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THE FREUDENTHAL SPACE FOR APPROXIMATE SYSTEMS OF COMPACTA AND SOME APPLICATIONS

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Abstract

In this paper we define a space $\sigma(\underline{\mathbf{X}})$ for approximate systems of compact spaces. The construction is due to H. Freudenthal for usual inverse sequences [**4**, p. 153–156]. We stablish the following properties of this space: (1) The space $\sigma(\underline{\mathbf{X}})$ is a paracompact space, (2) Moreover, if $\underline{\mathbf{X}}$ is an approximate sequence of compact (metric) spaces, then $\sigma(\underline{\mathbf{X}})$ is a compact (metric) space (Lemma 2.4). We give the following applications of the space $\sigma(\underline{\mathbf{X}})$: (3) If $\underline{\mathbf{X}}$ is an approximate system of continua, then $X = \lim \underline{\mathbf{X}}$ is a continuum (Theorem 3.1), (4) If $\underline{\mathbf{X}}$ is an approximate system of hereditarily unicoherent spaces, then $X = \lim \underline{\mathbf{X}}$ is hereditarily unicoherent (Theorem 3.6), (5) If $\underline{\mathbf{X}}$ is an approximate system of trees with monotone onto bonding mappings, then $X = \lim \underline{\mathbf{X}}$ is a tree (Theorem 3.13).

1. Introduction

Let $\mathcal{U}$ be any covering of a space X. For any subset Y of X we define $\mathrm{St}(Y,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap Y \neq \emptyset\}$.

Similarly, we define $\mathrm{St}\,\mathcal{U} = \{\mathrm{St}(U,\mathcal{U}) : U \in \mathcal{U}\}$. Inductively, for each positive integer n, $\mathrm{St}^n\,\mathcal{U} = \mathrm{St}(\mathrm{St}^{n-1}\,\mathcal{U})$, where $\mathrm{St}^1\,\mathcal{U} = \mathrm{St}\,\mathcal{U}$.

We say that a cover $\mathcal{V}$ is a *star refinement* of a cover $\mathcal{U}$ if the cover $\mathrm{St}\,\mathcal{V}$ is a refinement of $\mathcal{U}$.

An open cover $\mathcal{W}$ of a space X is *normal* [**3**, p. 379] if there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open covers of the space X such that $\mathcal{W}_1 = \mathcal{W}$ and $\mathcal{W}_{i+1}$ is a *star refinement* of $\mathcal{W}_i$ for $i = 1, 2, \dots$ A T_1 space X is paracompact iff each open cover of X is normal [**3**, Theorem 5.1.12]. A T_1 space X is normal iff each locally finite open cover of X is normal [**3**, p. 379].

The set of all normal covers of X is denoted by $\mathrm{Cov}(X)$.

If $\mathcal{U}$, $\mathcal{V} \in \mathrm{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \prec \mathcal{U}$. If f, $g : Y \to X$ are $\mathcal{U}$*-near* mappings, i.e. if for any $y \in Y$ there exists $U \in \mathcal{U}$ with $f(y)$, $g(y) \in U$, we write $(f, g) \prec \mathcal{U}$.

Approximate inverse systems were introduced by S. Mardešić and L. R. Rubin [**11**] for compacta and by S. Mardešić and Watanabe [**12**] for general topological spaces.

Definition 1.1. An *approximate inverse system* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ consists of the following data: A preordered set $(A, \leq)$ which is directed and has no maximal element; for each $a \in A$, a topological space X_a and a normal covering $\mathcal{U}_a$ of X_a (called the mesh of X_a) and for each pair $a \leq b$ from A, a mapping $p_{ab} : X_b \to X_a$. Moreover the following three conditions must be satisfied:

(A1) The mappings $p_{ab}p_{bc}$ and p_{ac} are $\mathcal{U}_a$-near, $a \leq b \leq c$, i.e. $(p_{ab}p_{bc}, p_{ac}) \prec \mathcal{U}_a$.
(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is $b \geq a$ such that $(p_{ac}p_{cd}, p_{ad}) \prec \mathcal{U}$, whenever $a \leq b \leq c \leq d$.
(A3) For each $a \in A$ and each normal cover $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is $b \geq a$ such $\mathcal{U}_c \prec p_{ac}^{-1}(\mathcal{U}) = \{p_{ac}^{-1}(U) : U \in \mathcal{U}\}$ for each $c \geq b$.

In the case of metric compact spaces we replace the normal coverings by real numbers [**11**].

If the spaces X_a are T_1 paracompact, then in the above definition one can use all open coverings on the spaces X_a, $a \in A$, since in this case each open cover is normal.

Definition 1.2. An *approximate map* $p = \{p_a : a \in A\} : X \to X_a$ into an approximate inverse system $\mathbf{X} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ is a collection of maps $p_a : X \to X_a$, $a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $\mathcal{U} \in \mathrm{Cov}(X_a)$ there is $b \geq a$ such that $(p_{ac}p_c, p_a) \prec \mathcal{U}$ for each $c \geq b$. (See [**12**].)

