ON SUBGROUPS OF $ZJ$ TYPE OF AN $\mathfrak{F}$-INJECTOR FOR FITTING CLASSES $\mathfrak{F}$ BETWEEN $\mathfrak{E}_{p^*p}$ AND $\mathfrak{E}_{p^*}\mathfrak{S}_p$

A. MARTÍNEZ PASTOR (*)

Abstract

Let $G$ be a finite group and $p$ a prime. We consider an $\mathfrak{F}$-injector $K$ of $G$, being $\mathfrak{F}$ a Fitting class between $\mathfrak{E}_{p^*p}$ and $\mathfrak{E}_{p^*}\mathfrak{S}_p$, and we study the structure and normality in $G$ of the subgroups $ZJ(K)$ and $ZJ^*(K)$, provided that $G$ verify certain conditions, extending some results of G. Glauberman (A characteristic subgroup of a $p$-stable group, Canad. J. Math. 20 (1968), 555-564).

1. Introduction and notation

In this paper we consider a finite group $G$ verifying certain conditions of stability and constraint, and we study the structure and normality in $G$ of the subgroups $ZJ(K)$ and $ZJ^*(K)$, being $K$ and $\mathfrak{F}$-injector of $G$ and $\mathfrak{F}$ a Fitting class such that $\mathfrak{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$, extending some results of Glauberman [6].

All groups in this paper are assumed to be finite. Given a fixed prime $p$, $\mathfrak{S}_p$ will denote the class of all $p$-groups, $\mathfrak{E}_{p^*}$, the class of all $p^*$-groups, $\mathfrak{E}_{p^*p}$ the class of all $p^*p$-groups and $\mathfrak{E}_{p^*}\mathfrak{S}_p$ that of all $p^*$-by-$p$-groups. The corresponding radicals in a group $G$ are denoted by $O_p(G)$, $O_{p^*}(G)$, $O_{p^*p}(G)$ and $O_{p^*p}(G)$ respectively. For all definitions we refer to Bender [3].

The notation for Fitting classes is taken from [4]. The remainder of the notation is standard and it is taken mainly from [7] and [8]. In particular, $E(G)$ is the semisimple radical of $G$ and $F^*(G) = F(G)E(G)$ the quasimilpotent radical of $G$. If $H$ is a subgroup of $G$, $C_G^*(H)$ is the generalized centralizer of $H$ in $G$ (see [3]). Note that $C_G^*(F^*(G)) \leq$}

(* ) Work supported by the CICYT of the Spanish Ministry of Education and Science, project PB90-0414-C03-01.
A group \( G \) is said to be \( \mathfrak{N} \)-constrained if \( C_G(F(G)) \leq F(G) \), that is, if \( E(G) = 1 \).

Moreover, \( \pi(G) \) is the set of primes dividing the order of \( G \), \( d(G) \) is the maximum of the orders of the abelian subgroups of \( G \), \( \mathfrak{A}(G) \) is the set of all abelian subgroups of order \( d(G) \) in \( G \) and \( J(G) \) is the subgroup generated by \( \mathfrak{A}(G) \), that is, the Thompson subgroup of \( G \). We set \( ZJ(G) = Z(J(G)) \).

In [6] G. Glauberman proves his well-known \( ZJ \)-Theorem and also introduces the subgroup \( ZJ^*(P) \) proving the following: “Let \( p \) be an odd prime and let \( P \) be a Sylow \( p \)-subgroup of a group \( G \). Suppose that \( C_G(O_p(G)) \leq O_p(G) \) and that \( SA(2;p) \) is not involved in \( G \). Then \( ZJ^*(P) \) is a characteristic subgroup of \( G \) and \( C_G(ZJ^*(P)) \leq ZJ^*(P) \)”.

On the other hand, Arad and Glauberman study in [2] the structure and normality of the subgroup \( ZJ(H) \), \( H \) being a Hall \( \pi \)-subgroup of a \( \pi \)-soluble group \( G \) with abelian Sylow \( 2 \)-subgroups and \( O_{\pi^*}(G) = 1 \).

Some related results were obtained by Arad in [1], by Ezquerro in [5] and by Pérez Ramos in [11] and [12].

Here we study the structure of the subgroups \( ZJ(K) \) and \( ZJ^*(K) \) where \( K \) is an \( \mathfrak{F} \)-injector of \( G \), being \( \mathfrak{F} \) a Fitting class such that \( \mathcal{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathcal{K} \), \( \mathcal{E}_{p^*} \mathcal{K} \), and we obtain that it depends only of \( G \). Also, we obtain some analogous to Glauberman’s \( ZJ \) and \( ZJ^* \) Theorems for such Fitting classes. Recall that such a Fitting class \( \mathfrak{F} \) is dominant in the class of all finite groups, so every finite group \( G \) has a unique conjugacy class of \( \mathfrak{F} \)-injectors (see [10]). Moreover, for such \( \mathfrak{F} \) every finite group is \( \mathfrak{F} \)-constrained in the sense of [9] (see [3]).

In the following \( \mathfrak{F} \) will be a Fitting class such that \( \mathcal{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathcal{E}_{p^*} \mathcal{K} \).

