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MULTIPLIER EXTENSION AND SAMPLING THEOREM ON HARDY SPACES

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Abstract

Extension by integer translates of compactly supported function for multiplier spaces on periodic Hardy spaces to multiplier spaces on Hardy spaces is given. Shannon sampling theorem is extended to Hardy spaces.

1. Introduction and statement of results

The purpose of this paper is to establish a natural extension from multiplier spaces $\tilde{M}(p)$ on periodic Hardy spaces $H^p(T)$ to multiplier spaces $M(p)$ on Hardy spaces $H^p(R)$ by integer translates of a function ϕ and to extend Shannon sampling theorem to Hardy spaces. It is the continuation of [**13**] on stability of integer translates of a function but with different interest. In [**13**], the following stability problem of integer translates of ϕ

$$(1)\qquad C^{-1}\|f\|_{H^p(Z)} \leq \left\|\sum_{n\in\mathbb{Z}} f(n)\phi(x-n)\right\|_{H^p(R)} \leq C\|f\|_{H^p(Z)}$$

was considered which arises in the interpolation of sequences by functions and plays an important role in multiresolution analysis, where $0 < p < \infty$, $f = \{f(n)\}_{n\in Z}$ is a tempered sequence, $H^p(Z)$ and $H^p(R)$ denotes Hardy spaces on Z and R respectively, and Z is the set of integers. A natural replacement of the norm in (1) when $p = \infty$ is the norm as multiplier operator on $H^p(T)$ and $H^p(R)$ respectively, which is an

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original inspiration to consider multiplier extension here. To this end, we introduce some notations.

Let Φ be a smooth function such that $\operatorname{supp}\hat{\Phi} \subset \{\frac{15}{40} \le |x| \le \frac{9}{10}\}$, $|\hat{\Phi}(x)| \ge C_0$ on $\{\frac{7}{20} \le |x| \le \frac{4}{5}\}$ and $\sum_{m\in Z}\hat{\Phi}(2^m x) = 1$ for $x \ne 0$, where C_0 is a positive constant and the Fourier transform is defined by $\hat{\Phi}(x) = \int e^{2\pi ixy}\Phi(y)\,dy$. Denote $\Phi_m(x) = 2^m\Phi(2^m x)$ for $m \in Z$. Now we define Hardy spaces $H^p(R)$ by

$$H^p(R) = \left\{ f \in S'(R);\ \|f\|_{H^p(R)} = \left\| \left(\sum_{m\in Z} |\Phi_m * f|^2 \right)^{\frac{1}{2}} \right\|_{L_p(R)} < +\infty \right\}$$

and define Hardy spaces $H^p(T)$ (c.f. [**3**]) by

$$H^p(T) = \left\{ f(x) = \sum_{k\in Z} f_k e^{2\pi ikx} \in S'(R);\right.$$
$$\left. |f_0| + \left\| \left(\sum_{m\ge -1} |\Phi_m * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(T)} < +\infty \right\}$$

$$= \left\{ f(x) = \sum_{k\in Z} f_k e^{2\pi ikx} \in S'(R);\right.$$

$$\left. \|f\|_{H^p(T)} = |f_0| + \left(\int_0^1 \left(\sum_{m\ge 0} \left| \sum_{\substack{2^m \le |k| \\ < 2^{m+1}}} f_k e^{2\pi ikx} \right|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} < +\infty \right\},$$

where we denote the space of tempered distributions by $S'(R)$, the norm of p-integrable functions on R and T by $\|\cdot\|_{L^p(R)}$ and by $\|\cdot\|_{L^p(T)}$, respectively and $T = R/Z$ denotes the torus. For a measurable function m on R, we say that m is a multiplier on $H^p(R)$ if

$$\|F\|_{H^p(R)} \le C_m \|f\|_{H^p(R)} \tag{2}$$

holds for any Schwartz function f, where $\hat{F} = m\hat{f}$. We denote the infinum C_m in (2) by $\|m\|_{M(p)}$. For a sequence $\tilde{m} = \{\tilde{m}(n)\}$, we say that $\tilde{m}$ is a multiplier on $H^p(T)$ if

$$\|G\|_{H^p(T)} \le C_{\tilde{m}} \|g\|_{H^p(T)} \tag{3}$$

holds for every trigonometric polynomial $g(x) = \sum_{k\in Z} g_k e^{2\pi ikx}$, where $G(x) = \sum_{k\in Z} \tilde{m}(k) g_k e^{2\pi ikx}$. Also we denote $\|\tilde{m}\|_{\tilde{M}(p)}$ the infinum $C_{\tilde{m}}$ in (3).

