der Put's basis, but with the space $C(\mathbb{Z}_p \to K)$ replaced by $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$, then there exist elements $a_k$ of $K$ such that $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where the series on the right-hand-side is uniformly convergent. We are able to give an expression for the coefficients $a_k$.

In Theorem 2 of section 3, we prove the following result:

Define $r_n(x) = \sum_{j=0}^{n} c_{n,j} e_j(x)$, $c_{n,j} \in K$, $c_{n,n} \neq 0$ ($(e_n(x))$ as in Theorem 1 below).

Then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$ if and only if for all $n \|r_n\| = 1$ and $|c_{n,n}| = 1$.

In Theorem 3 of section 3, we give an extension of Theorem 2: Let $(r_n(x))$ be such a sequence which forms a normal basis for $C(V_q \to K)$, and let $(s_n(x))$ be a sequence such that $s_n(x) = \sum_{j=0}^{n} d_{n,j} r_j(x)$, $d_{n,j} \in K$, $d_{n,n} \neq 0$. Then $(s_n(x))$ forms a normal basis for $C(V_q \to K) \iff \|s_n\| = 1, |d_{n,n}| = 1 \iff |d_{n,j}| \leq 1, |d_{n,n}| = 1$.

Acknowledgement. I thank professor Van Hamme for the advice he gave me during the preparation of this paper.

2. Proof of the first theorem

We start with some lemmas and some definitions.

Definition.
If $b$ and $x$ are elements of $\mathbb{Z}_p$, $b \equiv 1 \pmod{p}$, then we put $b^x = \lim_{n \to x} b^n$.

The mapping: $\mathbb{Z}_p \to \mathbb{Z}_p : x \to b^x$ is continuous.

For more details, we refer the reader to [1, section 32].

Notation.
Take $m \geq 1$, $m$ the smallest integer such that $q^m \equiv 1 \pmod{p}$.
We have $1 \leq m \leq p - 1$ and $(q^m)^x$ is defined for all $x$ in $\mathbb{Z}_p$. 
Definition.

Let \( k \) be a natural number prime to \( p \). We denote by \( \mathbb{Z}_p(k) \) the projective limit \( \mathbb{Z}_p(k) = \lim_\rightarrow (\mathbb{Z}/kp^j\mathbb{Z}) \cong (\mathbb{Z}/k\mathbb{Z}) \times \mathbb{Z}_p. \)

In the following lemma we use the fact that \( \mathbb{Z}_p(m) = (\mathbb{Z}/m\mathbb{Z}) \times \mathbb{Z}_p \) to denote an element of \( \mathbb{Z}_p(m) \) as \( x = (r, y) \). Also, if \( n \in \mathbb{N}, n = r + mk \) \((0 \leq r < m)\) then the map \( n \rightarrow (r, k) \) imbeds \( \mathbb{N} \) in \( \mathbb{Z}_p(m) \).

Lemma 1.

The mapping \( \varphi : \mathbb{Z}_p(m) \rightarrow V_q : (r, y) \rightarrow aq^r(q^m)^y \) is a homeomorphism.

The proof of this lemma can be found in [2, p. 377].

Corollary.

If \( q \equiv 1 \pmod{p} \), i.e. \( m = 1 \), then the mapping: \( \mathbb{Z}_p \rightarrow V_q : x \rightarrow aq^x \) is a homeomorphism.

Let \( \beta \) be an element of \( \mathbb{Z}_p \{0\} \). We want to know the valuation of the \( p \)-adic integer \((q^m)^\beta - 1\). Therefore we need two lemmas:

The following lemmas (2 and 3) are proved in [3]:

Lemma 2.

Let \( \alpha \) be an element of \( \mathbb{Z}_p \), \( \alpha \equiv 1 \pmod{p^r} \), \( \alpha \neq 1 \pmod{p^{r+1}} \) \( r \geq 1. \)

If \((p, r) \neq (2, 1)\), \( \beta \in \mathbb{Z}_p \{0\} \) then \( \alpha^\beta \equiv 1 \pmod{p^{r+\text{ord}_p \beta}} \), \( \alpha^\beta \neq 1 \pmod{p^{r+1+\text{ord}_p \beta}} \).

Corollary.

Let \( q^m \equiv 1 \pmod{p^{k_0}} \), \( q^m \neq 1 \pmod{p^{k_0+1}} \). If \((p, k_0) \neq (2, 1)\), \( \beta \in \mathbb{Z}_p \{0\} \) then \((q^m)^\beta \equiv 1 \pmod{p^{k_0+\text{ord}_p \beta}}, (q^m)^\beta \neq 1 \pmod{p^{k_0+1+\text{ord}_p \beta}} \).