Definition 1.3. Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system and let $p = \{p_a : a \in A\} : X \to X_a$ be an approximate map. We say that p is a *limit* of $\underline{\mathbf{X}}$ provided it has the following *universal* property [**12**, p. 592]:

(UL) For any approximate map $q = \{q_a : a \in A\} : Y \to X_a$ of a space Y there exists a unique map $g : Y \to X$ such that $p_a g = q_a$ for any $a \in A$.

Remark 1.4. If $p : X \to \underline{\mathbf{X}}$ is a limit of $\underline{\mathbf{X}}$, then the space X is determined up to a unique homeomorphism. Therefore, we often speak of the limit X of $\underline{\mathbf{X}}$ and we write $X = \lim \underline{\mathbf{X}}$.

Definition 1.5. Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called a *thread* of $\underline{\mathbf{X}}$ provided it satisfies the following condition:

(L) $\quad (\forall\, a \in A)(\forall \mathcal{U} \in \mathrm{Cov}(X_a))(\exists\, b \geq a)(\forall\, c \geq b) p_{ac}(x_c) \in \mathrm{st}(x_a, \mathcal{U}).$

Remark 1.6. If X_a is a $T_{3.5}$ space, then the sets $\mathrm{st}(x_a, \mathcal{U})$, $U \in \mathrm{Cov}(X_a)$, form a basis of the topology at the point x_a. Therefore, for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

(L)* $\quad (\forall\, a \in A) \lim\{p_{ac}(x_c) : c \geq a\} = x_a.$

The following theorem shows that the set of threads is a limit of $\underline{\mathbf{X}}$.

Theorem 1.7. *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system. Let $X \subset \prod X_a$ be the set of all threads of $\underline{\mathbf{X}}$ and let $p_a : X \to X_a$ be the restriction $p_a = \pi_a | X$ of the projection $\pi_a : \prod X_a \to X_a$, $a \in A$. Then $p = \{p_a : a \in A\} X \to \underline{\mathbf{X}}$ is a limit of $\underline{\mathbf{X}}$.*

Proof: See [**12**, Theorem (1.14)]. ∎

The *canonical limit* of $\underline{\mathbf{X}}$ is the set of all threads of $\underline{\mathbf{X}}$ [**12**, p. 593].

Theorem 1.8. *For any approximate inverse system $\underline{\mathbf{X}}$ the canonical limit* $\lim \underline{\mathbf{X}}$ *is closed in $\prod X_a$. Moreover, if all X_a are compact and non-empty, then* $\lim \underline{\mathbf{X}}$ *is compact and non-empty.*

Proof: See the proof of Lemma (1.16) and Theorem (4.1) of [**12**]. ∎

Lemma 1.9. *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system of Tychonoff spaces, let X be the canonical limit of $\underline{\mathbf{X}}$ and let $B \subseteq A$ be a cofinal subset of A. Then the collection $\mathcal{B}$ of all sets of the form $p_b^{-1}(U_b)$, where $b \in B$ and $V_b \subseteq X_b$ is open, is a basis of the topology for X.*

Proof: See [**12**, (1.18) Lemma]. ∎

Theorem 1.10. *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system of compact Hausdorff spaces with limit X. For each closed $F \subseteq X$ we have*

$$F = \bigcap\{p_a^{-1}(p_a(F)) : a \in A\}.$$

Proof: It is obvious that $F \subseteq p_a^{-1}(p_a(F))$ for each $a \in A$. Thus, $F \subseteq \cap\{p_a^{-1}(p_a(F)) : a \in A\}$. If $x \notin F$, then, by Lemma 19 we infer that there exists an $a \in A$ and an open set $U_a \subseteq X_a$ such that $x \in p_a^{-1}(V_a) \subseteq X - F$. This means that $p_a(x) \notin p_a(F)$ and $x \notin p_a^{-1}(p_a(F))$. ∎

2. The Freudenthal space $\sigma(\underline{\mathbf{X}})$

The following construction is similar to the construction due to H. Freudenthal [**4**, p. 153] for usual inverse sequences. For any usual inverse system see [**10**].

Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system of compact Hausdorff spaces with limit X and the projections $p_a : X = \lim \underline{\mathbf{X}} \to X_a$. The Freudenthal space $\sigma(\underline{\mathbf{X}})$ associated to $\underline{\mathbf{X}}$ is the set

$$\sigma(\underline{\mathbf{X}}) = X \bigcup \left(\bigcup \{X_a : a \in A\} \right) \tag{1}$$

where all X_a and their limit X are considered as being disjoint sets [**10**], in which a topology is defined as follows. If U_a is an open set in X_a, let

$$U_a^* = \bigcup \{p_{ab}^{-1}(U_a) : b \geq a\} \bigcup p_a^{-1}(U_a). \tag{2}$$