The classical result of de Leeuw [**7**] on multiplier said the restriction to the integer lattice of a continuous multiplier on $L^p(R)$ is a multiplier on $L^p(T)$ when $1 < p < \infty$. In 1992, Liu [**8**] extended the above conclusion to Hardy spaces. The multiplier extension was considered by Jodeit [**6**], Berkson and Gillespie [**2**]. Let ϕ be a continuous function with compact support. Denote the space of sequences by S and the linear span of integer translates of ϕ by $S(\phi) = \{\sum_{n\in Z} C(n)\phi(x-n);\ \{C(n)\} \in S\}$. Define a natural map $\phi*'$ from S to $S(\phi)$ by

$$\phi*' : S \ni \{C(n)\} \longmapsto \sum_{n\in Z} C(n)\phi(x-n) \in S(\phi).$$

We say that the integer translates of ϕ are *globally linearly independent* if $\phi*'$ is one-to-one. Denote the restriction of $\phi*'$ on $\tilde{M}(p)$ by I. Berkson and Gillespie [**2**] proved that I maps $\tilde{M}(p)$ to $M(p)$ boundedly under the hypotheses $1 < p < \infty$ and $\phi = \chi_{[-\frac{1}{2},\frac{1}{2}]} * \Lambda_0$, where $\chi_{[-\frac{1}{2},\frac{1}{2}]}$ is the characteristic function of $\left[-\frac{1}{2},\frac{1}{2}\right]$, $*$ denotes the convolution operator and Λ_0 is a bounded variation function supported in $\left[-\frac{1}{2},\frac{1}{2}\right]$. In this paper, we will prove

Theorem 1. *Let $0 < p < \infty$ and ϕ have compact support. If $\int |\hat{\phi}(x)|^{\min(1,p)}\,dx < +\infty$, then I maps $\tilde{M}(p)$ to $M(p)$ boundedly.*

We improve Berkson and Gillespie's result since under their hypotheses $|\hat{\phi}(x)| \le C(1+|x|)^{-2}$ and $\int |\hat{\phi}(x)|\,dx < \infty$. Applying to Bochner-Riesz summation operator B_δ, we reproved that B_δ maps $H^p(R^n)$ to $H^p(R^n)$ when $\delta > \frac{n}{p} - \frac{n+1}{2}$ and $0 < p \le 1$ [**11**], [**12**] when we let $\phi(x) = (1-|x|^2)_+^\delta$.

To consider the inverse of Theorem 1, we introduce the paraproduct P_h and show that P_h maps $H^p(T)$ to $H^p(R)$.

Theorem 2. *Let ϕ be a continuous function with compact support. If $\phi*' : S \to S(\phi)$ is one-to-one, then I has bounded inverse $I^{-1} : M(p) \cap S(\phi) \to \tilde{M}(p)$.*

In the proof of Theorem 1 and Theorem 2, Lemma 2 plays an important role. If we assume Lemma 2 is true, or $m(x) = \sum_{n\in Z} C(n)\phi(x-n) \in L^\infty$ implies $\{C(n)\} \in l^\infty$, then it suffices to assume I is one-to-one in Theorem 2. In particular Theorem 2 can be written as that I has bounded inverse $I^{-1} : M(p) \cap S(\phi) \to \tilde{M}(p)$ provided I has bounded inverse $I^{-1} : M(2) \cap S(\phi) \to \tilde{M}(2)$ and ϕ is a continuous function with compact support, where $0 < p < \infty$. The continuity condition on ϕ can be dropped in one spatial dimension since for any distribution ϕ on R such that $\phi *'$ is one-to-one there exists a univariate spline B_k such that $\psi = B_k * \phi$ is continuous and $\psi *'$ is one-to-one. But I do not know how to construct this modifier B_k in high spatial dimensions. By Fourier transform characterization of global linear independence in [**9**], the box spline and Daubechies' scaling function satisfy the condition on ϕ in Theorem 2.

Shannon sampling theorem [**10**] plays an important role in signal analysis. It says a function with its Fourier transform supported in $\left[-\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon\right]$ for some $0 < \varepsilon < \frac{1}{2}$ has its $L^p(R)$ norm comparable to its $l^p(Z)$ norm of its restriction to integer lattices Z where $1 < p < \infty$. In 1990, R. Torres [**14**] extended the above conclusion to Besov spaces. Let $\{\tilde{\Phi}_m\}_{m\ge 0}$ be a family of sequences such that $\hat{\tilde{\Phi}}_m$ is smooth, $\operatorname{supp}\hat{\tilde{\Phi}}_m(\xi) \subset \{2^{-m-2} \le |\xi| \le 2^{-m}\}$, $|\hat{\tilde{\Phi}}_m(\xi)| \ge C_0$ on T_m, where $\hat{\tilde{\Phi}}_m(\xi) = \sum_{n\in Z} \tilde{\Phi}_m(n)e^{2\pi i n\xi}$, $T_0 = \left\{\frac{3}{8} \le |\xi| \le 1\right\}$ and $T_m = \left\{\frac{3}{8}2^{-m} \le |\xi| \le \frac{9}{10}2^{-m}\right\}$. Define (c.f. [**13**] or [**14**])

$$H^p(Z) = \left\{ \{f(n)\}_{n\in Z};\ \left(\sum_{n\in Z}\left(\sum_{m\ge 0} |\tilde{\Phi}_m * f(n)|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} < +\infty \right\},$$

where $f = \{f(n)\}$ is a tempered sequence and $\tilde{\Phi}_m * f(n) = \sum_{k\in Z} \tilde{\Phi}_m(n-k)f(k)$.