In Lemma 2 we excluded the case where \((p, r) = (2, 1)\). This case will be handled in the following lemma:

Lemma 3.

Let \( \alpha \) be an element of \( \mathbb{Z}_2 \), \( \alpha \equiv 3 \pmod{4} \). Define a natural number \( n \) by \( \alpha = 1 + 2 + 2^2 \varepsilon, \varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots, \varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{n-1} = 1, \varepsilon_n = 0. \)

If \( \beta \in \mathbb{Z}_2 \{0\}, \text{ord}_2 \beta = 0 \) then \( \alpha^\beta \equiv 1 \pmod{2}, \alpha^\beta \neq 1 \pmod{4}. \)
If $\beta \in \mathbb{Z}_2 \setminus \{0\}$, $\text{ord}_2 \beta = k \geq 1$ then $\alpha^\beta \equiv 1 \pmod{2^{n+2+\text{ord}_2 \beta}}$, $\alpha^\beta \not\equiv 1 \pmod{2^{n+3+\text{ord}_2 \beta}}$.

**Corollary.**

If $q \equiv 3 \pmod{4}$, we define a natural number $N$ by $q = 1 + 2 + 2^2 \varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$.

If $\beta \in \mathbb{Z}_2 \setminus \{0\}$, $\text{ord}_2 \beta = 0$ then $q^\beta \equiv 1 \pmod{2}$, $q^\beta \not\equiv 1 \pmod{4}$.

If $\beta \in \mathbb{Z}_2 \setminus \{0\}$, $\text{ord}_2 \beta = k \geq 1$ then $q^\beta \equiv 1 \pmod{2^{N+2+\text{ord}_2 \beta}}$, $q^\beta \not\equiv 1 \pmod{2^{N+3+\text{ord}_2 \beta}}$.

We remark that it is possible to write each $x$ and element of $V_q$ in the following way: $x = aq^i (q^m)^{\alpha_x}$ where $i_x$ is a natural number, $0 \leq i_x < m$, and where $\alpha_x$ is an element of $\mathbb{Z}_p$. This immediately follows from Lemma 1. This leads us to the following definition:

**Definition.**

We now define a sequence of functions $e_k$ in the following way. Write $k(\in \mathbb{N})$ in the form $k = i + mj$, $0 \leq i < m$, $(i, j \in \mathbb{N})$. The functions $e_k$ are defined by

$$e_k(x) = e_{i+mj}(x) = 1 \text{ if } x = aq^i (q^m)^{\alpha_x} \text{ where } i_x = i, j < \alpha_x.$$  

$$= 0 \text{ otherwise.}$$

Let us use the notation $B(b, r^-)$ for the 'open' disc with radius $r$ and with center $b$, i.e. $B(b, r^-) = \{x \in \mathbb{V}_q | |x-b| < r\}$, and $B(b, r)$ for the 'closed' disc with radius $r$ and with center $b$, i.e. $B(b, r) = \{x \in \mathbb{V}_q | |x-b| \leq r\}$.

In the following lemmas we will show that the functions $e_k(x)$ are characteristic functions of discs. There exists a $k_0$ such that $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$. We distinguish two cases: $(p, k_0) \neq (2, 1)$ (Lemma 4), and $(p, k_0) = (2, 1)$ i.e. $q \equiv 3 \pmod{4}$ (Lemma 5). If we use the same notation in Lemmas 4 and 5 as in the definition, we have

**Lemma 4.**

Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ and suppose $(p, k_0) \neq (2, 1)$.

If $0 \leq i < m$ then $e_i(x)$ is the characteristic function of the closed disc $B(aq^i, p^{-k_0})$, and if $0 \leq i < m$, $j \geq 1$ then $e_k(x) = e_{i+jm}(x)$ is the characteristic function of the open disc $B\left(aq^i (q^m)^j, \left(\frac{p^{-k_0}}{j}\right)^{-}\right)$.  

**Proof:**

Let \( j = \sum_{i=0}^{s} a_i p^i \) be the Hensel development of \( j \in \mathbb{N}_0 \), with \( a_s \) different from zero.

If we use the notation \( x = a q^{iz} (q^m)^{\alpha_x} (0 \leq i_x < m) \) for an element \( x \) of \( V_q \), we will show the following:

a) if \( 0 \leq i < m : |x - a q^i| \leq \frac{p^{-k_0}}{j} \) and only if \( i_x = i \).

b) if \( 0 \leq i < m, j \geq 1 : |x - a q^i (q^m)^j| < \frac{p^{-k_0}}{j} \) if and only if \( i_x = i \), \( j < \alpha_x \).