Now, we define a topology T on $\sigma(\underline{\mathbf{X}})$ by a base [**3**, p. 27] $\mathcal{B}$ which consists of all open sets U_a in all X_a and all U_a^* for all open sets $U_a \subseteq X_a$, $a \in A$. Since the sets $p_a^{-1}(U_a)$ form a basis for X, it follows that $\mathcal{B}$ is a cover of $\sigma(\underline{\mathbf{X}})$. By virtue of [**3**, p. 27] we need to prove that for each $x \in \sigma(\underline{\mathbf{X}})$ and each pair B, $C \in \mathcal{B}$ with $x \in B \cap C$ there is a $D \in \mathcal{B}$ such that $x \in D \subseteq B \cap C$. It suffices to prove this statement if B is some U_a^* and C is some U_b^*. If x is a point of X_c, then x is contained in a set $p_{ac}^{-1}(U_a) \cap p_{bc}^{-1}(U_b)$ which is open in X_c and thus belongs to $\mathcal{B}$. If x is a point of X, then

$$z \in p_a^{-1}(U_a) \bigcap p_b^{-1}(U_b) \tag{3}$$

i.e., $x_a = p_a(x) \in U_a$, and $x_b = p_b(x) \in U_b$. Choose $\mathcal{V}_a \in \mathrm{Cov}(X_a)$, $\mathcal{V}_b \in \mathrm{Cov}(X_b)$ such that

$$\mathrm{St}(x_a, \mathcal{V}_a) \subseteq U_a \text{ and } \mathrm{St}(x_b, \mathcal{V}_b) \subseteq U_b. \tag{4}$$

Take $\mathcal{W}_a \in \mathrm{Cov}(X_a)$, $\mathcal{W}_b \in \mathrm{Cov}(X_b)$ such that $\mathrm{St}^2\, \mathcal{W}_a \prec \mathcal{V}_a$, $\mathrm{St}^2\, \mathcal{W}_b \prec \mathcal{V}_b$ and $c \in A$ such that $c \geq a$, b, (A2) and (A3) hold for a, b, $\mathcal{W}_a$, $\mathcal{W}_b$ and (L) holds for x, a, b, $\mathcal{W}_a$, $\mathcal{W}_b$. Put

$$V_c = \mathrm{St}(x_c, \mathcal{U}_c). \tag{5}$$

Since $x \in p_c^{-1}(V_c) \subseteq V_c^*$, the proof will be complete if we show that

$$V_c^* \subseteq U_a^* \bigcap U_b^*. \tag{5.1}$$

We first prove that

$$p_c^{-1}(V_c) \subseteq p_a^{-1}(U_a) \bigcap p_b^{-1}(U_b). \tag{6}$$

Consider a point $y = (y_a) \in p_c^{-1}(V_c)$. By (5) there is a $U_1 \in \mathcal{U}_c$ such that

$$x_c,\, y_c \in U_1. \tag{7}$$

By the choice of c (property (A3)) $\mathcal{U}_c \prec p_{ac}^{-1}(\mathcal{W}_a)$ and $\mathcal{U}_c \prec p_{bc}^{-1}(\mathcal{W}_b)$. This means that there is a $W_1 \in \mathcal{W}_a$ and $W_2 \in \mathcal{W}_b$ such that $U_1 \subseteq p_{ac}^{-1}(W_1)$ and $U_1 \subseteq p_{bc}^{-1}(W_2)$. Thus, (7) implies

$$p_{ac}(x_c),\, p_{ac}(y_c) \in W_1 \text{ and } p_{bc}(x_c),\, p_{bc}(y_c) \in W_2. \tag{8}$$

By the choice of c (property (L)), there are $W_3 \in \mathcal{W}_a$, $W_4 \in \mathcal{W}_b$ such that

$$x_a,\, p_{ac}(x_c) \in W_3 \text{ and } x_b,\, p_{bc}(x_c) \in W_4. \tag{9}$$

Since $y \in p_b^{-1}(U_b) \subseteq X$, there is a $d \geq c$ satisfying (L) for y, a, $\mathcal{W}_a$ and for y, b, $\mathcal{W}_b$. Thus, there exist a $W_5 \in \mathcal{W}_a$, $W_6 \in \mathcal{W}_b$ and $U_4 \in \mathcal{U}_c$ such that

$$p_{ad}(y_d),\, y_a \in W_5 \text{ and } p_{bd}(y_d),\, y_b \in W_6 \tag{10}$$

and

$$p_{cd}(y_d),\, y_c \in U_4. \tag{11}$$

By the choice of c (property (A3)), $\mathcal{U}_c \prec p_{ac}^{-1}(\mathcal{W}_a)$ and $\mathcal{U}_c \prec p_{bc}^{-1}(\mathcal{W}_b)$. Hence, there exist a $W_7 \in \mathcal{W}_a$ and $W_8 \in \mathcal{W}_b$ such that $U_4 \subseteq p_{ac}^{-1}(W_7)$ and $U_4 \subseteq p_{bc}^{-1}(W_8)$. By (11) we have

$$p_{ac}p_{cd}(y_d),\, p_{ac}(y_c) \in W_7 \text{ and } p_{bc}p_{cd}(y_d),\, p_{bc}(y_c) \in W_8. \tag{12}$$