Theorem 3. *Let $0 < p < +\infty$. If $f \in S'(R)$ with $\operatorname{supp}\hat{f} \subset \left[-\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon\right]$ for some $0 < \varepsilon < \frac{1}{2}$, then the inequality*

$$C^{-1}\|\{f(n)\}\|_{H^p(Z)} \le \|f\|_{H^p(R)} \le C\|\{f(n)\}\|_{H^p(Z)}$$

holds for some constant C dependent of ϵ and p only.

For simplicity in the exposition we restrict ourselves to one spatial dimension, all results can be extended to high spatial dimensions. The results of Theorem 1 and 3 can be extended to spaces of Triebel-Lizorkin type trivially. The big letter C will denote different constant at different occurance.

2. Some lemmas

To prove our theorems, we will use the following fundamental lemmas.

Lemma 1. c.f. [**15**]. *Let $f \in S'(R)$ have its Fourier transform contained in a compact set. Therefore*

$$|\psi * f(x)| \leq CM(|f|^r)^{\frac{1}{r}}(x) \tag{4}$$

holds for every Schwartz function ψ and $0 < r < 1$, where M denotes Hardy-Littlewood maximal operator and the constant C depends on the seminorm of ψ, r and the radius R for which $\hat{f}$ is supported in the ball with radius R and center zero.

Proof of Lemma 1: Without loss of generality we assume $x = 0$, $M(|f|^r)(0) < +\infty$ and $\operatorname{supp} \hat{f} \subset \left[-\frac{1}{4}, \frac{1}{4}\right]$ by dilation invariance. Write

$$f(x) = \sum_{n \in Z} f(n)\varphi(x-n), \tag{5}$$

for some Schwartz function φ such that $\operatorname{supp} \hat{\varphi} \subset \left[-\frac{3}{8}, \frac{3}{8}\right]$. Hence $|f * \psi(0)|^r \leq \sum_{n \in Z} |f(n)|^r \ (1+|n|)^{-3}$. To prove (4), we first prove $\sum_{n \in Z} |f(n)|^r (1+|n|)^{-3} < +\infty$. Recall that $f \in S'(R)$ and $\operatorname{supp} \hat{f} \subset \left[-\frac{1}{4}, \frac{1}{4}\right]$. Therefore $|f(n)| \leq C(1+|n|)^N$ for some constants C and N. On the other hand we have

$$|f(n)|^r \leq |f(n+\delta)|^r + C|\delta|^r \sum_{m \in Z} |f(m)|^r (1+|m-n|)^{-(N+3)}$$

by (5) and

$$\sum_{|n| \leq 2^k} |f(n)|^r (1+|n|)^{-3} \leq C_{\delta_0} + C_N |\delta_0|^r \sum_{|n| \leq 2^{k+1}} |f(n)|^r (1+|n|)^{-3}$$

by summing over $|n| \le 2^k$ and integrating over $|\delta| < \delta_0 < 1$, where δ_0 is chosen later and C_N is independent of δ_0. Denote $A_k = \sum_{2^{k-1}<|n|\le 2^k} |f(n)|^r(1+|n|)^{-3}$. Therefore $\sum_{j=1}^k A_j \le 2C_{\delta_0} + 2C_N|\delta_0|^r A_{k+1}$. Conversely if we assume $\sum_{j=1}^{k_0} A_j \ge 4C_{\delta_0}$ for some k_0, then $\sum_{j=1}^{k_0+s} A_j \ge \left(1+\frac{1}{4C_N|\delta_0|^r}\right)^s C_{\delta_0}$, which contradicts $\sum_{j=1}^{k_0} |A_j| \le C\sum_{j=1}^{2^{k_0+s}}(1+|n|)^N \le C2^{sN}$ provided δ_0 is chosen small enough. This proved $\sum_{n\in Z} |f(n)|^r(1+|n|)^{-3} < +\infty$.