We first prove a). If \( i_x = i \), then \( |x - a q^i| = |a q^{iz} (q^m)^{\alpha_x} - a q^i| = \frac{1}{(q^m)^{\alpha_x} - 1} \leq \frac{p^{-k_0}}{j} \) by the corollary to Lemma 2.

If \( i_x \neq i \), then

\[
|x - a q^i| = |a q^{iz} (q^m)^{\alpha_x} - a q^i| = \max\{|a q^{iz} (q^m)^{\alpha_x} - a q^{iz}|, |a q^{iz} - a q^i|\} = 1,
\]

since \( |a q^{iz} (q^m)^{\alpha_x} - a q^{iz}| \leq \frac{p^{-k_0}}{j} \), \( |a q^{iz} - a q^i| = 1 \). This proves a).

Now we prove b).

Suppose \( i_x = i, j < \alpha_x \). Then \( |x - a q^i (q^m)^j| = |(q^m)^{\alpha_x - j} - 1| \leq \frac{p^{-k_0} - (s+1)}{j} \) by the corollary following Lemma 2, since \( j \) is an initial part of \( \alpha_x \). Since \( j \) is strictly smaller than \( p^{(s+1)} \), we conclude that \( |x - a q^i (q^m)^j| < \frac{p^{-k_0}}{j} \).

For the converse, suppose \( |x - a q^i (q^m)^j| < \frac{p^{-k_0}}{j} \). Then we must have that \( i_x \) equals \( i \), since otherwise \( |x - a q^i (q^m)^j| = 1 \):

\[
|x - a q^i (q^m)^j| = |a q^{iz} (q^m)^{\alpha_x} - a q^i (q^m)^j| = \max\{|a q^{iz} (q^m)^{\alpha_x} - a q^{iz}|, |a q^{iz} - a q^i|, |a q^i - a q^i (q^m)^j|\} = 1
\]

since \( |a q^{iz} (q^m)^{\alpha_x} - a q^{iz}| \leq \frac{p^{-k_0}}{j} \), \( |a q^i - a q^i (q^m)^j| \leq \frac{p^{-k_0}}{j} \) (corollary to Lemma 2) and \( |a q^{iz} - a q^i| = 1 \) if \( i_x \) is different from \( i \).

So we have \( |(q^m)^{\alpha_x - j} - 1| < \frac{p^{-k_0}}{j} \) and from this it follows that \( |(q^m)^{\alpha_x - j} - 1| \leq \frac{p^{-k_0} - (s+1)}{j} \) since \( j \) is at least \( p^s \). This means that \( \text{ord}_p (\alpha_x - j) \) is at least \( s + 1 \) (again by the corollary to Lemma 2) and so we conclude that \( j \) is an initial part of \( \alpha_x \).

**Lemma 5.**

If \( q \equiv 3 \pmod{4} \), with \( q = 1 + 2 + 2^2 \varepsilon, \) where \( \varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots \), \( \varepsilon_0 = \varepsilon_1 = \cdots = \varepsilon_{N-1} = 1, \varepsilon_N = 0 \), then \( \varepsilon_0 (x) \) is the characteristic
function of $V_q$, and $e_j(x)$ is the characteristic function of the open disc $B\left( a_\alpha^j, \left( \frac{2^{-(N+2)}}{j} \right)^{1/2} \right)$ if $j \geq 1$.

Proof:
In this case $m$ equals one and we use the notation $x = a_\alpha x$ for an element $x$ of $V_q$.

It is clear that $e_0(x)$ is the characteristic function of $V_q$.

If $j$ is at least one, we prove: $|x - a_\alpha^j| < \frac{2^{-(N+2)}}{j}$ if and only if $j < \alpha_x$.

Suppose $j < \alpha_x$. Then $|x - a_\alpha^j| = \left| q^{\alpha_x - j} - 1 \right| \leq 2^{-(N+2)-(s+1)}$ (corollary following Lemma 3), and since $j$ is strictly smaller than $2^{s+1}$, we conclude $|x - a_\alpha^j| < \frac{2^{-(N+2)}}{j}$.

For the converse, suppose $|x - a_\alpha^j| < \frac{2^{-(N+2)}}{j}$. Then $\left| q^{\alpha_x - j} - 1 \right| < \frac{2^{-(N+2)}}{j}$ and so $\left| q^{\alpha_x - j} - 1 \right| \leq 2^{-(N+2)-(s+1)}$ since $j$ is at least $2^s$. By the corollary to Lemma 3, we have that $\text{ord}_2(\alpha_x - j)$ is at least $s + 1$ and so $j$ is an initial part of $\alpha_x$.