By the choice of c (property (A2)), we also have a $W_9 \in \mathcal{W}_a$ and $W_{10} \in \mathcal{W}_b$ such that

$$p_{ac}p_{cd}(y_d),\, p_{ad}(y_d) \in W_9 \text{ and } p_{bc}p_{cd}(y_d),\, p_{bd}(y_d) \in W_{10}. \tag{13}$$

Now, (9), (8), (12), (13), (10), $\mathrm{St}^2\,\mathcal{W}_a \prec \mathcal{V}_a$ and $\mathrm{St}^2\,\mathcal{W}_a \prec \mathcal{V}_b$ yield a $V' \in \mathcal{V}_a$ and a $V'' \in \mathcal{V}_b$ such that $x_a, y_a \in W_1 \cup W_3 \cup W_5 \cup W_7 \cup W_9 \subseteq V'$ and $x_b, y_b \in W_2 \cup W_4 \cup W_6 \cup W_8 \cup W_{10} \subseteq V''$. This and (4) imply

$p_a(y) = y_a \in \mathrm{St}(x_a, \mathcal{V}_a) \subseteq U_a$ and $p_b(y) = y_b \in \mathrm{St}(x_b, \mathcal{V}_b) \subseteq U_b$. This means that $y \in p_a^{-1}(U_a) \cap p_b^{-1}(U_b)$, i.e., (6) is proved. It remains to prove

$$p_{cd}^{-1}(V_c) \subseteq p_{ad}^{-1}(U_a) \bigcap p_{bd}^{-1}(U_b) \quad \forall\, d \geq c. \tag{14}$$

Let $z_d \in p_{cd}^{-1}(V_c)$. By (5) there is a $U_{11} \in \mathcal{U}_c$ such that

$$x_c,\, p_{cd}(z_d) \in U_{11}. \tag{15}$$

By the choice of c (property (A3)) there is a $W_{11} \in \mathcal{W}_a$ and a $W_{12} \in \mathcal{W}_b$ such that $U_{11} \subseteq p_{ac}^{-1}(W_{11})$ and $U_{11} \subseteq p_{bc}^{-1}(W_{12})$. Thus, (15) implies

$$p_{ac}(x_c),\, p_{ac}(p_{cd}(z_d)) \in W_{11} \text{ and } p_{bc}(x_c),\, p_{bc}(p_{cd}(z_d)) \in W_{12}. \tag{16}$$

By (A2) we infer there are $W_{13} \in \mathcal{W}_a$ and $W_{14} \in \mathcal{W}_b$ such that

$$p_{ac}p_{cd}(z_d),\, p_{ad}(z_d) \in W_{13} \text{ and } p_{bc}p_{cd}(z_d),\, p_{bd}(z_d) \in W_{14}. \tag{17}$$

From (9), (16) and (17) it follows $x_a,\, p_{ad}(z_d) \in \mathrm{St}\,\mathcal{V}_a$ and $x_b,\, p_{bd}(z_d) \in \mathrm{St}\,\mathcal{V}_b$. By (4) $p_{ad}(z_d) \in U_a$ and $p_{bd}(z_d) \in U_b$. We infer that $z_d \in p_{ad}^{-1}(U_a) \cap p_{bd}^{-1}(U_b)$ and (14) is proved. Hence, we have $x \in V_c^* \subseteq U_a^* \cap U_b^*$, i.e., (5.1) is proved. This means that $\mathcal{B}$ is a basis for some topology T on $\sigma(\underline{\mathbf{X}})$.

Now, we will prove that T is a Hausdorff topology. Let x, y be a pair of distinct points in $\sigma(\underline{\mathbf{X}})$. If x, $y \notin \lim \underline{\mathbf{X}}$, then there exists a pair a, $b \in A$ such that $x \in X_a$, $y \in X_b$. If $a = b$, then x and y have disjoint neighborhoods since X_a is a Hausdorff space. If $a \neq b$, then X_a and X_b are disjoint neighborhoods (in $\sigma(\underline{\mathbf{X}})$) of x and y respectively. Now, suppose that $x \in \lim \underline{\mathbf{X}}$ and $y \notin \lim \underline{\mathbf{X}}$. Let $y \in X_b$ for some $b \in A$. By virtue of Lemma 1.9 there is a $c > b$ and an open set U_c such that $p_c^{-1}(U_c)$ is a neighborhood of x in $\lim \underline{\mathbf{X}}$. It is clear that X_b and V_c^* are disjoint neighborhoods of y and x in $\sigma(\underline{\mathbf{X}})$. Finally, let x, $y \in \lim \underline{\mathbf{X}}$. Since $\lim \underline{\mathbf{X}}$ is a Hausdorff space, there are open (in $\lim \underline{\mathbf{X}}$) disjoint sets U and V such that $x \in U$ and $y \in V$. By virtue of Lemma 1.9 there exists a $b \in A$ and open sets U_b and V_b such that $x \in p_b^{-1}(U_b) \subseteq U$ and $y \in p_b^{-1}(V_b) \subseteq V$. It follows that U_b and V_b are disjoint since U and V are disjoint. Hence, U_b^* and V_b^* are disjoint. Thus, $\sigma(\underline{\mathbf{X}})$ is a Hausdorff space.