2. Preliminary results

**Remark 1.**

Let \( K \) be an \( \mathfrak{F} \)-injector of a group \( G \). By [10] we know that

\[
K = (O_{p^*}(G)p)_{\mathfrak{F}}
\]

where \( P \) is a Sylow \( p \)-subgroup of \( G \). Moreover, \( O_{p^*}(K) = O_{p^*}(G) \), so \( O_{p^*}(K) = O_{p^*}(G) \) and \( O_{p^*}(F(K)) = O_{p^*}(F(G)) \). On the other hand, since \( F^*(G) \leq K \), we have \( E(K) = E(G) \).

**Remark 2.**

Suppose that \( K \) is an \( \mathcal{E}_{p^*} \mathcal{K} \)-group, that is, \( K = O_{p^*}(K)S \) where \( S \) is a Sylow \( p \)-subgroup of \( K \). Since \([O_{p^*}(K), O_p(K)] = 1\), it is clear that \( K \)
acts nilpotently on $O_p(K)$, i.e. $K = C^*_K(O_p(K))$. In particular, we can deduce that

$$C^*_K(E(K)O_p(F(K))) = C^*_K(F^*(K)) \leq F(K).$$

**Lemma 2.1.**

Let $G$ be a group and let $K$ be an $E_p \cdot C_p$-subgroup of $G$ containing $F^*(G)$. Then $\pi(ZJ(K)) \leq \pi(F(G)) = \pi(F(K))$. Moreover if the prime $p$ belongs to $\pi(F(G))$ then $p \in \pi(ZJ(K))$.

**Proof:**

Since $\pi(F(K)) = \pi(Z(F(K)))$ and $Z(F(K)) \leq C(G,F^*(G)) \leq F(G)$, the first statement can be easily obtained. On the other hand if $p \in \pi(F(G))$ and $P$ is a Sylow $p$-subgroup of $K$ we have $1 \neq Z(P) \cap O_p(K) \leq Z(K) \leq ZJ(K)$ since $K = PO_{p^*}(K)$, and so the result holds. ■

**Lemma 2.2.**

Let $G$ be a group and let $K$ be an $E_p \cdot C_p$-subgroup of $G$ containing $O_p(G)$. Let $B$ be a nilpotent normal subgroup of $G$ and let $A$ be any nilpotent subgroup of $K$. Then $A O_p(B)$ is nilpotent.

**Proof:**

By the Remark 2 $A$ acts nilpotently on $O_p(B) \leq O_p(K)$, so the result follows. ■

Next we will deal with the subgroup $ZJ^*(K)$ of an arbitrary group $K$ and its properties:

**Definition 2.3.** [5].

For any group $K$ define two sequences of characteristic subgroups of $K$ as follows. Set $ZJ^0(K) = 1$ and $K_0 = K$. Given $ZJ^i(K)$ and $K_i$, $i \geq 0$, let $ZJ^{i+1}(K)$ and $K_{i+1}$ the subgroups of $K$ that contain $ZJ^i(K)$ and satisfy:

$$ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K))$$

$$K_{i+1}/ZJ^i(K) = C_{K_i/ZJ^i(K)}(ZJ^{i+1}(K)/ZJ^i(K)).$$

Let $n$ be the smallest integer such that $ZJ^n(K) = ZJ^{n+1}(K)$, then $ZJ^n(K) = ZJ^{n+r}(K)$ and $K_n = K_{n+r}$ for every $n \geq 0$. Set $ZJ^*(K) = ZJ^n(K)$ and $K^*_n = K_n$. 

Example.

In general, the subgroups $ZJ(K)$ and $ZJ^*(K)$ of a group $K$ are different. To see this, we can consider, as an example, the group $K = [Q_8 \times C_3]S_3$ generated by the elements $a, b, c, x, y$ with the following relations:

$$a^4 = 1, \quad a^2 = b^2, \quad a^b = a^{-1}, \quad c^3 = 1, \quad a^c = a, \quad b^c = b, \quad x^3 = y^2 = 1, \quad x^y = y^{-1},$$

$$a^x = ba, \quad b^x = a^{-1}, \quad c^x = c, \quad a^y = b, \quad b^y = a, \quad c^y = c^{-1}.$$

Then we can get check that $d(K) = 18$, $Z(K) = Z(Q_8) = \langle a^2 \rangle$, $ZJ(K) = Z(Q_8) \times C_3$, $K_1 = [Q_8 \times C_3] \langle x \rangle = J(K)$ and $ZJ^*(K) = ZJ^2(K) = K_2 = [Q_8 \times C_3]$.

Remark 3.

For every group $K$:

i) $ZJ(K_i/ZJ^i(K)) = ZJ(K_{i+1}/ZJ^i(K)) = Z(K_{i+1}/ZJ^i(K))$, for every $i \geq 0$.

ii) $Z(J_i) \leq Z(K_{i+1})$, for every $i \geq 0$.

Lemma 2.4.

For any group $K$ and for every $i \geq 0$:

i) $ZJ^i(K)$ is nilpotent.

ii) $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$.