Corollary.
The functions $(e_k(x))$ are continuous functions on $V_q$.

In the following theorem we prove that the sequence $(e_k(x))$ forms a normal basis for $C(V_q \to K)$. This implies that if $f$ is an element of $C(V_q \to K)$, there exists elements $a_k$ of $K$ such that $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where the right-hand-side is uniformly convergent. We are able to give an expression for the coefficients $a_k$. The proof of this theorem is analogous to the proof of Theorem 62.2 in [1].

Theorem 1.
The functions $(e_k(x))$ form a normal basis for $C(V_q \to K)$. If $f$ is an element of $C(V_q \to K)$ then $f$ can be written as a uniformly convergent series $f(x) = \sum_{k=0}^{\infty} a_k e_k(x)$ where

\begin{align*}
(*) & \quad a_k = f(a q^k) & \text{if } 0 \leq k < m \\
& \quad a_k = a_{i+jm} = f(a q^i(q^m)^j) - f(a q^i(q^m)^{j-}) & \text{if } 0 \leq i < m, j > 0.
\end{align*}

Proof:
Let $f$ be an element of $C(V_q \to K)$, and let $a_k$ be defined as $a_k = f(a q^k)$ if $0 \leq k < m$, $a_k = a_{i+jm} = f(a q^i(q^m)^j) - f(a q^i(q^m)^{j-})$ if $0 \leq i < m, j > 0$.
We first prove that \( a_k \) tends to zero if \( k \) tends to infinity: for all \( \varepsilon > 0 \), there exists a \( J \) such that \( k > J \) implies \( |a_k| < \varepsilon \). To prove this, we distinguish two cases:

i) Let \( q^m \equiv 1 \pmod{p^{k_0}} \), \( q^m \not\equiv 1 \pmod{p^{k_0+1}} \), with \( (p, k_0) \neq (2,1) \).

Since the function \( f \) is continuous on \( V_q \), it is uniformly continuous on \( V_q \), and so there exist an \( S \), such that \( |x - y| \leq p^{-(k_0+S)} \) implies \( |f(x) - f(y)| < \varepsilon \). We then put \( J = p^S \).

If \( k > J \), and \( k \) equals \( i+jm \) with \( 0 \leq i < m \), then we have that \( j \geq p^S \) and so (corollary to Lemma 2) \( |aq^j(q^m)^j - aq^i(q^m)^j| = |(q^m)^j - 1| \leq p^{-(k_0+S)} \) and this implies that \( |a_k| = |f(aq^i(q^m)^j) - f(aq^i(q^m)^j)| < \varepsilon \).

ii) Let \( q \equiv 3 \pmod{4}, q = 1 + 2 + 2^2 \varepsilon, \varepsilon = \varepsilon_0 + \varepsilon_12 + \varepsilon_22^2 + \ldots, \varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1, \varepsilon_N = 0 \). We remark that \( m \) equals one in this case.

Since the function \( f \) is continuous on \( V_q \), it is uniformly continuous on \( V_q \), and so there exist an \( S \), such that \( |x - y| \leq 2^{-(N+2+S)} \) implies \( |f(x) - f(y)| < \varepsilon \). We then put \( J = 2^S \).

If \( k > J \), then (corollary to Lemma 3) \( |q^k - q^{k-1}| = |q^{k-k-1}| \leq 2^{-(N+2+S)} \) and this implies that \( |a_k| = |f(q^k) - f(q^{k-1})| < \varepsilon \).

We conclude that \( a_k \) tends to zero if \( k \) tends to infinity.

If \( f \) is an element of \( C(V_q \to K) \), we introduce a function \( g(x) \) defined by \( g(x) = \sum_{k=0}^{\infty} a_k e_k(x) \) with \( a_k \) as in (\(*\)). Since \( \|a_k e_k\| \leq |a_k| \to 0 \), the series on the right-hand-side converges uniformly, so the function \( g \) is continuous as a uniformly limit of continuous functions. We can prove that \( g(aq^k) = f(aq^k) \) if \( 0 \leq k < m \) and that \( g(aq^i(q^m)^j) - g(aq^j(q^m)^j) = f(aq^i(q^m)^j) - f(aq^j(q^m)^j) \) for \( 0 \leq i < m, j > 0 \). Then we have \( g(aq^k) = f(aq^k) \) for all natural numbers \( k \) and by continuity, we conclude that \( f(x) = g(x) \).