Furthermore by (5) we have

$$|f(n)|^r \le |f(n+\delta)|^r + C|\delta|^r \sum_{m\in Z} |f(m)|^r(1+|n-m|)^{-3}$$

for some constant C independent of f and by integrating over $|\delta| \le \delta_0$ for some sufficiently small $\delta_0 > 0$ we get

$$\sum_n |f(n)|^r(1+|n|)^{-3} \le C\int_R |f(x)|^r(1+|x|)^{-3}\,dx \le CM(|f|^r)(0).$$

Therefore Lemma 1 is proved. ■

Lemma 2. *Let the integer translates of the continuous function ϕ be globally linearly independent. If $m(x) = \sum_{n\in Z} C(n)\phi(x-n) \in M(p)$, then $\{C(n)\} \in l^\infty$ and $\|\{C(n)\}\|_{l^\infty(Z)} \le C\|m\|_{M(p)}$.*

Proof of Lemma 2: First we prove

$$\|m\|_{L^\infty(R)} \le C\|m\|_{M(p)}. \tag{6}$$

Obviously (6) is true when $1 < p < \infty$ since Marcinkiewicz real interpolation, $\|m\|_{M(2)} = \|m\|_{L^\infty(R)}$ and $\|m\|_{M(p)} = \|m\|_{M(p')}$ where $p' = \frac{p}{p-1}$. Hence the matter reduces to proving (6) for $0 < p \le 1$. For $f \in H^p(R)$, we have the atomic decomposition $f(x) = \sum_{k=0}^\infty \lambda_k a_k(x)$ with $C^{-1}\|f\|_{H^p(R)} \le (\sum_{k=0}^\infty |\lambda_k|^p)^{\frac{1}{p}} \le C\|f\|_{H^p(R)}$, where a_k are $(p,2,s)$ atoms and $s \ge \frac{1}{p}-1$. We call that a is an $(p,2,s)$ atom if there exists an interval I such that $\operatorname{supp} a \subset I$, $\|a\|_{L^2(R)} \le |I|^{\frac{1}{2}-\frac{1}{p}}$ and $\int x^\alpha a(x)\,dx = 0$ for $0 \le \alpha \le s$. It is easy to show $\hat a(x)$ is continuous, $|\hat a(x)| \le C|I|^{1-\frac{1}{p}}$ and $|\hat a(x)| \le C|x|^{s+1}|I|^{s+2-\frac{1}{p}}$. Hence $|\hat a(x)| \le C|x|^{\frac{1}{p}-1}$ and $|\hat f(x)| \le \sum_{k=0}^\infty |\lambda_k|\,|\hat a_k(x)| \le C(\sum_{k=0}^\infty |\lambda_k|^p)^{\frac{1}{p}}|x|^{\frac{1}{p}-1}$. Denote $f_t(x) = t^{-\frac{1}{p}}f\left(\frac{x}{t}\right)$. Therefore $f_t(x) = \sum_{k=0}^\infty \lambda_k(a_k)_t(x)$ and $C^{-1}(\sum_{k=0}^\infty |\lambda_k|^p)^{\frac{1}{p}} \le$

$\|f_t\|_{H^p(R)} \le C(\sum_{k=0}^{\infty} |\lambda_k|^p)^{\frac{1}{p}}$. Recall that $m \in M(p)$ for $0 < p \le 1$. Hence

$$\|(m\hat{f}_t)^{\vee}\|_{H^p(R)} \le C\|f_t\|_{H^p(R)} \le C\|f\|_{H^p(R)}$$

and

$$|m(x)t^{(1-\frac{1}{p})}\hat{f}(tx)| \le C\|f\|_{H^p(R)}|x|^{\frac{1}{p}-1}$$

for all $t > 0$. Therefore $|m(x)\hat{f}\left(\frac{x}{|x|}\right)| \le C\|f\|_{H^p(R)}$ for every $x \ne 0$ when we let $t = |x|^{-1}$ and (6) is proved for $0 < p \le 1$ by choosing $f \in H^p(R)$ such that $\hat{f} \equiv 1$ on the unit sphere.

Second we prove

$$\|\{C(n)\}\|_{l^\infty(Z)} \le C\|m\|_{L^\infty(R)}. \tag{7}$$

By [**1**, Theorem 1.3], there exists a local algebraic dual $\{\Lambda_n\}$ of $\{\phi(x-n)\}$, which says $\Lambda_n\phi(x-k) = \delta_{nk}$ and there exists a bounded set K such that $\Lambda_n f = 0$ when $f \in S(\phi)$ and $\operatorname{supp} f \cap (K+n) = \emptyset$, where we define the Kronekker symbol δ_{nk} by $\delta_{nn} = 1$ and $\delta_{nk} = 0$ when $n \ne k$. Recall that ϕ is continuous. Hence there exist finite points $x_i \in K$ and weights $C(x_i)$ such that $\Lambda_n f = \sum_i C(x_i)f(x_i+n)$ for every $f \in S(\phi)$. This shows $|C(n)| = |\Lambda_n m| \le C\|m\|_{L^\infty}$ for every $n \in Z$. Therefore (7) holds and Lemma 2 is proved by combining (6) and (7). ■