Proof:

i) By induction on $i$, assume that $ZJ^i(G)$ is nilpotent, for every group $G$. By ([5, Prop. II 3.6]) we have that $ZJ^{i+1}(K)/ZJ^i(K) = ZJ^i(K_1/ZJ^1(K))$, so this is a nilpotent group. Now, by the previous remark, $ZJ^i(K) \leq Z(K_i) \leq Z(J_i)$, and $ZJ^{i+1}(K) \leq K_i$, hence $ZJ^{i+1}(K)$ is nilpotent.

ii) By induction on $i$. The assertion is clear for $i = 0$. Assume now that $F(K_i/ZJ^i(K)) = F(K_i)/ZJ^i(K)$. We have:

$$F(K_{i+1}/ZJ^{i+1}(K)) \cong F(K_{i+1}/ZJ^i(K))/ZJ^{i+1}(K)/ZJ^i(K))$$

and since $ZJ^{i+1}(K)/ZJ^i(K) = Z(K_{i+1}/ZJ^i(K))$, it follows

$$F(K_{i+1}/ZJ^i(K))/ZJ^{i+1}(K)/ZJ^i(K)) = F(K_{i+1}/ZJ^i(K))/ZJ^{i+1}(K)/ZJ^i(K).$$

But applying the inductive hypothesis we have:

$$F(K_{i+1}/ZJ^i(K)) = F(K_i/ZJ^i(K)) \cap K_{i+1}/ZJ^i(K) = F(K_i)/ZJ^i(K) \cap K_{i+1}/ZJ^i(K) = F(K_{i+1}/ZJ^i(K))$$

and so we can conclude that $F(K_{i+1}/ZJ^{i+1}(K)) = F(K_{i+1})/ZJ^{i+1}(K)$. 

3. The structure of the $ZJ$-subgroup and the $ZJ^*$-subgroup

In this section we will study the structure of the subgroups $ZJ(K)$ and $ZJ^*(K)$ being $K$ an $\mathcal{E}_p^r \mathcal{S}_p$-subgroup of a group $G$ containing $O_p(G)$ and satisfying that $O_p^r(K) = O_p^r(G)$, properties that hold for an $\mathfrak{S}$-injector of $G$, as we have seen.

**Theorem 3.1.**

Let $G$ be an $\mathfrak{N}$-constrained group and let $K$ be an $\mathcal{E}_p^r \mathcal{S}_p$-subgroup of $G$ containing $O_p(G)$ and such that $O_p^r(K) = O_p^r(G)$. Assume that at least one of the following conditions hold:

i) $O_p^r(F(G)) \leq ZJ(K)$,

ii) $F(G)$ is abelian,

iii) $d(K)$ is odd and $O_2(G)$ is abelian.

Then:

a) $\{O_p(A)|A \in \mathfrak{A}(K)\} = \mathfrak{A}(O_p(K))$.

b) $O_p(ZJ(K)) = ZJ(O_p(K))$.

c) $\{O_p^r(A)|A \in \mathfrak{A}(K)\} = \mathfrak{A}(O_p^r(G))$.

d) $O_p^r(ZJ(K)) = ZJ(O_p^r(G))$.

In particular, if we assume $O_p^r(F(G)) \leq ZJ(K)$ then for every $A \in \mathfrak{A}(K)$

$O_p^r(A) = O_p^r(ZJ(K)) = O_p^r(F(G))$.

Moreover the prime numbers divisors of $d(K)$, $|ZJ(K)|$, $|F(K)|$ and $|F(G)|$ coincide.

**Proof:**

Let $A \in \mathfrak{A}(K)$. Since $F^*(G) \leq K$ we know that $E(K) = E(G) = 1$, so $K$ is an $\mathfrak{N}$-constrained group. Leading from our assumptions we can obtain that $AF(G)$ is nilpotent (if we assume i) Lemma 2.2 applies; if we assume ii) or iii) Proposition 1 of [2] applies). Moreover, since $O_p^r(K) = O_p^r(G)$ we have $O_p^r(F(K)) = O_p^r(F(G))$.

a) Let $A \in \mathfrak{A}(K)$. Since $AF(G)$ is nilpotent $O_p(A)$ centralizes $O_p^r(F(G))$ and so applying Remark 2 we obtain

$O_p(A) \leq C_K(O_p^r(F(K))) \leq F(K)$

so $O_p(A) \leq O_p(K)$.

Let $B \in \mathfrak{A}(O_p(K))$. Since $AO_p(K)$ is nilpotent by Lemma 2.2, $O_p^r(A)$ centralizes $O_p(K)$, so $O_p^r(A)B$ is an abelian subgroup of $K$ and then

$|O_p^r(A)B| \leq |A| = |O_p^r(A)O_p(A)|$. 
Hence $d(O_p(K)) \leq |O_p(A)|$. Since $O_p(A) \leq O_p(K)$ the equality $d(O_p(K)) = |O_p(A)|$ holds.

Thus, for every $B \in \mathfrak{A}(O_p(K))$, $O_p'(A) \times B \in \mathfrak{A}(K)$. So we have

$$\{O_p(A)|A \in \mathfrak{A}(K)\} = \mathfrak{A}(O_p(K)).$$

b) This follows easily from a):

$$O_p(ZJ(K)) = O_p(\cap\{A|A \in \mathfrak{A}(K)\})$$

$$= \cap\{O_p(A)|A \in \mathfrak{A}(K)\} = ZJ(O_p(K)).$$

c) Let $A \in \mathfrak{A}(K)$. By a) we know that $O_p(A) \leq O_p(K)$. On the other hand, since $K$ is an $\mathcal{E}_p \cdot \mathcal{G}_p$-group we have $O_p'(A) \leq O_p(K) = O_p'(G)$.

