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NILPOTENT SUBGROUPS OF THE GROUP OF FIBRE HOMOTOPY EQUIVALENCES

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Abstract

Let $\xi = (E, p, B, F)$ be a Hurewicz fibration. In this paper we study the space $\mathcal{L}_G(\xi)$ consisting of fibre homotopy self equivalences of ξ inducing by restriction to the fibre a self homotopy equivalence of F belonging to the group G. We give in particular conditions implying that $\pi_1(\mathcal{L}_G(\xi))$ is finitely generated or that $\mathcal{L}_1(\xi)$ has the same rational homotopy type as $\mathrm{aut}_1(F)$.

Let $\xi = (E, p, B, F)$ be a Hurewicz fibration where B and F are compactly generated spaces. The set of free (not necessarily fibre) homotopy classes of free fibre homotopy equivalences of ξ into itself is a group $\mathcal{L}(\xi)$, for the multiplication induced by the composition of maps.

Recall that a fibre homotopy equivalence $f : E \to E$ induces an homotopy equivalence of $p^{-1}(b)$ for each $b \in B$ (A theorem of Dold ([**4**, Theorem 6.3]) asserts that the converse is true if B is a CW complex). There exists thus a natural map

$$R : \mathcal{L}(\xi) \longrightarrow \mathrm{Aut}\, F,$$

where $\mathrm{Aut}\, F$ denotes the group of free homotopy classes of free homotopy equivalences of the space F into itself.

Our purpose in this paper is the study of the groups $\mathcal{L}_G(\xi) = R^{-1}(G)$ and the spaces $L_G(\xi)$ where G is some subgroup of $\mathrm{Aut}\, F$. Here $\mathrm{aut}\, X$ is the monoid of free homotopy equivalences of the space X into itself, $\mathrm{aut}_G\, X$ is the submonoid of $\mathrm{aut}\, X$ consisting of the path components belonging to G, and $L_G(\xi)$ is the space of fibre homotopy self-equivalences of ξ inducing by restriction to the fibre an element of $\mathrm{aut}_G\, F$: $\pi_0(L_G(\xi)) = \mathcal{L}_G(\xi)$.

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When G is reduced to the identity $\{1\}$, we obtain the (connected) monoid $\mathrm{aut}_1 F$ of self-equivalences homotopic to the identity, and the monoid $L_1(\xi)$ of fibre self-equivalences of ξ inducing by restriction to the fibre a map homotopic to the identity.

The monoids $L_G(\xi)$ and $\mathrm{aut}_G F$ are H-spaces, so that all their components have the same homotopy type. The study of the homotopy type of $L_G(\xi)$ is therefore reduced to the consideration of

(a) the map $\pi_0(R) : \pi_0(L_G(\xi)) = \mathcal{L}_G(\xi) \to \pi_0(\mathrm{aut}_G F) = G$.

and

(b) the restriction map $R_1 : L_1(\xi) \to \mathrm{aut}_1 F$.

Our main problems can be stated as follows :

1. On what conditions is the group $\mathcal{L}_1(\xi)$ finitely generated or finite (rigidity of the fibration) [cf. Theorem 4, below].
2. On what conditions is the map R_1 a homotopy equivalence [cf. for instance Theorem 5 below].

We first show that the group $\mathcal{L}_1(\xi)$ and the groups $\pi_i(L_1(\xi))$, $i \geq 1$, are finitely generated groups when the base B is a simply connected finite CW complex and the fibre F has the homotopy type of a simply connected finite type CW complex. In the particular case the base is a sphere, the result is more precise. We have indeed :

Theorem 1. *If $\xi = (E, p, S^n, F)$ is a fibration with clutching function $\alpha : S^{n-1} \to \mathrm{aut}_G F$, then there exists an exact sequence of groups*

$$\pi_1(\mathrm{aut}_G F) \xrightarrow{\partial_\alpha} \pi_n(\mathrm{aut}_G F) \xrightarrow{L} \mathcal{L}_G(\xi) \xrightarrow{\pi_0(R)} G_\alpha \to 1,$$

where

(1) *∂_α is the Samelson product by $\{\alpha\} \in \pi_{n-1}(\mathrm{aut}_G F)$.*