Lemma 3. ([**4**] *or* [**5**, Theorem A.1]). *Let $1 < p < +\infty$ and $1 < q \le +\infty$. Therefore the following Fefferman-Stein vector-valued maximal inequality*

$$\left\|\left(\sum_{k\in Z}|Mf_k|^q\right)^{\frac{1}{q}}\right\|_{L^p(R)} \le C\left\|\left(\sum_{k\in Z}|f_k|^q\right)^{\frac{1}{q}}\right\|_{L^p(R)}$$

holds where M denotes the Hardy-Littlewood maximal operator on R as usual.

Let h and η be two Schwartz functions such that $\operatorname{supp}\hat{\eta} \subset \left\{|x| \le \frac{1}{20}\right\}$ and $\hat{\eta}(x) = 1$ on $\left\{|x| \le \frac{1}{40}\right\}$. Let Φ_m be as in the definition of $H^p(R)$. For $f \in H^p(T)$ we introduce a new type of *paraproduct operator* P_h defined by

$$P_h f(x) = \sum_{m\ge 0} (\eta_m * h)(x)(\Phi_m * f)(x), \tag{8}$$

where $\eta_m(x) = 2^m\eta(2^m x)$.

Lemma 4. *Let P_h be defined by (8) and h be a Schwartz function. Then P_h maps $H^p(T)$ to $H^p(R)$,*

$$\|P_h f\|_{H^p(R)} \leq C\|f\|_{H^p(T)}.$$

Proof: Observe that $|\eta_m * h(x)| \leq C_N(1+|x|)^{-N}$ for every $N \geq 0$ and some C_N independent of m. Also observe that $\operatorname{supp}((\eta_m*h)(\Phi_m*f))^\wedge \subset \left\{\frac{11}{40}2^m \leq |x| \leq \frac{19}{20}2^m\right\}$ and $\Phi_m * f(x+k) = \Phi_m * f(x)$ for all $k \in Z$. Therefore

$$\begin{aligned}\|P_h f\|^p_{H^p(R)} &\leq C\int_R \left(\sum_{m\geq 0} |(\eta_m * h)(x)(\Phi_m * f)(x)|^2\right)^{\frac{p}{2}} dx \\ &\leq C\sum_{k\in Z}\int_0^1 \left(\sum_{m\geq 0}(1+|k|)^{-\max(4,\frac{4}{p})}|\Phi_m * f(x)|^2\right)^{\frac{p}{2}} dx \\ &\leq C\|f\|^p_{H^p(T)}\end{aligned}$$

and Lemma 4 is proved. ■

3. Proof of theorems

Proof of Theorem 1: Let $\{C(n)\}$ be a multiplier on $H^p(T)$ and $\operatorname{supp}\phi \subset \left[-\frac{1}{4}M, \frac{1}{4}M\right]$ for some $M > 1$. Denote $m(x) = \sum C(n)\phi(x-n)$. Let Φ_k be as in the definition of $H^p(R)$. Write $f = f_0 + f_1 + f_2$, where $f_0 = \sum_{k\leq 2M_1}\Phi_k * f$ and $f_i = \sum_{k\geq M_1}\Phi_{2k+i} * f$ for $i = 1, 2$, where M_1 is a positive integer such that $2^{M_1} \geq 20M$. Observe that

$$m(x)(\Phi_k * f)^\wedge(x) = \sum_{|l|\leq 2^{2M_1}+M} C(l)\phi(x-l)(\Phi_k * f)^\wedge(x)$$

for $k \leq 2M_1$. Write $\phi^\vee(x) = \sum_{n\in Z}\phi^\vee\left(\frac{n}{M}\right)\psi\left(x-\frac{n}{M}\right)$ for some $\psi \in S(R)$ with $\operatorname{supp}\hat{\psi} \subset \left\{|x| \leq \frac{3}{8}\right\}$, where $\phi^\vee$ denotes inverse Fourier transform. Therefore

$$\int_R \left(\sum_{k \le 2M_1} |(m(\Phi_k * f)^\wedge)^\vee(x)|^2 \right)^{\frac{p}{2}} dx$$

$$\le C \sum_{|l| \le 2^{2M_1}+M} |C(l)|^p \left(\sum_{n \in Z} \left| \phi^\vee \left(\frac{n}{M} \right) \right|^{\min(p,1)} \right)^{\max(p,1)}$$

$$\int_R \left(\sum_{k \le 2M_1} |\psi * |(\Phi_k * f)|(x)|^2 \right)^{\frac{p}{2}} dx$$