Let $B \in \mathfrak{A}(O_p'(G))$. Since $[O_p'(G), O_p(K)] = 1$, $O_p(A)$ centralizes $B$ so $O_p(A)B$ is an abelian subgroup of $K$ and then

$$|O_p'(A)B| \leq |A| = |O_p(A)O_p'(A)|.$$  

Hence $d(O_p'(G)) \leq |O_p'(A)|$. Since $O_p'(A) \leq O_p'(G)$ it follows $d(O_p'(G)) = |O_p'(A)|$. Therefore, for every $B \in \mathfrak{A}(O_p'(G))$, $O_p(A) \times B \in \mathfrak{A}(K)$. This proves c).

d) This follows from c) as in b).

If we assume $O_p'(F(G)) \leq ZJ(K)$ then it is clear that $O_p'(ZJ(K)) = O_p'(F(K)) = O_p'(F(G))$. Let $A \in \mathfrak{A}(K)$. Since $ZJ(K) = \cap\{A|A \in \mathfrak{A}(K)\}$ and $AF(G)$ is nilpotent we obtain that $O_p'(A) \leq C_G(F(G)) \leq F(G)$ and so the equality $O_p'(F(G)) = O_p'(ZJ(K)) = O_p'(A)$ holds.

Now since $F^*(G) \leq K$ we can apply Lemma 2.1 and our assumptions to obtain $\pi(ZJ(K)) = \pi(F(G)) = \pi(F(K))$. Moreover, if $A \in \mathfrak{A}(K)$ it is clear that $\pi(ZJ(K)) \subseteq \pi(A) = \pi(d(K))$. On the other hand, if $q$ is a prime number such that $q \neq p$ and $q \in \pi(A)$, then $q \in \pi(F(G))$, by the foregoing assertion. Finally, if we assume that $p \in \pi(A)$, then $p \in \pi(F(K)) = \pi(F(G))$ because of a), and so the result follows. ■

Corollary 3.2.

Let $G$ be an $\mathfrak{A}$-constrained group, $H$ an $\mathcal{E}_p \cdot \mathcal{G}_p$-injector of $G$ and $K = H_3$ its associated $3$-injector of $G$. If one of the following conditions holds:

i) $O_p'(F(G)) \leq ZJ(K)$,

ii) $F(G)$ is abelian,

iii) $d(K)$ is odd and $O_2(G)$ is abelian,
then

\[ \text{ZJ}(K) = \text{ZJ}(O_p^*(G)) \times \text{ZJ}(O_p(H)) = \text{ZJ}(H). \]

So, in particular, \( \text{ZJ}(K) \) does not depend on the Fitting class \( \mathfrak{F} \).

**Proof:**

Given \( A \) in \( \mathfrak{A}(H) \), by Remark 2 we see that \( O_p(A) \leq O_p(H) = O_p(K) \). On the other hand, due to the structure of the injectors considered here, one has \( O_p'(A) \leq O_p'(H) = O_p'(H) = O_p'(G) \leq K \). Therefore \( \mathfrak{A}(H) = \mathfrak{A}(K) \). Then apply Theorem 3.1 parts b) and d) to the subgroups \( H \) and \( K \). ■

**Corollary 3.3.**

If \( G \) is an \( \mathfrak{N} \)-constrained group and \( K \) and \( \mathfrak{F} \)-injector of \( G \) such that \( O_p'(F(G)) \leq Z(K) \), then

\[ K = O_p'(F(G)) \times P \]

where \( P \) is a Sylow \( p \)-subgroup of \( G \). In particular,

\[ \mathfrak{A}(K) = \{ O_p'(F(G))A | A \in \mathfrak{A}(P) \}. \]

**Proof:**

Since \( K = PO_p^*(G) \), \( P \) a Sylow \( p \)-subgroup of \( K \) and \( O_p'(F(G)) \leq Z(K) \), due to 6.11 in [3], we can write \([P, O_p(G)] = 1\). Now by \( \mathfrak{N} \)-

constrained, \( K \) is nilpotent and hence it is an \( \mathfrak{E}_p \cdot \mathfrak{S}_p \)-injector of \( G \) (see [10]); therefore \( P \) is a Sylow \( p \)-subgroup of \( G \) and \( K = O_p'(F(G)) \times P \). ■

Our next goal is to study the structure of the \( ZJ^* \)-subgroup.

**Theorem 3.4.**

Let \( G \) be an \( \mathfrak{N} \)-constrained group. Let \( K \) be an \( \mathfrak{E}_p \cdot \mathfrak{S}_p \)-subgroup of \( G \) containing \( O_p(G) \) and such that \( O_p^*(K) = O_p^*(G) \). Assume that \( O_p'(F(G)) \leq ZJ(K) \). Denote \( P = O_p(K) \). Then for every \( i \geq 1 \), \( O_p'(ZJ^i(K)) = O_p'(F(K_i)) = O_p'(F(G)), K_i \) is a nilpotent group and

\[ O_p(ZJ^i(K)) = ZJ^i(P) \quad O_p(K_i) = P_i \]

with the notation given in Definition 2.3. In particular \( O_p(ZJ^*(K)) = ZJ^*(P), O_p(K_*) = P_* \) and

\[ ZJ^*(K) = ZJ^*(P) \times O_p'(F(G)). \]
Proof:
Since $O_p'(ZJ(K)) \leq O_p'(ZJ^i(K)) \leq O_p'(F(K)) \leq O_p'(F(G))$, the first statement is clear.