(2) *G_α is the stabilizer of $\{\alpha\}$ in G for the natural action of G on $\pi_{n-1}(\mathrm{aut}_G F)$, $G_\alpha = \{g \in G | g \cdot \alpha = \alpha\}$.*

In case $G = \mathrm{Aut}\, X$, this result has been obtained by K. Tsukiyama ([**21**]), as a corollary of a result of D. Gottlieb ([**9**]). Theorem 1 is obtained in a similar way from a slight modification of the quoted result of D. Gottlieb.

The interest of the above generalization of Tsukiyama's result lies in

Theorem 2. *Under the hypothesis of Theorem 1, if we suppose that F is a nilpotent space and that G acts unipotently on each $H_i(F; \mathbb{Z})$, then $\mathcal{L}_G(\xi)$ is a nilpotent group.*

Theorem 2 follows from Theorem 1 and Theorem 3.3 of ([**6**]). Indeed, Theorem 3.4 of ([**6**]) states that under our conditions the group G is nilpotent.

As a consequence of Theorem 2, we obtain after 0-localization the exact sequence

$$\pi_1(\mathrm{aut}_G F)\otimes\mathbb{Q}\xrightarrow{\partial_\alpha\otimes\mathbb{Q}}\pi_n(\mathrm{aut}_G F)\otimes\mathbb{Q}\xrightarrow{L\otimes\mathbb{Q}}\widehat{\mathcal{L}_G(\xi)}\xrightarrow{\widehat{R}}\widehat{G_\alpha}\to 1,$$

where $\widehat{\mathcal{L}_G(\xi)}$ and $\widehat{G_\alpha}$ respectively denote the Malcev completions of the nilpotent groups $\mathcal{L}_G(\xi)$ and G_α.

Our next result gives a complete description of this exact sequence in terms of a Sullivan model of F (see ([**20**], [**11**]) for basic notions in rational homotopy theory).

Let $(\wedge X, d)$ be a minimal model for F with a fixed K.S. basis $(x_i)_{i\in I}$. A derivation θ of $(\wedge X, d)$ is locally nilpotent (rel. (x_i)) if we have

$$\theta(x_i)\in\wedge(\bigoplus_{j<i}x_j\mathbb{Q}).$$

Denote by $\mathrm{Der}_*\wedge X$ the graded Lie algebra of derivations of $(\wedge X, d)$. This is a $\mathbb{Z}$-graded Lie algebra. The differential $D=[d,-]$ makes $\mathrm{Der}_*\wedge X$ into a graded differential Lie algebra. We define the sub differential Lie algebra L_* by :

$L_{-i}=\mathrm{Der}_{-i}(\wedge X)$,	$i\geq 1$
$L_j=0$	$j\geq 1$
L_0	is the subspace of $\mathrm{Der}_0(\wedge X)$ consisting of cycles which are locally nilpotent with respect to the fixed K.S. basis.

Theorem 3. *Let $\xi=(E,p,S^n,F)$ be a unipotent fibration with fibre a nilpotent space F, and let G be a maximal subgroup of* $\mathrm{Aut}\,F$ *acting unipotently on $H_*(F;\mathbb{Z})$. If G is torsion free, then we have the exact sequence*

$$H_{-1}(L_*,D)\xrightarrow{\partial_\eta}H_{-n}(L_*,D)\xrightarrow{\lambda}\widehat{\mathcal{L}_G(\xi)}\xrightarrow{\rho}\exp(H_0(L_*,D)_\eta)\to 1,$$

where

(a) *η is a derivation of degree $(-n+1)$ which is determined by the classifying map k of ξ. Moreover $D(\eta)=0$.*
(b) *∂_η is the Lie bracket by the homology class of η.*
(c) $H_0(L_*,D)_\eta=\{\gamma\in H_0(L_*,D)\mid[\gamma,\eta]=0\}$.
(d) $\exp(H_0(L_*,D)_\eta)$ *denotes the Malcev group associated to the locally nilpotent Lie algebra $H_0(L_*,D)_\eta$.*

Note that the torsion free hypothesis on G is not difficult to satisfy. For instance, if X is a rational space, then $\operatorname{Aut} X$ is a torsion free group ([**3**, Theorem 2.5]).