A *net in a topological space* X [**3**, p. 73] is an arbitrary function from a non-empty directed set D to the space X. Nets will be denoted by $\mathcal{N} = \{x_d : d \in D\}$. A point $x \in X$ is called a *limit* of a net $\mathcal{N} = \{x_d : d \in D\}$ if for every neighborhood U of x there is a $d_0 \in D$ such that $x_d \in U$ for each $d \geq d_0$. We say that the net $\mathcal{N}$ *converges* to x. A point $x \in X$ is called a *cluster* point of a net $\mathcal{N} = \{x_d : d \in D\}$ if for every neighborhood U of x and every $d_0 \in D$ there exists a $d \geq d_0$ such that $x_d \in U$.

Lemma 2.1. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of non-empty compact Hausdorff spaces with limit* X*.*

1. *If* A' *is a cofinal subset of* A*, then each family* $\mathcal{N} = \{x_a : x_a \in X_a, a \in A'\}$ *is a net in* $\sigma(\underline{\mathbf{X}})$ *which has at least one cluster point* x *(in the topology* T*) such that* $x \in X \subseteq \sigma(\underline{\mathbf{X}})$*.*
2. *Each point* $x \in X$ *is the limit (in the topology* T*) of the net* $\{p_a(x_a) : a \in A\}$*.*

Proof: For each $a \in A$ we consider the net $\mathcal{N}_a = \{p_{ab}(x_b) : b \in A', b \geq a\}$. From the compactness of X_a it follows that the set C_a of all cluster points of $\mathcal{N}_a$ is non-empty. Clearly, each C_a is closed and compact in X_a. First, we prove

(a) *For each* $a \in A$ C_a *is a non-empty subset of* $p_a(X)$*.*

If we suppose that some $c_a \in C_a \backslash p_a(X)$, then c_a and $p_a(X)$ respectively, have disjoint neighborhoods U and V. By virtue of the property (B3) [**12**, p. 606, 615] there is a $b \geq a$ such that $p_{ac}(X_c) \subseteq V$ for each $c \geq b$, $c \in A'$. This is impossible since there exists $c \geq b$ such that $p_{ac}(x_c) \in U$ (c_a is a cluster point of the net $\mathcal{N}_a$).

From (a) it easily follows that

(b) *For each* $a \in A$ *the set* $p_a^{-1}(C_a)$ *is non-empty.*

By (b) there is $y^a \in p_a^{-1}(C_a) \subseteq \lim \underline{\mathbf{X}}$, $a \in A'$. Since $\lim \underline{\mathbf{X}}$ is compact, there is a cluster point $y \in \lim \underline{\mathbf{X}}$ of the net $\mathcal{Y} = \{y^a : a \in A'\}$. Let us prove

(c) $p_a(y) \in C_a$, $a \in A$.

It suffices to prove that for each neighborhood U_a of $p_a(y)$ and each b_0 there exists a $d \geq b_0$ such that $p_{ad}(x_d) \in U_a$. Let $\mathcal{U}$ be a normal cover of X_a such that

$$\mathrm{St}^2(p_a(y), \mathcal{U}) \subseteq U_a. \tag{18}$$

Let $U_1 \in \mathcal{U}$ be such that $p_a(y) \in U_1$. Then $p_a^{-1}(U_1)$ is a neighborhood of y. The set B of all $b \in A'$ with $y^b \in p_a^{-1}(U_1)$ is cofinal in A' since y is a cluster point of $\mathcal{Y}$. By virtue of (AS) the set $B' \subseteq B$ of all $b \in B$, $b \geq b_0$, such that

$$(p_a, p_{ab}p_b) \prec \mathcal{U} \tag{19}$$

is cofinal in A. Similarly, by (A2), the set $B'' \subseteq B'$ of all $b \in B'$ such that

$$(p_{ac}, p_{ab}p_{bc}) \prec \mathcal{U}, \quad c \geq b \tag{20}$$

is cofinal in A. Let $b \in B''$. Then $y^b \in p_a^{-1}(U_1)$. Thus

(21) $$p_a(y),\, p_a(y^b) \in U_1.$$

By virtue of (19) it follows

(22) $$p_a(y^b),\, p_{ab}p_b(y^b) \in U_2 \in \mathcal{U}.$$

This and (21) imply

(23) $$p_{ab}p_b(y^b) \in \mathrm{St}(p_a(y),\mathcal{U}).$$

Now, $p_b(y^b) \in C_b$ since $y^b \in p_b^{-1}(C_b)$. We infer that $p_{ab}^{-1}(\mathrm{St}(p_a(y),\mathcal{U}))$ is a neighborhood of $p_b(y^b)$. Since $p_b(y^b)$ is a cluster point of $\mathcal{N}_a$ there is a $d \geq b \geq b_0$, $d \in A'$ such that $p_{bd}(x_d) \in p_{ab}^{-1}(\mathrm{St}(p_a(y),\mathcal{U}))$. This means that $p_{ab}(p_{bd}(x_d)) \in \mathrm{St}(p_a(y),\mathcal{U})$. Using (20), $p_{ad}(x_d) \in \mathrm{St}^2(p_a(y),\mathcal{U})$. Thus, by (18)