Notice that $O_p'(F(G)) \leq ZJ(K) \leq Z(K_1)$, so $O_p'(K_1) \leq C_G(F(G)) \leq F(G)$. Hence $O_p'(K_1) = O_p'(F(K)) \leq Z(K_1)$ and $K_1$ is a nilpotent group. Now apply that for every $i \geq 1, K_i \leq K_1$.

We will prove that $O_p(ZJ^i(K)) = ZJ^i(P)$ and $O_p(K_i) = P_i$ by induction on $i$. By Proposition 3.2 we have $ZJ(P) = O_p(ZJ(K))$. On the other hand $P = O_p(K)$ centralizes $O_p'(ZJ(K))$, so $C_P(ZJ(P)) \leq C_K(ZJ(K))$ and then we obtain

$$O_p(K_1) = P \cap K_1 = P \cap C_K(ZJ(K)) = C_P(ZJ(P)) = P_1.$$

Thus, the statement is clear for $i = 1$.

Now suppose that $O_p(ZJ^i(K)) = ZJ^i(P)$ and $O_p(K_i) = P_i$. Applying Lemma 2.4 and the fact that $O_p'(F(K_i)) = O_p'(ZJ^i(K))$, we get that $K_i/ZJ^i(K) = F(K_i)/ZJ^i(K)$ is a $p$-group. Then it follows that

$$K_i/ZJ^i(K) = P_iZJ^i(K)/ZJ^i(K) \cong P_i/ZJ^i(K) \cap P_i = P_i/ZJ^i(P)$$

by the inductive hypothesis. Thus

$$ZJ^{i+1}(K)/ZJ^i(K) = ZJ(K_i/ZJ^i(K)) \cong ZJ(P_i/ZJ^i(P)) = ZJ^{i+1}(P)/ZJ^i(P).$$

and since $ZJ^{i+1}(K) = ZJ^i(K)(ZJ^{i+1}(K) \cap P_i)$ we can conclude

$$O_p(ZJ^{i+1}(K)) = ZJ^{i+1}(K) \cap O_p(K_i) = ZJ^{i+1}(K) \cap P_i = ZJ^{i+1}(P).$$

Now we will prove that $O_p(K_{i+1}) = P_{i+1}$. It is clear that $O_p(K_{i+1}) \leq O_p(K_i) = P_i$ and

$$[O_p(K_{i+1}), ZJ^{i+1}(P)] \leq [O_p(K_{i+1}), ZJ^{i+1}(K)] \leq O_p(K_{i+1}) \cap ZJ^i(K) = ZJ^i(P).$$

Hence by the definition of $P_{i+1}$ it follows that $O_p'(K_{i+1}) \leq P_{i+1}$. On the other hand, $P_{i+1} \leq P_i \leq K_i$ and since $O_p'(F(G)) \leq ZJ(K) \leq Z(K_i)$, we have

$$[P_{i+1}, ZJ^{i+1}(K)] = [P_{i+1}, ZJ^{i+1}(P)] \leq ZJ^i(P) \leq ZJ^i(K).$$

Thus, by the definition of $K_{i+1}$ we obtain $P_{i+1} \leq K_{i+1}$. Now, since $O_p(K_{i+1})$ is the Sylow $p$-subgroup of $K_{i+1}$ the result follows. ■
Corollary 3.5.

Let $G$ be an $\mathcal{H}$-constrained group. Let $H$ be an $\mathfrak{E}_p \mathfrak{G}_p$-injector of $G$ and assume that $O_p'(F(G)) \leq ZJ(H)$. Let $K = H_{\mathfrak{F}}$ be an $\mathfrak{F}$-injector of $G$. Then

$$ZJ^*(K) = O_p'(F(G)) \times ZJ^*(O_p(H)) = ZJ^*(H).$$

In particular, $ZJ^*(K)$ does not depend on $\mathfrak{F}$.

Proof:

Because of Corollary 3.2 we have $ZJ(K) = ZJ(H)$. Now Theorem 3.4 is applied, keeping in mind that $O_p(K) = O_p(H)$. ■

4. The normality of the ZJ-subgroup and the ZJ*-subgroup

In this section we prove some results related to the normality of the ZJ-subgroup and the normality and self-centrality of the ZJ*-subgroup of an $\mathfrak{F}$-injector $K$ of a group $G$, provided that $G$ verifies certain conditions of stability. Concretely, we will use the following version of $p$-stability:

Definition 4.1.

A group $G$ is said to be $p$-stable if whenever $A$ is a subnormal $p$-subgroup of $G$ and $B$ is a $p$-subgroup of $N_G(A)$ satisfying $[A, B, B] = 1$, then

$$B \leq O_p(N_G(A) \mod C_G(A)).$$

Proposition 4.2.

Let $G$ be a $p$-stable group. Let $K$ be an $\mathfrak{E}_p \mathfrak{G}_p$-subgroup of $G$ containing the $\mathfrak{E}_{p^\infty}$-radical of $G$, $O_{p^\infty}(G)$. If $N$ is an abelian normal subgroup of $K$ then $N \trianglelefteq G$ and $N \leq F(G)$. In particular $ZJ(K) \leq F(G)$.