On the other hand, if X is a finite type virtually nilpotent CW complex, then $\operatorname{Aut} X$ is finitely generated ([**5**]).

Using rational homotopy, we can make precise the structure of $\mathcal{L}_G(\xi)$ in two interesting cases.

It is well known that fibrations ξ with fibre an homogeneous space K/H with $\operatorname{rank} K = \operatorname{rank} H$ have special properties. We know that the Serre spectral sequence of ξ with rational coefficients collapses at the E_2-term. Here we show that the space of self-equivalences of ξ is very small. More precisely,

Theorem 4. *Let $\xi : (E, p, B, F)$ be a fibration where all spaces are simply connected and of the homotopy type of finite CW complexes. We suppose that F is an homogeneous space, $F = K/H$ with K and H compact connected Lie groups of the same rank, and that $H^{2n+1}(B; \mathbb{Z})$ is a finite group for $n \geq 0$. Let G be a maximal subgroup of* $\operatorname{Aut} F$ *acting unipotently on $H_*(F; \mathbb{Z})$.*
Then,

(a) *the group $\mathcal{L}_G(\xi)$ is a finite group.*
(b) *the space $L_1(\xi)$ is a connected finite dimension H-space, and for $n > 1$, we have*

$$\dim .\pi_{2n-1}(L_1(\xi)) \otimes \mathbb{Q} = \sum_{p \leq n} \dim .H^{2p}(B; \mathbb{Q}) \otimes \pi_{2n-2p}(B_{\operatorname{aut}_1 F}).$$

Remark that (a) means that two self-equivalences of ξ inducing homotopic restrictions to the fibre F localized at 0 are already homotopic, after localization at 0.

In a similar way, we obtain

Theorem 5. *Let $\xi : (E, p, B, F)$ be a fibration where all spaces are simply connected and of the homotopy type of finite CW complexes. We suppose that there exists an integer n such that $\pi_q(F)$ is finite for $q > n$ and $\tilde{H}^q(B; \mathbb{Z})$ is finite for $q \leq n$. Let G be a maximal subgroup of* $\operatorname{Aut} F$ *acting unipotently on $H_*(F; \mathbb{Z})$. Then,*

(a) *The restriction map $\hat{R} : \widehat{\mathcal{L}_G(\xi)} \to \hat{G}$ is injective. This implies that $L_1(\xi)$ is a connected H-space.*
(b) *The restriction R induces a rational homotopy equivalence*

$$L_1(\xi) \to \operatorname{aut}_1 F.$$

1. Proof of Theorem 1

We consider the fibre sequence

$$\operatorname{aut}^\bullet X \to \operatorname{aut} X \xrightarrow{e} X$$

where e is the evaluation map. Taking the classifying space of the monoids $\operatorname{aut}^\bullet X$ and $\operatorname{aut} X$, we get a fibration sequence (up to homotopy)

$$\mathcal{U} : X \to B_{\operatorname{aut}^\bullet X} \xrightarrow{u} B_{\operatorname{aut} X}$$

which is universal for Hurewicz fibrations with fibre X, ([**6**, Proposition 4.1]).

By analogy with the theory of fibre bundles, we consider $\operatorname{Aut} F$ as the "structural group" of a Hurewicz fibration $\xi = (E, p, B, F)$ and we shall say that the structural group of ξ can be reduced to $G \subset \operatorname{Aut} F$ if ξ admits a classifying map $k : B \to B_{\operatorname{aut} F}$ such that the image of the map $\pi_1(k) : \pi_1(B) \to \pi_1(B_{\operatorname{aut} F}) \cong \pi_0(\operatorname{aut} F)$ is contained in G. This is only a useful analogy because the classifying map does not factor at all through the classifying space B_G. In fact we can form the monoid $\operatorname{aut}_G F$ of self-equivalences of F whose homotopy classes belong to G. In case of a G-reduction the classifying map k factors through the space $B_{\operatorname{aut}_G F}$ ([**17**], [**6**, Proposition 4.2]). The fibration

$$(\mathcal{U}_G) : F \to B_{\operatorname{aut}^\bullet_G F} \to B_{\operatorname{aut}_G F}$$

is a universal fibration for fibrations with fibre F whose "structural group" can be reduced to G.