$$\le C \|\{C(l)\}\|^p_{l^\infty(Z)} \int_R \left(\sum_{k \le 2M_1} (M(\Phi_k * f)^r(x))^{\frac{2}{r}} \right)^{\frac{p}{2}} dx$$

$$\le C \|m\|^p_{\tilde{M}(p)} \|f\|^p_{H^p(R)},$$

where $0 < r < \min(p,1)$. The first inequality follows Hölder inequality and $\sum_{n \in Z} |a_n| \le (\sum_{n \in Z} |a_n|^p)^{\frac{1}{p}}$ for $0 < p \le 1$, the second inequality follows from Lemma 1 and

$$\sup_{|y| \le 2} \sum_{n \in Z} |\phi^\vee(y+n)|^{\min(p,1)} \le C \int |\hat{\phi}(x)|^{\min(p,1)} \, dx$$

(see [**13**, Lemma 6]), and the third inequality follows from Lemma 2 and Lemma 3. For $k \ge 2M_1$, write $\phi(x)(\Phi_k * f)^\wedge(x+n) = \sum_{l \in Z} C_{k,n}(l) e^{2\pi i l x/M} \hat{\eta}(x/M)$ for some Schwartz function η with $\operatorname{supp} \hat{\eta} \subset \left\{ |x| \le \frac{3}{8} \right\}$. Therefore we get

$$(m(\Phi_k * f)^\wedge)^\vee(x) = \sum_{l \in Z} \sum_{2^{k-2} \le |n| < 2^k} C(n) C_{k,n}(l) e^{-2\pi i n x} \eta(Mx - l)$$

since $2^{M_1} \ge 20M$ and

$$\sum_{2^{k-2} \le |n| < 2^k} C_{k,n}(l) e^{-2\pi i n x}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(y) \left(\sum_{n \in Z} (f * \Phi_k)^\wedge(y+n) e^{-2\pi i n x} \right) e^{-2\pi i l y} \, dy$$

$$= \sum_{n \in Z} (f * \Phi_k)(x+n) \hat{\phi} \left(x + n - \frac{l}{M} \right)$$

by Poisson summation formula. Hence

$$\|(m\hat{f}_i)^\vee\|^p_{H^p(R)} \le C\int_R \left(\sum_{k\ge M_1} |(m(\Phi_{2k+i}*f)^\wedge)^\vee(x)|^2\right)^{\frac{p}{2}} dx$$

$$\le C\sum_{l\in Z}\int_R \left(\sum_{k\ge M_1}\left|\sum_{\substack{2^{2k+i-2}\le\\ |n|<2^{2k+i}}} C(n)C_{2k+i,n}(l)e^{-2\pi inx}\right|^2\right)^{\frac{p}{2}}$$

$$\times (1+|Mx-l|)^{-2}\,dx$$

$$\le C\sum_{l\in Z}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k\ge M_1}\left|\sum_{\substack{2^{2k+i-2}\le\\ |n|<2^{2k+i}}} C(n)C_{2k+i,n}(l)e^{-2\pi inx}\right|^2\right)^{\frac{p}{2}} dx$$

$$\le C\|\{C(n)\}\|^p_{\tilde{M}(p)}\sum_{l\in Z}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k\ge M_1}\left|\sum_{\substack{2^{2k+i-2}\le\\ |n|<2^{2k+i}}} C_{2k+i,n}(l)e^{-2\pi inx}\right|^2\right)^{\frac{p}{2}} dx$$

$$\le C\|\{C(n)\}\|^p_{\tilde{M}(p)}\sum_{l\in Z}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{k\ge M_1}\left(\sum_{n\in Z}|f*\Phi_{2k+i}(x+n)|\right.\right.$$

$$\left.\left.\times\left|\hat{\phi}\left(x+n-\frac{l}{M}\right)\right|\right)^2\right)^{\frac{p}{2}} dx$$

$$\le C\|\{C(n)\}\|^p_{\tilde{M}(p)}\left(\int|\hat{\phi}(x)|^{\min(p,1)}\,dx\right)^{\max(p,1)}$$

$$\times\left(\sum_{n\in Z}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{k\ge M_1}|f*\Phi_{2k+i}(x+n)|^2\right)^{\frac{p}{2}} dx\right)$$

$$\le C\|\{C(n)\}\|^p_{\tilde{M}(p)}\|f\|^p_{H^p(R)},$$

and Theorem 1 is proved. ■

Proof of Theorem 2: Let $m(x)=\sum_{n\in Z} C(n)\phi(x-n)$ be a multiplier of $H^p(R)$ and $f(x)=\sum_{l\in Z}\hat{f}(l)e^{2\pi ilx}$ is a trigonometric polynomial.