(24) $$p_{ad}(x_d) \in U_a.$$

We infer that $p_a(y) \in C_a$, i.e., $y \in p_a^{-1}(C_a)$ for each $a \in A$.

(d) *The point y is a cluster point (in the topology T) of $\mathcal{N}$.*

This follows from (24) since $x_d \in p_{ad}^{-1}(U_a)$. This means that for each neighborhood U_a^* of y and each $b_0 \in A$ there is a $d \geq b_0$, $d \in A'$, such that $x_d \in U_a^*$.

The proof of Lemma 2.1 is complete since the second statement easily follows from the definition of the topology T on $\sigma(\underline{\mathbf{X}})$. ■

Lemma 2.2. *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system of compact Hausdorff spaces. If U is a neighborhood of $X = \lim \underline{\mathbf{X}}$ in $\sigma(\underline{\mathbf{X}})$, then there exists $a \in A$ such that $X_b \subseteq U$ for each $b \geq a$.*

Proof: Since X is compact and since the sets (2) form a basis for the neighborhoods of the points of X, one can find $\{U_{a_i}^* : i = 1, \dots, n\}$ such that

(25) $$V = \bigcup \{U_{a_i}^* : i = 1, \dots, n\}$$

and $X \subseteq V \subseteq U$. In order to complete the proof, it suffices to find an $a \in A$, $a \geq a_1, \dots, a_n$ such that

(26) $$X_a \subseteq V$$

since then we have

(27) $$X_b \subseteq V \subseteq U, \quad b \geq a.$$

Suppose that no $a \in A$ satisfies (26). This means that for each $a \in A$ there is $x_a \in X_a - V$. We obtain a net $\{x_a : a \in A\}$ in $\sigma(\underline{\mathbf{X}})$ which has no cluster point in $V \supseteq X$. This contradicts Lemma 2.1. The proof is complete. ■

Lemma 2.3. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of compact Hausdorff spaces. Then* $\sigma(\underline{\mathbf{X}})$ *is paracompact. Moreover, if* $\underline{\mathbf{X}}$ *is an approximate sequence, then* $\sigma(\underline{\mathbf{X}})$ *is compact.*

Proof: Let $\mathcal{V} = \{V_\mu\}$ be any cover of $\sigma(\underline{\mathbf{X}})$. Since X is compact, there is a finite subcollection, consisting of sets $V_{\mu(1)}, \dots, V_{\mu(n)}$ which cover X. Let V be the union of this subcollection. By virtue of Lemma 2.2 there is an $a \in A$ such that all X_b, $b \geq a$, are in V. Let us recall that the set $X_a^* = (\cup\{X_b : b \geq a\} \cup X$ is of type (2) with $U_a = X_a$ and it is open in $\sigma(\underline{\mathbf{X}})$. Now consider the following collection $\mathcal{U}$ of open sets of $\sigma(\underline{\mathbf{X}})$: take first the open sets $X_a^* \cap V_{\mu(1)}, \dots, X_a^* \cap V_{\mu(n)}$ for members of $\mathcal{U}$. Furthermore, for each $b \in A - \{c : c \in A,\, c \geq a\}$ consider the open covering $\{X_b \cap V_\mu\}$ of X_b and take members of a finite subcovering as new members of $\mathcal{U}$. This is possible since X_b is compact and open in $\sigma(\underline{\mathbf{X}})$. The family $\mathcal{U}$ of open sets of $\sigma(\underline{\mathbf{X}})$ is a star-finite covering of $\sigma(\underline{\mathbf{X}})$ which refines the covering $\mathcal{V}$. Moreover, $\mathcal{U}$ is a locally finite refinement of $\mathcal{V}$. The proof of paracompactness is complete. If $\underline{\mathbf{X}}$ is an approximate sequence, then we obtain a finite subcovering since the set $A - \{c : c \in A,\, c \geq a\}$ is finite. The proof is complete. ■

Theorem 2.4. *Let* $\underline{\mathbf{X}} = \{X_n, \epsilon_n, p_{mn}, N\}$ *be an approximate inverse sequence of compact metric spaces* X_n. *Then* $\sigma(\underline{\mathbf{X}})$ *is a compact metric space.*

Proof: Each space X_n has a countable base $\mathcal{B}_n$ [**3**, 4.1.15 Theorem]. It follows that the family $\mathcal{B}^* = \{U^* : U \in \mathcal{B}_n : n \in N\}$ is countable. It is obvious that the union $\mathcal{B} = \{\mathcal{B}_n : n \in N\} \cup \mathcal{B}^*$ is a countable base for topology T. Thus $\sigma(\underline{\mathbf{X}})$ is metrizable [**3**, p. 351]. ■

We close this section with the following theorem which is similar to the theorem for usual inverse systems of compact Hausdorff spaces due to S. Mardešić [**10**, Theorem 4] (see Theorem 4.2 of [**12**]).