Example. Let $B = S^n$. A Hurewicz fibration $\xi = (E, p, B, F)$ is determined, up to fibre homotopy, by the homotopy class $\{\alpha\}$ of a clutching function $\alpha : S^{n-1} \to \operatorname{aut} F$. In this case the structural group of ξ can be reduced to G if and only if for some point p in S^{n-1} the class $[d(p)]$ belongs to G.

Henceforth we shall fix a Hurewicz fibration $\xi = (E, p, B, F)$ whose base is a CW complex and with classifying map $k : B \to B_{\operatorname{aut}_G F}$.

Because Hurewicz fibrations give rise to a homotopy functor ([**1**]), and from ([**19**, Chapitre 7, Section 7, Theorem 11]), we can choose k as an inclusion and ξ as the restriction of $(\mathcal{U}_G)$ to B.

Let $L^*(\xi, \mathcal{U}_G)$ be the space of fibre preserving maps from E to $B_{\operatorname{aut}^\bullet_G F}$ which carry each fibre of ξ into a fibre of $\mathcal{U}_G$ by a homotopy equivalence. Let $L^*(\xi, \mathcal{U}_G; k)$ be the set of maps in $L^*(\xi, \mathcal{U})$ with the additional property that every map $f \in L^*(\xi, \mathcal{U})$ covers a map $B \to B_{\operatorname{aut}_G F}$ which is homotopic to k.

We denote by $L(B, B_{\mathrm{aut}_G F}; k)$ the component of k in the space of maps from B to $B_{\mathrm{aut}_G F}$ and by

$$\Phi : L^*(\xi, \mathcal{U}_G; k) \to L(B, B_{\mathrm{aut}_G F}; k)$$

the map that associates to every $f \in L^*(\xi, \mathcal{U}_G)$ the map $g \in L(B, B_{\mathrm{aut}_G F})$ covered by f.

Following the lines of the proof given by D. Gottlieb in the case $B_{\mathrm{aut}\, F}$ ([**9**, Theorem 1]), we obtain

Proposition 1. *Let $F \to E \to B$ be a fibration whose base is a CW complex and with classifying map k. If Φ is defined as above, then :*

(1) $\Phi^{-1}(k) \cong L_G(\xi)$.

(2) $L_G(\xi) \to L^*(\xi, \mathcal{U}_G; k) \xrightarrow{\Phi} L(B, B_{\mathrm{aut}_G F}; k)$ *is a principal fibration with a left action of $L_G(\xi)$ on $L^*(\xi, \mathcal{U}_G)$ given by composition of maps.*

(3) *If E is compactly generated, then $\pi_i(L^*(\xi, \mathcal{U}_G; k))$ is trivial for all $i \geq 0$.*

This implies immediately :

Corollary 1. *If E is compactly generated, then*

$$\mathcal{L}_G(\xi) = \pi_0(L_G(\xi)) = \pi_1(L(B, B_{\mathrm{aut}_G F}; k)).$$

and

$$\pi_i(L_G(\xi)) \cong \pi_{i+1}(L(B, B_{\mathrm{aut}_G F}; k)), \quad i \geq 1.$$

In the particular case when $B = \{*\}$, we have a fibration

$$\Phi : L^*(\xi, \mathcal{U}_G; *) \to B_{\mathrm{aut}_G F}$$

with fibre $\mathrm{aut}_G F$. Therefore we recover

Corollary 2. *If F is compactly generated, then*

$$\pi_i(B_{\mathrm{aut}_G F}) \cong \pi_{i-1}(\mathrm{aut}_G F), \quad i \geq 1.$$

Corollary 3. *If B is a simply connected finite CW complex and F has the homotopy type of a simply connected finite type CW complex, then the groups $\pi_i(\mathcal{L}_1(\xi))$, $i \geq 1$, are finitely generated.*

Proof: Denote by F_0 the rationalisation of the space F. The induced map $\pi_n(\mathrm{aut}_1 F) \to \pi_n(\mathrm{aut}_1(F_0))$ is finite to one for $n \geq 1$ ([**12**, (5.4)]).