Proof:

First notice that $O_{p^\infty}(G) \leq K$ implies $O_p(K) = O_{p^\infty}(G)$ (see [3, 4.22]). Thus, $O_p'(N) \leq O_p'(G) \leq O_{p^\infty}(G) \leq K$, and so $O_p'(N) \leq O_{p^\infty}(G)$.

On the other hand, it holds $[O_p(G), O_p(N), O_p(N)] = 1$ and so applying the $p$-stability of $G$ we have:

$$O_p(N)C_G(O_p(G))/C_G(O_p(G)) \leq O_p(G/C_G(O_p(G))),$$

$$= C_G^*(O_p(G))/C_G(O_p(G)).$$
(see [3, 3.8]). Then we obtain
\[ O_p(N) \leq C_G^*(O_p(G)) \cap C_G(E(G)O_p'(F(G))) \leq C_G^*(F^*(G)) \leq F(G) \]
so \( O_p(N) \leq O_p\cdot p(G) \) and the result follows. ■

**Theorem 4.3.**

Let \( G \) be a \( p \)-stable group, \( p \) and odd prime and assume that \( O_p(G) \neq 1 \). If \( K \) is an \( \mathcal{Z} \)-injector of \( G \) then
\[ 1 \neq O_p(ZJ(K)) \leq G. \]
Moreover, if \( O_p'(F(G)) \leq ZJ(K) \), then \( 1 \neq ZJ(K) \leq G. \)

**Proof:**

First note that \( O_p(ZJ(K)) \leq G \) implies \( O_p(ZJ(K)) \) char \( G \), because of the conjugacy of the \( \mathcal{Z} \)-injectors.

By Proposition 4.2, we know that \( O_p(ZJ(K)) \leq O_p(G) \), and by Lemma 2.1 \( O_p(ZJ(K)) \neq 1 \). Now, to obtain the theorem it is enough to prove that if \( B \) is a normal \( p \)-subgroup of \( G \), then \( B \cap O_p(ZJ(K)) \) is normal in \( G \).

Assume the result false and suppose that \( G \) is a minimal counterexample. Suppose that \( B \) is a normal \( p \)-subgroup of \( G \) of least order such that \( B \cap O_p(ZJ(K)) \) is not normal in \( G \).

Set \( Z = O_p(ZJ(K)) \) and let \( B^* \) be the normal closure of \( B \cap Z \) in \( G \), then \( B \cap Z = B^* \cap Z \) and by our minimal choice of \( B \) we obtain \( B = B^* \).

Moreover, since \( B' < B \) we have that \( B' \cap Z \) is a normal subgroup of \( G \). Thus, for any \( g \) in \( G \) we have \( [(B \cap Z)^g, B] = [B \cap Z, B]^g \leq B' \cap Z \).
Since \( B \) is generated by all such \( (B \cap Z)^g \), it follows that \( B' \leq Z \). In particular \( B \cap Z \) centralizes \( B' \), and applying the foregoing argument we get \( [B, B, B] = 1 \).

Let \( A \in \mathfrak{A}(K) \). By Lemma 2.2 we know that \( AB \) is nilpotent, so there exists some positive integer \( n \) such that \( [B, A; n] = 1 \). Moreover, since \( p \) is an odd prime \( [A, B'] \leq B' \) has odd order.

Now by Glauberman's replacement Theorem ([1, Corollary 2.8]) we can conclude that there exists an element \( A \) in \( \mathfrak{A}(K) \) such that \( B \leq N_G(A) \), and therefore \( [B, A, A] = 1 \).

In particular, \( [B, O_p(A), O_p(A)] = 1 \). Since \( G \) is \( p \)-stable we have:
\[ O_p(A)C/C \leq O_p(G/C) = T/C \leq G/C \]
where \( C = C_G(B) \) and \( T = C_G^*(B) \). Moreover, since \( O_p'(A) \leq C_G(B) \) we get
\[ A \leq T. \]
If $T = G$, then $G/C$ is a $p$-group, so $KC$ is a subnormal subgroup of $G$. Since $KC$ normalizes $B \cap Z$, $KC < G$. Let $M$ be a normal proper subgroup of $G$ such that $KC \leq M$. Clearly $M$ verifies the hypothesis of the theorem, $K$ being an $\mathfrak{Z}$-injector of $M$, so by our minimal choice of $G$, we get $Z \leq M$, and then $Z \operatorname{char} M$. Therefore, $Z \leq G$, contrary to our choice of $G$.

Thus, we have $T < G$. Since $A \leq K \cap T$, it follows that $\mathfrak{A}(K \cap T) \subseteq \mathfrak{A}(K)$, $J(K \cap T) \leq J(K)$ and $ZJ(K) \leq ZJ(K \cap T)$. It is clear that $T$ verifies the hypothesis of the theorem, being $K \cap T$ an $\mathfrak{Z}$-injector of $T$. Thus, by the minimal choice of $G$, $O_p(ZJ(K \cap T)) \operatorname{char} T$ and then $O_p(ZJ(K \cap T)) \leq G$. Since $B$ is the normal closure of $B \cap Z$ in $G$ we obtain $B \leq O_p(ZJ(K \cap T))$. In particular, $B$ is abelian.