Observe that $|\hat{f}(k)| \le C_k \|f\|_{H^p(T)}$. Therefore we assume $\hat{f}(0) = 0$ without loss of generality. Write

$$(9) \quad (m(\cdot)(P_h f)(\cdot)^\wedge)^\vee(x)$$
$$= \sum_{n,k\in Z}\sum_{m\ge 0} C(n+k)\hat{f}(k)\hat{\Phi}(2^{-m}k)e^{2\pi ikx}\int \hat{\eta}_m(\xi)\hat{h}(\xi)\phi(\xi-n)e^{-2\pi ix\xi}\,d\xi$$
$$= \sum_{n\in Z}\sum_{m\ge 0}\Phi_m*(\tau_n f)(x)\int_{-\frac12}^{\frac12} e^{2\pi iny}\sum_{k\in Z}\hat{\phi}(y+k)(\eta_m*h-h)(x-y-k)\,dy$$
$$+\sum_{n\in Z}(\tau_n f)(x)\int_{-\frac12}^{\frac12} e^{2\pi iny}\sum_{k\in Z}\hat{\phi}(y+k)h(x-y-k)\,dy,$$

where $(\tau_n f)(x) = \sum_{k\in Z} C(n+k)\hat{f}(k)e^{2\pi ikx}$ and the second equality follows from Poission summation formula $\sum_{n\in Z} f(n) = \sum_{n\in Z}\hat{f}(n)$ and $\sum_{m\ge 0}\hat{\Phi}_m = 1$ on $\{|\xi|\ge 1\}$. Let h_0 be a smooth function with compact support such that

$$\sum_{k\in Z}\hat{\phi}(y+k)h_0(-y-k) = 1 \text{ on } \left[-\frac{1}{2},\frac{1}{2}\right],$$

since for every $y \in \left[-\frac12,\frac12\right]$ there exists $k \in Z$ for which $\hat{\phi}(y+k) \ne 0$ by the Fourier transform characterization of global linear independence of integer translates of ϕ [**R**]. Denote $h_i(x) = h_0(x - 2^{-N}i)$ and the characteristic function on $[2^{-N}i, 2^{-N}(i+1))$ by χ_i, where N is chosen later and $-2^{N-1} \le i \le 2^{N-1}-1$. Therefore multiplying χ_i, on the two sides of (9), we get

$$|\chi_i(x)|\,|\tau_0 f(x)| \le |\chi_i(x)(m(P_{h_i}f)^\wedge)^\vee(x)|$$
$$+ C\sum_{n\in Z}\chi_i(x)|\tau_n f(x)|2^{-\frac{N}{2}}(1+|n|)^{-2/\min(p,1)}$$
$$+ CA_{m_0}|\chi_i(x)|\sum_{n\in Z}(1+|n|)^{-2/\min(p,1)}\left(\sum_{m\ge 0}|\Phi_m*(\tau_n f)(x)|^2\right)^{\frac12},$$

where $A_{m_0}(h) = \sup_{-2^{N-1}\le i\le 2^N-1}\sum_{|k|\le\frac{2}{p}+2}(\sum_{m\ge m_0}\|\eta_m*\left(\frac{\partial}{\partial x}\right)^k h_i - \left(\frac{\partial}{\partial x}\right)^k h_i\|_{L^\infty}^2)^{\frac12}$ and $\hat{f}(k) = 0$ for $|k| \le 2^{m_0}$. It is easy to prove $A_{m_0}(h) \le C2^{-m_0}$. Recall that (8), $m \in M(p)$ and $|\hat{f}(k)| \le C_k\|f\|_{H^p(T)}$ for $|k| \le 2^{m_0}$. Hence

$$\|\tau_0 f\|_{L^p(T)}^p \le C_{N,m_0}\|m\|_{M(p)}^p\|f\|_{H^p(T)}^p$$
$$+ C(2^{-N/2}+2^{-m_0})\sum_{n\in Z}\|\tau_n f\|_{H^p(T)}^p(1+|n|)^{-2},$$

holds for some constant C independent of N and m_0 by Lemma 4. Define the Hilbert transform H by

$$Hf(x) = \sum_{k>0} f_k e^{2\pi ikx} - \sum_{k<0} f_k e^{2\pi ikx}$$

for $f(x) = \sum_{k\in Z} f_k e^{2\pi ikx}$. Therefore H maps $H^p(T)$ to $H^p(T)$ and

$$\|f\|_{H^p(T)} \le C\|Hf\|_{L^p(T)} + C\|f\|_{L^p(T)}.$$

In high spatial dimensions, we can use Riesz transforms to replace Hilbert transform [**8**]. Hence we have

$$\|\tau_0 f\|^p_{H^p(T)} \le C_{N,m_0}\|m\|^p_{M(p)}\|f\|^p_{H^p(T)} + C(2^{-N/2} + 2^{-m_0}) \sum_{n\in Z} \|\tau_n f\|^p_{H^p(T)}(1+|n|)^{-2}.$$