Theorem 2.5. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of compact Hausdorff spaces and let* $f : X \to R$ *be a mapping of their limit into a simplicial complex. Then there exists an* $a \in A$ *such that for each* $b \geq a$ *one can define a mapping* $f_b : X_b \to R$ *with the property that* $f_b p_b$ *is homotopic to* f.

3. Applications

In this section we give some applications of the space $\sigma(\underline{\mathbf{X}})$. We start with

Theorem 3.1. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of Hausdorff continua. The space* $X = \lim \underline{\mathbf{X}}$ *is a continuum.*

Proof: By virtue of 1.8 X is a compact. Suppose that X is not connected. There is a pair F, G of non-empty closed (in X) disjoint subsets of X. Since X is closed in $\sigma(\underline{\mathbf{X}})$, the sets F and G are closed in the normal space $\sigma(\underline{\mathbf{X}})$ (Lemma 2.3). There are two disjoint open (in $\sigma(\underline{\mathbf{X}})$) sets U and V which contain F and G. By virtue of Lemma 2.2 there is $a \in A$ such that X_b is contained in $U \cup V$ for each $b \geq a$. We shall prove that X_b intersects U and V for sufficiently large b. If x is a point of F, then there is an $a_1 \in A$ such that for each $c \geq a_1$ there is an open set $U_c \subseteq X_c$ for which U_c^* is a neighborhood of x contained in U. Hence, if $b \geq a$, then X_b intersects U. Similarly, there is $a_2 \in A$ such that X_b intersects V for each $b \geq a_2$. Thus, there is a $b \in A$ such that X_b intersects both U and V and is contained in $U \cup V$. This is impossible since X_b is connected. ■

In the sequel we use the notion of a net of sets in the sense of [**13**] or [**7**, p. 343].

A *net of sets* $\{A_n : n \in D\}$ of a topological space X is a function [**13**] defined on a directed set D which assigns to each $n \in D$ a subset A_n of X.

If $\{A_n : n \in D\}$ is a net of subsets of X, then:

1. The *limit inferior* $\mathrm{Li}\, A_n$ is the set of all points $x \in X$ such that for every neighborhood U of x there exists $n_0 \in D$ such that U intersect A_n for each $n \geq n_0$.
2. The *limit superior* $\mathrm{Ls}\, A_n$ is the set of all points $x \in X$ such that for every neighborhood U of x and each $n_0 \in N$ there is $n \geq n_0$ such that U intersect A_n.

A net $\{A_n : n \in D\}$ is said to be *topologically convergent* (to a set A) if $\mathrm{Ls}\, A_n = \mathrm{Li}\, A_n$ $(= A)$ and in this case the set A will be denoted by $\mathrm{Lim}\, A_n$.

Lemma 3.2. *Let* $\{C_n : n \in D\}$ *be the net of subsets of a space* X*. Let* U *be a neighborhood of* $\mathrm{Ls}\, C_n$ *such that* $X \backslash U$ *is compact. Then there is a* $m \in D$ *such that* $C_p \subseteq U$ *for each* $p \geq m$*.*

Proof: Suppose, on the contrary, that for each $m \in D$ there is a $p \in D$ such that $Z_p = C_p \backslash U$ is non-empty. Let z_p be any point of Z_p and let P be the set of all such $p \in D$. The net $\{z_p : p \in P\}$ has a cluster point z in $X \backslash U$. This is impossible since $z \in \mathrm{Ls}\, C_n \subseteq U$. The proof is complete. ■

Lemma 3.3. *Let $\{C_n : n \in D\}$ be the net of connected sets C_n of a normal space X such that $\operatorname{Li} C_n \neq \emptyset$. If for each neighborhood U of $\operatorname{Ls} C_n$ the set $X \backslash U$ is compact, then $\operatorname{Ls} C_n$ is connected.*

Proof: Suppose that $\operatorname{Ls} C_n$ is disconnected. This means that there are disjoint closed nonempty subsets F and G of $\operatorname{Ls} C_n$ such that $\operatorname{Ls} C_n = F \cup G$. The sets are closed in X since $\operatorname{Ls} C_n$ is closed in X. From the normality of X it follows that there are two disjoint open sets U and V such that $F \subseteq U$ and $G \subseteq V$. This means that $\operatorname{Ls} C_n \subseteq U \cup V$. Let $\operatorname{Li} C_n \cap U \neq \emptyset$. By virtue of Lemma 3.2 there is an $m \in D$ such that $C_p \subseteq U \cup V$ for each $p \geq m$. Clearly, there is some $p \geq m$ such that C_p intersects U (since $\operatorname{Li} C_n \cap U \neq \emptyset$) and C_p intersects V (since $V \cap \operatorname{Ls} C_n \neq \emptyset$). This means that $C_p \subseteq U \cup V$ and $U \cap C_p \neq \emptyset$, $V \cap C_p \neq \emptyset$. This contradicts the connecteness of C_p. ∎