On the other hand, denoting by M the Sullivan minimal model of F, we have a sequence of group isomorphisms $\pi_n(\mathrm{aut}_1(F_0)) \cong \pi_n((\mathrm{aut}_1(F))_0) \cong \pi_n(\mathrm{aut}_1(M))$ ([**12**, 3.11]). As $\pi_n(\mathrm{aut}_1(M))$ is finitely generated, the same is true for $\pi_n(\mathrm{aut}_1(F))$ for $n \geq 1$. We now make use of the Federer spectral sequence ([**7**]) converging to $\pi_*(L(B, B_{\mathrm{aut}_1 F}, k))$. It is easy to see that $E^2_{p,q} = H^q(B, \pi_{p+q}(B_{\mathrm{aut}_1 F}))$ is finitely generated abelian so that $E^\infty_{p,q}$ is finitely generated abelian. Since an extension of finitely generated abelian groups is a finitely generated abelian group, the groups $\pi_n(L(B, B_{\mathrm{aut}_1 F}, k))$ are finitely generated.

Consider now the evaluation map

$$e : L(S^n, B_{\mathrm{aut}_G F}) \to B_{\mathrm{aut}_G F}.$$

This is a Hurewicz fibration and the fibre is the space of based maps $L_\bullet(S^n, B_{\mathrm{aut}_G F})$. It results from ([**22**, Theorem 3.2]) that the homotopy exact sequence associated to this fibration is isomorphic to the exact sequence

$$\to \pi_{i+1}(B_{\mathrm{aut}_G F}) \xrightarrow{[k,-]} \pi_{n+i}(B_{\mathrm{aut}_G F}) \xrightarrow{T} \pi_i(L(S^n, B_{\mathrm{aut}_G F}); k) \xrightarrow{e_*} \pi_i(B_{\mathrm{aut}_G F})$$

where $[k, -]$ denotes the Whitehead bracket and $T = \tau \circ \pi_*(j)$ where j is the canonical injection

$$j : L_\bullet(S^n, B_{\mathrm{aut}_G F}) \to L(S^n, B_{\mathrm{aut}_G F}),$$

and τ the natural isomorphism

$$\pi_{n+i}(Y) = [S^i \wedge S^n, Y] = \pi_i(L_\bullet(S^n, Y)) \cong \pi_i(L_\bullet(S^n, Y), k), \quad i \geq 1.$$

The natural isomorphism

$$\partial_Y : \pi_i(Y) \to \pi_{i-1}(\Omega Y)$$

transforms the Whitehead product into the Samelson product, up to a sign, and $\pi_*(e)$ into $R : \mathcal{L}_G(\xi) \to \mathrm{Aut}_G F = G$. Then, using corollaries 1 and 2 above, we deduce the exact sequence of groups

$$\pi_1(\mathrm{aut}_G F) \xrightarrow{\partial_k} \pi_n(\mathrm{aut}_G F) \xrightarrow{\gamma} \mathcal{L}_G(\xi) \xrightarrow{R} G,$$

with $\gamma = \partial_{L(S^n, B_{\mathrm{aut}_G F})} \circ T \circ \partial^{-1}_{B_{\mathrm{aut}^\bullet F}}$. Now by ([**13**, Theorem 2.2]), we know that the image of R is precisely G_α. ■