Observe that $\sup_{n\in Z} \|\tau_n f\|_{H^p(T)} < +\infty$ when f is a trigonometric polynomial and $\{C(n)\} \in l^\infty$ by Lemma 2. Therefore by choosing N and m_0 large enough we get

$$\sup_{n\in Z} \|\tau_n f\|_{H^p(T)} \le C\|m\|^p_{M(p)}\|f\|_{H^p(T)}$$

for every trigonometric polynomial f. Theorem 2 is proved. ■

Proof of Theorem 3: First the right inequality. Since $\operatorname{supp}\hat{f} \subset \left[-\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon\right]$ for some $0<\varepsilon<\frac{1}{2}$. Write $f(x) = \sum_{n\in Z} f(n)\psi(x-n)$ where ψ is a Schwartz function such that $\hat{\psi}(x) = 1$ on $\left[-\frac{1}{2}+\frac{\epsilon}{4}, \frac{1}{2}-\frac{\epsilon}{4}\right]$ and $\operatorname{supp}\hat{\psi} \subset \left[-\frac{1}{2}+\frac{\epsilon}{8}, \frac{1}{2}-\frac{\epsilon}{8}\right]$. Observe that

$$\begin{aligned}\Phi_m * f(x) &= \sum_{n\in Z} f(n) \int \hat{\Phi}(2^{-m}\xi)\bar{\hat{\psi}}(\xi)e^{-i2\pi(x-n)\xi}\,d\xi \\ &= \int (\varphi^*_{-m} f)^\wedge(\xi)e^{-2\pi ix\xi}\,d\xi,\end{aligned}$$

where we denote $(\varphi^*_m f)^\wedge(\xi) = \hat{\Phi}(2^m\xi)\bar{\hat{\psi}}(\xi)\hat{f}(\xi)$ for $m \in Z$ and $\hat{f}(\xi) = \sum_{n\in Z} f(n)e^{2\pi in\xi}$. Observe that

$$\Phi_m * f(x) = \sum_{n\in Z} \varphi^*_{-m}(f)(n)g(x-n)$$

where g is a Schwartz function with $\operatorname{supp}\hat{g} \subset \left[-\frac{1}{2}+\frac{\xi}{16}, \frac{1}{2}-\frac{\xi}{16}\right]$ when $m \leq 0$, $\Phi_m * f = 0$ when $m \geq 1$ and Φ is chosen approciately. Therefore by Lemma 1 we get

$$\left\| \left(\sum_{m\in Z} |\Phi_m * f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(R)} \leq \left\| \left(\sum_{m\geq 0} \left(\sum_{n\in Z} \varphi_m^*(f)(n) g(x-n) \right)^2 \right)^{\frac{1}{2}} \right\|_{L^p(R)}$$

$$\leq C \left\| \left(\sum_{m\geq 0} |\varphi_m^*(f)(n)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(Z)}$$

$$\leq C \|\{f(n)\}\|_{H^p(Z)}.$$

Now the left inequality. By the procedure used as in the proof of Lemma 1, it suffices to show $\sum_k (\sum_{m\leq 0} |\tilde{\Phi}_m * f(k)|^2)^{\frac{p}{2}} < +\infty$. Since $f \in S'(R)$, we get $|\tilde{\Phi}_m * f(k)| \leq C_{N,m}(2^{-m}+|k|)^N$ for some N and all $m \geq 0$. As in the proof of Lemma 1, we get $\sum_{k\in Z} |\tilde{\Phi}_m * f(k)|(1+|k|)^{-N_1} < +\infty$ and $\sum_{k\in Z} |\tilde{\Phi}_m * f(k)|(1+|k|)^{-N_1} \leq C \sum_{k\in Z} |\tilde{\Phi}_m * f(k+\delta)|(1+|k+\delta|)^{-N_1}$ for some C independent of f and m, where N_1 are chosen later. Therefore

$$\left(\sum_{m\geq 0} |\tilde{\Phi}_m * f(k)|^2 \right)^{\frac{p}{2}} \leq C \int \left(\sum_{m\geq 0} |\tilde{\Phi}_m * f(x)|^2 \right)^{\frac{p}{2}} dx (1+|k|)^{N_1}$$

for some $N_1 \leq \frac{2}{p} + 2$. Still by the procedure used as in the proof of Lemma 2, we get $\sum_{k\in Z} (\sum_{m\geq 0} |\tilde{\Phi}_m * f(k)|^2)^{\frac{p}{2}} < +\infty$ and $\sum_{k\in Z} (\sum_{m\geq 0} |\tilde{\Phi}_m * f(k)|^2)^{\frac{p}{2}} \leq C \|f\|^2_{H^p(R)}$. Hence the left inequality and Theorem 3 is proved. ■

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