If $J(K) = J(K \cap T)$ then $O_p(ZJ(K)) = O_p(ZJ(K \cap T)) \leq G$, contrary to the choice of $G$. Thus, there exists an element $A_1 \in \mathfrak{A}(K)$ such that $A_1$ is not a subgroup of $T$. Then we must have $[B, A_1, A_1] \neq 1$. Among all such $A_1$, choose $A_1$ such that $|A_1 \cap B|$ is maximal. As $B$ does not normalize $A_1$, by Thompson’s replacement Theorem ([1, Theorem 2.5], there exists an element $A_2$ in $\mathfrak{A}(K)$ such that $A_1 \cap B < A_2 \cap B$ and $A_2$ normalizes $A_1$. The maximal choice of $A_1$ implies that $[B, A_2, A_2] = 1$ and $A_2 \leq T$. Hence, $B \leq ZJ(K \cap T) \leq A_2 \leq N_G(A_1)$ and this is the last contradiction.

Finally, if in addition we assume $O_p(F(G)) \leq ZJ(K)$, then $O_p(F(G)) = ZJ(K)$ and the result follows. ■

**Corollary 4.4** (compare with Glauberman’s $ZJ$-Theorem [6]).

Let $G$ be a $p$-stable group such that $C_G(O_p(G)) \leq O_p(G)$, $p$ and odd prime. If $P$ is a Sylow $p$-subgroup of $G$ then $ZJ(P) \leq G$.

**Proof:**

Leading from our assumptions we have $O(p^*F(G)) = O_p(G) = 1$, so $P$ is actually an $\mathfrak{E}_p^*, \mathfrak{G}_p^*$-injector of $G$ and Theorem 4.3 applies. ■

**Theorem 4.5.**

Let $p$ be an odd prime and $K$ an $\mathfrak{Z}$-injector of a group $G$, being $\mathfrak{Z}$ a $Z$-extensible and $Q_2$-closed Fitting class. Assume that $SA(2,p)$ is not involved in $G$ and that $O_{p'}(F(G)) \leq ZJ(K)$. Then $ZJ^i(K)$ is a characteristic subgroup of $G$ for every $i \geq 0$.

**Proof:**

Assume the result to be false and let $G$ be a minimal counterexample. Since $SA(2,p)$ is not involved in $G$, we know that $G$ is $p$-stable (using Definition 4.1 above, proceed as in [6]). Therefore applying Theorem 4.3
we have $ZJ(K)$ char $G$. Because of the choice of $G$ we can assume $1 \neq ZJ(K)$.

Set $C = C_G(ZJ(K))$. Assume that $C < G$. Then for every $i \geq 0$ we have $ZJ^i(K \cap C)$ char $C$, and so $ZJ^i(K \cap C) \leq G$. Now since $J(K) \leq K \cap C$, it follows that $J(K) = J(K \cap C)$ and $ZJ(K) = ZJ(K \cap C)$. Also $K_1 = C_K(ZJ(K)) = C_K \cap C(ZJ(K \cap C))$ and applying induction on $i$ we can obtain $ZJ^i(K) = ZJ^i(K \cap C) \leq G$, contrary to the choice of $G$.

Therefore $C = G$ and then $ZJ(K) = ZG$. Since $|G/Z(G)| < G$ and $K/Z(G)$ is an $\mathfrak{F}$-injector of $G/Z(G)$ we obtain $ZJ^i(K) \leq G$, for every $i \geq 0$. Now since $K_1 = C_K(ZJ(K)) = K$, using ([5, Prop. II.3.6]) we can deduce $ZJ^i(K/Z(G)) = ZJ^{i+1}(K)/Z(G)$, and so $ZJ^{i+1}(K)$ char $G$ for every $i \geq 0$, which is the last contradiction. ■

Remark 4.

Recall that for any group $K$, $C_K(ZJ^*(K)) \leq K^*$ and $K^*/C_K(ZJ^*(K))$ is nilpotent (by [5, Prop. II 3.7]). Using this facts it is easy to see that for any group $K$ the following statements are equivalent:

i) $C_K(ZJ^*(K)) \leq ZJ^*(K)$  
ii) $K^* = ZJ^*(K)$.

Also, we know that $C_K(K^*) \leq C_K(ZJ^*(K)) \leq K^*$, using ([5, Prop. II 3.7]).

Remark 5.

Let $K$ be an $\mathfrak{F}$-injector of a group $G$. Then $K$ is also an $\mathfrak{F}$-injector of any subgroup of $G$ containing $K$ (see [10]). In particular, $K$ is an $\mathfrak{F}$-injector of $N_G(K^*)$, and so by the previous remark $Z(K^*) = C_K(K^*) = C_G(K^*) \cap K$ is an $\mathfrak{F}$-injector of $C_G(K^*)$. Thus if $x \in C_G(K^*)$, since $\langle x, Z(K^*) \rangle$ is an abelian subgroup of $N_G(K^*)$ with $Z(K^*) \leq \langle x, Z(K^*) \rangle \leq C_G(K^*)$, we can conclude that $Z(K^*) = \langle x, Z(K^*) \rangle$. Therefore, we have proved that $C_G(K^*) \leq K^*$.

Proposition 4.6.