2. Proof of Theorem 3

Let us consider the cochains $\mathcal{C}^*(L_*)$ on the differential graded Lie algebra L_* defined in the introduction,

$$\mathcal{C}^*(L_*) = (\wedge s(L_*^\vee), d),$$

where $L_*^\vee$ denotes the graded vector space dual to L

$$(L_*^\vee)^i = Hom(L_{-i}, \mathbb{Q}).$$

By ([**20**, section 11]), $(\wedge sL_*^\vee, d)$ is a (non minimal) model of $B_{\mathrm{aut}_G F}$ when G is a maximal subgroup of $\mathrm{Aut}\, F$ acting unipotently on $H_*(F; \mathbb{Q})$. Thus, if F is a nilpotent compactly generated space, Corollary 2 together with ([**20**, Theorem 10.1]) give the isomorphism

$$\pi_i(\mathrm{aut}_G F) \cong H_i(L_*, D), \quad i \geq 1.$$

If L is a locally nilpotent Lie algebra over $\mathbb{Q}$, we denote by $\exp(L)$ the divisible group associated to L by the Campbell-Hausdorff formula

$$x \cdot y = x + y + \frac{1}{2}[x, y] + \cdots .$$

Let G be a finitely generated torsion free nilpotent group. In ([**14**]), Malcev constructs a Lie algebra L_G over the rationals such that G naturally embeds into $\exp(L_G)$. The group $\hat{G} = \exp(L_G)$ is called the Malcev completion of G ([**16**], [**14**]).

Let now X be a nilpotent space. The action of $\pi_1(X)$ onto $\pi_n(X)$ can be described, modulo the isomorphism $\pi_r(X) \cong \pi_{r-1}(\Omega X)$, by the map

$$\mu : \pi_0(\Omega X) \times \pi_{n-1}(\Omega X) \to \pi_{n-1}(\Omega X),$$

$$\mu(g, \alpha) = g \cdot \alpha(t) \cdot g^{-1}.$$

Such a space X admits a 0-localization X_0, which satisfies $\pi_1(X_0) = \widehat{\pi_1(X)}$, $\pi_i(X_0) = \pi_i(X) \otimes \mathbb{Q}, i \geq 2$. Moreover, the action of $\pi_1(X)$ on $\pi_*(X)$ induces an action of the Lie algebra $L_{\pi_1(X)}$ on $\pi_n(X) \otimes \mathbb{Q}$ which is given by the bracket in the Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ ([**2**]).

We now return to the particular case, $\Omega X = \mathrm{aut}_G F$. Let η be a derivation that represents α. We then have

$$\exp(H_0(L_*, D)_\eta) = \widehat{G_\alpha}.$$

3. Proof of Theorems 4 and 5

The rational homotopy groups $\pi_i(L(B, B_{\mathrm{aut}_G F}, k)) \otimes \mathbb{Q}$, $i > 1$ and the Malcev completion of the nilpotent group $\pi_1(L(B, B_{\mathrm{aut}_G F}, k))$ can be computed by rational homotopy theory and more precisely by Haefliger's work on mapping spaces ([**10**]). In fact, if $f : S \to T$ is a continuous map between nilpotent finite type CW complexes, then there exists a complex (D_*, ∂),

$$D_n = \oplus_p \left[H^p(S; \mathbb{Q}) \otimes \pi_{n+p}(T_0)\right]$$

such that

(i) $H_q(D_*, \partial) \cong \pi_q(L(S, T; f)) \otimes \mathbb{Q}$, for $q > 1$.

(ii) $H_1(D_*, \partial) = \pi_1(L\widehat{(S, T;} f))$.

The differential ∂ depends on the map f and the construction is described in ([**10**], [**8**]).

Proof of Theorem 4: When F is an homogeneous space $G = K/H$, with $\operatorname{rank} K = \operatorname{rank} H$, Shiga and Tezuka ([**18**]) prove that

$$\pi_{2r}(\mathrm{aut}_G F) \otimes \mathbb{Q} = 0, r \geq 1.$$

This implies :

$$D_{2n+1} = \bigoplus_{2p \leq 2n+1} H^{2p}(B; \mathbb{Q}) \otimes \pi_{2n+1+2p}(B_{\mathrm{aut}_G F}) = 0,$$

and thus $\partial = 0$. Therefore,

$$\begin{cases} \pi_{2n}(L(B, B_{\mathrm{aut}_G F}; k)) \otimes \mathbb{Q} = D_{2n}, & n \geq 0 \\ \pi_{2n+1}(L(B, B_{\mathrm{aut}_G F}; k)) \otimes \mathbb{Q} = 0 \end{cases}$$