So we have \( f(x) = \sum_{k=0}^{\infty} a_k e_k(x) \), with \( a_k \) as in (\(*\)).

It is clear that \( \|f\| \leq \max_{0 \leq k} \{|a_k|\} \), but we also have \( |f(aq^k)| \leq \|f\| \) and \( |(aq^i(q^m)^j) - f(aq^j(q^m)^j)| \leq \|f\| \), so we conclude \( \|f\| = \max_{0 \leq k} \{|a_k|\} \).

Finally we prove the uniqueness of the coefficients.

If \( f(x) = \sum_{k=0}^{\infty} a_k e_k(x) = \sum_{k=0}^{\infty} b_k e_k(x) \), then \( \sum_{k=0}^{\infty} (a_k - b_k) e_k(x) = 0 \). So \( \max_{0 \leq k} \{|a_k - b_k|\} = 0 \), from which it follows that \( a_k = b_k \) for all \( k \). This proves the theorem. ■
3. More bases for $C(V_q \to K)$

We can make more normal bases, using the basis $(e_k(x))$ of Theorem 1:

Theorem 2.

Let $(e_n(x))$ be as above, and define $r_n(x) = \sum_{j=0}^{n} c_{n;j} e_j(x)$, $c_{n;j} \in K$, $c_{n;0} \neq 0$. Then $(r_n(x))$ forms a normal basis for $C(V_q \to K)$ if and only if $\|r_n\| = 1$ and $|c_{n;n}| = 1$ for all $n$.

The proof of this theorem will not be given here, since it is analogous to the proof of Theorem 2 in [3].

Remark.

An analogous result can be found on the space $C(\mathbb{Z}_p \to K)$, if we replace the sequence $(e_n(x))$ by the van der Put basis $(g_n(x))$ from the introduction.

We can extend Theorem 2 to the following:

Theorem 3.

Let $(r_n(x))$ be a sequence as found in Theorem 2, which forms a normal basis for $C(V_q \to K)$, and let $(s_n(x))$ be a sequence such that $s_n(x) = \sum_{j=0}^{n} d_{n;j} r_j(x)$, $d_{n;j} \in K$, $d_{n;0} \neq 0$.

Then the following are equivalent:

i) $(s_n(x))$ forms a normal basis for $C(V_q \to K)$.

ii) $\|s_n\| = 1$, $|d_{n;n}| = 1$.

iii) $|d_{n;j}| \leq 1$, $|d_{n;n}| = 1$.

Proof:

i) $\equiv$ ii) follows from Theorem 2, using the expression $r_n(x) = \sum_{j=0}^{n} c_{n;j} e_j(x)$, and ii) $\equiv$ iii) follows from the fact that $(r_n(x))$ forms a normal basis. 

Examples.

1) If a sequence $(r_n(x))$, as found in Theorem 2, forms a normal basis of $C(V_q \to K)$, then so does $(s_n(x))$, where $s_n(x) = r_0(x) + r_1(x) + \cdots + r_n(x)$: apply iii).

2) If we put for $0 \leq i < m$,

$$
r_i(x) = 1 \quad \text{if } x = a q^i z (q^m)^{\alpha z} \quad \text{where } i_x = i
$$

$$
= 0 \quad \text{otherwise},
$$
Anf for \( k \geq m \) we put

\[
\tau_k(x) = \tau_{i+m_j}(x) (0 \leq i < m) = 1 \quad \text{if} \quad x = aq^i (q^m)^{\alpha_x}
\]

where \( i_x = i, \ j \neq \alpha_x \).

\[
= 0 \quad \text{otherwise}.
\]

Then \( (\tau_n(x)) \) forms a normal basis for \( C(V_q \rightarrow K) \). We can apply iii) since \( \tau_i(x) = e_i(x) \) for \( 0 \leq i < m \), \( \tau_k(x) = e_i(x) - e_k(x) \) for \( k = i+m_j, \ 0 \leq i < m, \ j > 0 \). If \( f \in C(V_q \rightarrow K) \), then there exists a uniformly convergent expansion of the form

\[
f(x) = \sum_{k=0}^{\infty} c_k \tau_k(x),
\]

where

\[
c_k = c_{i+jm}
\]

\[
= f(aq^i (q^m)^{\alpha_x}) - f(aq^i (q^m)^{j}) \quad \text{if} \quad 0 \leq i < m, \ j > 0, \ \text{and}
\]

\[
c_i = f(aq^i) - \sum_{j=1}^{\infty} c_{i+jm} \quad \text{if} \quad 0 \leq i < m.
\]

References