Let $K$ be an $\mathfrak{F}$-injector of a group $G$ and assume $O_{p'}(F(G)) \leq ZJ(K)$. Then the following are equivalent:

i) $G$ is an $\mathfrak{N}$-constrained group.
ii) $K^* = ZJ^*(K)$.
iii) $C_G(ZJ^*(K)) \leq ZJ^*(K)$. 

Proof:

First notice that, applying Lemma 2.1, since $K_*/ZJ^*(K)$ is an $\mathfrak{S}_p\mathfrak{G}_p$-group, $ZJ(K_*/ZJ^*(K)) = 1$ implies $O_p(K_*/ZJ^*(K)) = 1$. Now applying Lemma 2.4 and the fact that $O_p^e(F(G)) \leq ZJ(K)$ we obtain that $F(K_*/ZJ^*(K)) = F(K_*)/ZJ^*(K)$ is a $p$-group and so we conclude $ZJ^*(K) = F(K_*)$.

i) $\Rightarrow$ ii) Since $F(G) \leq K$ it follows that $C_K(F(K)) \leq F(K)$, and so on $C_{K_*}(F(K_*)) \leq F(K_*)$. Bearing in mind that $ZJ^*(K) = F(K_*)$ and $C_K(ZJ^*(K)) = C_{K_*}(ZJ^*(K))$, ii) follows from Remark 4.

ii) $\Rightarrow$ iii) Since $ZJ^*(K)$ is nilpotent we have $E(G) \leq C_G(ZJ^*(K)) \leq ZJ^*(K)$, and then $E(G) = 1$, that is, $G$ is an $\mathfrak{M}$-constrained group. ■

Corollary 4.7.

Let $p$ be an odd prime and $K$ an $\mathfrak{H}$-injector of an $\mathfrak{N}$-constrained group $G$, being $\mathfrak{F}$ a $Z$-extensible and $QZ$-closed Fitting class. Assume that $SA(2,p)$ is not involved in $G$ and that $O_p^e(F(G)) \leq ZJ(K)$. Then $ZJ^*(K)$ is a characteristic subgroup of $G$ and $C_G(ZJ^*(K)) \leq ZJ^*(K)$.

Recall that both the classes $\mathfrak{E}_p^*\mathfrak{G}_p$ and $\mathfrak{E}_p^*\mathfrak{G}_p$ are $Z$-extensible and $QZ$-closed Fitting classes (see [3] and [10]), so the previous result applies for such classes. Moreover, as in the case of the $ZJ$-theorem we can also recover the Glauberman’s $ZJ^*$-Theorem quoted at the beginning as a consequence of the above corollary.

5. Final remarks

Remark 6.

There exist $\mathfrak{M}$-constrained groups $G$ such that $O_p^e(F(G)) \leq ZJ(K)$, being $K$ an $\mathfrak{E}_p^*\mathfrak{G}_p$-injector of $G$, verifying that $SA(2,p)$ is not involved in $G$, $p$ odd, and however with $O_p^e(G) \neq 1$.

Proof:

It is enough to take the group $G = SA(3,3) = [N]H$, with $N \cong C_3 \times C_3 \times C_3$ and $H \cong SL(3,3)$ and the prime $p = 13$. Really, $G$ is an $\mathfrak{M}$-constrained group with $O_p^e(F(G)) = N$, an $\mathfrak{E}_p^*\mathfrak{G}_p$-injector of $G$ is $K = O_p^e(G)P = NP$ where $P \cong C_{13}$, and $ZJ(K) = N$. Moreover, it is clear that $SA(2,13)$ is not involved in $G$, bearing the orders in mind. ■
Remark 7.
In [2] and [12], the authors consider a \( \pi \)-soluble group \( G \) with abelian Sylow 2-subgroups and \( O_{\pi'}(G) = 1 \), and they study the structure of the subgroup \( ZJ(H) \), where \( H \) is a Hall \( \pi \)-subgroup of \( G \), or \( H \) is an \( \mathfrak{Z} \)-injector of \( G \) for certain Fitting classes \( \mathfrak{Z} \), respectively. Recall that such a group is an \( \mathfrak{N} \)-constrained group (see [2]), and moreover it is a \( p \)-stable group for any prime number \( p \) (see [12]).

Moreover, since the \( p \)-nilpotent groups are \( \mathfrak{Z}_p \cdot \mathfrak{S}_p \)-groups, we can easily generalizes Lemma 4 of [2], as follows:

"Let \( G \) be a group and let \( P \) be a \( p \)-subgroup of \( K = O_{p^*} \cdot p(G) \). Assume that \( P \) centralizes \( E(G)O_{p'}(F(G)) \). Then \( P \leq O_p(G) \)."

For the proof, let \( K = O_{p^*} \cdot p(G) \); since \( F^*(K) = F^*(G) \), applying Remark 2 it follows that \( P \leq C_K(E(K)O_{p'}(F(K))) \leq F(K) \), and hence \( P \leq O_p(K) = O_p(G) \).

References

5. L. M. Ezquerro, \( \mathfrak{Z} \)-estabilidad, constricción y factorización de grupos finitos, Tesis doctoral, Univ. de Valencia, 1983.

Departamento de Matemática Aplicada
E. U. Informática
Universidad Politécnica de Valencia
Camino de Vera s/n
Valencia
SPAIN

Primera versió rebuda el 15 de Març de 1994,
darrera versió rebuda el 10 de Maig de 1994