Now Corollary 1 implies that $\widehat{\mathcal{L}_G(\xi)} = 0$.The rationalization $(L_1(\xi))_0$ of $L_1(\xi)$ is a finite dimensional rational H-space, with

$$\pi_{2n}((L_1(\xi))_0) = 0$$
$$\pi_{2n-1}((L_1(\xi))_0) = \bigoplus_{p \leq n} H^{2p}(B; \mathbb{Q}) \otimes \pi_{2n+2p}(B_{\mathrm{aut}\, F})$$

This proves Theorem 4. ■

Proof of Theorem 5: We now suppose that $\tilde{H}_q(B; \mathbb{Z})$ is finite for $q \leq n$ and that $\pi_q(F)$ is finite for $q > n$. This implies that

$$D_1 = H^0(B; \mathbb{Q}) \otimes \pi_1(\widehat{B_{\mathrm{aut}_G F}}) \cong \widehat{G},$$

and

$$D_q \cong H^0(B;\mathbb{Q}) \otimes \pi_{q+1}(\operatorname{aut}_G F) \cong \pi_{q+1}(\operatorname{aut}_G F) \otimes \mathbb{Q}, \text{ for } q > 1.$$

In particular, $\hat{R} : \widehat{\mathcal{L}_G(\xi)} \to \hat{G}$ is injective, $L_1(\xi)$ is a connected space and the evaluation map

$$e : L(B, B_{\operatorname{aut}_1 F}; k) \to B_{\operatorname{aut}_1 F}$$

is a rational homotopy equivalence. The commutativity of the following diagram together with Proposition 1 implies now that $L_1(\xi) \to \operatorname{aut}_1 F$ is also a rational homotopy equivalence.

$$\begin{array}{ccc} L(B, B_{\operatorname{aut}_1 F}; k) & \xrightarrow{e} & B_{\operatorname{aut}_1 F} \\ \uparrow & & \uparrow \\ L^*(\xi, \mathcal{U}_G; k) & \xrightarrow{ev} & L^*(F, B_{\operatorname{aut}^\bullet_G F}) \\ \uparrow & & \uparrow \\ L_1(\xi) & \xrightarrow{R} & \operatorname{aut}_1 F \end{array} \quad \blacksquare$$

Using rational homotopy we can make explicit computations.

Proposition 2. *Let $\xi : E \to B$ be a fibration with fibre F. We suppose that B and F are simply connected finite type CW complexes and that there exists an integer N such that $\pi_{>N}(F) \otimes \mathbb{Q} = 0$, then*

1) *$\pi_n(L_1(\xi))$ is a finite group for $n > N$.*
2) *We have isomorphisms*

$$\pi_N(L_1(\xi)) \otimes \mathbb{Q} \overset{\pi_N(R)}{\longrightarrow} \pi_N(\operatorname{aut}_1(F)) \otimes \mathbb{Q} \overset{\pi_N(ev)}{\longrightarrow} \pi_N(F) \otimes \mathbb{Q}.$$

Proof: The rational homotopy groups of the space $\operatorname{aut}_1(F)$ are isomorphic to the homology groups of the space of derivations of the Sullivan minimal model of F ([**20**]). It is then clear that $\pi_{>N}(\operatorname{aut}_1(F)) \otimes \mathbb{Q} = 0$ and that the evaluation map $ev : \operatorname{aut}_1(F) \to F$ induces an isomorphism on $\pi_N(-) \otimes \mathbb{Q}$. As B is simply connected, this implies that the vector spaces D_n are zero for $n > N$ and for $n = N - 1$. Therefore we have the isomorphisms $\pi_N(L_1(\xi) \otimes \mathbb{Q} \cong D_N = H^0(B;\mathbb{Q}) \otimes \pi_N(\operatorname{aut}_1(F))$. $\blacksquare$

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