Lemma 3.4. *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system of compact Hausdorff spaces. Let $\{C_a : a \in A,\, C_a \subseteq X_a\}$ be a net of continua such that $\operatorname{Li} C_a \subseteq \sigma(\underline{\mathbf{X}})$ is non-empty. Then $\operatorname{Ls} C_a$ is a non-empty subcontinuum of $X = \lim \underline{\mathbf{X}} \subseteq \sigma(\underline{\mathbf{X}})$.*

Proof: It is clear that $\operatorname{Li} C_a \subseteq \operatorname{Ls} C_a \subseteq X$. Suppose that $\operatorname{Ls} C_a$ is disconnected. We infer that there is a pair F, G of disjoint nonempty closed subsets of $\operatorname{Ls} C_a$ such that $\operatorname{Ls} C_a = F \cup G$. The sets F and G are closed in X and in $\sigma(\underline{\mathbf{X}})$. There are disjoint open sets of $\sigma(\underline{\mathbf{X}})$ (since $\sigma(\underline{\mathbf{X}})$ is normal) such that $F \subseteq U$ and $G \subseteq V$. Let $\operatorname{Li} C_a \cap U \neq \emptyset$. We claim that there is an $a \in A$ such that $C_b \subseteq U \cup V$ for each $b \geq a$. In the opposite case we obtain a net $\mathcal{N} = \{x_b : b \in A',\, x_b \in C_b \backslash (U \cup V),\, b \geq a\}$ where A' is cofinal in A. By virtue of Lemma 2.1 the net $\mathcal{N}$ has a cluster point x in X. Clearly, $x \notin U \cup V$. This is impossible since $x \in \operatorname{Ls} C_a$. Thus, there is an $a \in A$ such that $C_b \subseteq U \cup V$, $b \geq a$. It is clear that there is a $b \geq a$ such that C_b intersects U (since $\operatorname{Li} C_a \cap U \neq \emptyset$) and V (since V contains a point of $\operatorname{Ls} C_a$). But, this is impossible since C_b is connected and $C_b \subseteq U \cup V$. The proof is complete. ∎

Lemma 3.5. *Let $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ be an approximate inverse system of non-empty compact Hausdorff spaces with limit X. For each closed $F \subseteq X$ we have the net $\mathcal{N}(F) = \{p_a(F) : a \in A\}$ and, for each $a \in A$, the net $\mathcal{N}_a(F) = \{p_{ab}p_b(F) : b \geq a\}$ such that*

1. *$p_a(F) = \operatorname{Lim} \mathcal{N}_a(F)$,*
2. *$F = \operatorname{Lim} \mathcal{N}(F)$.*

Proof: From the definition of thread it follows that $p_a(F) \subseteq \operatorname{Li} \mathcal{N}_a(F)$. On the other hand, from property (B2) [**12**, p. 601, 615] we infer that if $x \notin p_a(F)$, then $x \notin \mathrm{Ls}\,\mathcal{N}_a(F)$. Thus, $p_a(F) \supseteq \mathrm{Ls}\,\mathcal{N}_a(F) \supseteq \mathrm{Li}\,\mathcal{N}_a(F)$. Therefore, $\mathrm{Lim}\,\mathcal{N}_a(F) = p_a(F)$. From 2 of Lemma 2.1 we have $F \subseteq \mathrm{Li}\,\mathcal{N}(F)$. On the other hand, for each point $y \in X \backslash F$ there is a $b \in A$ such that $p_b(y)$ and $p_b(F)$ have disjoint neighborhoods U_b and V_b. It follows that $U_b^* \cap p_c(F) = \emptyset$ for each $c \geq b$. This means that $y \notin \mathrm{Ls}\,\mathcal{N}$, i.e., $\mathrm{Ls}\,\mathcal{N} \subseteq F$. Finally, we have $F = \mathrm{Ls}\,\mathcal{N} = \mathrm{Li}\,\mathcal{N} = \mathrm{Lim}\,\mathcal{N}$ and the proof is complete. ∎

We say that a space X is *hereditarily unicoherent* if for each pair C, D of closed connected subsets of X the intersection $C \cap D$ is connected. For continua this definition is equivalent (see [**1**, p. 187]) to the following:

D1. A Hausdorff continuum is hereditarily unicoherent if every two points of it can be joined by exactly one irreducible continuum between them.

Theorem 3.6. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of hereditarily unicoherent compact Hausdorff spaces. Then* $X = \lim \underline{\mathbf{X}}$ *is hereditarily unicoherent.*

Proof: Let C, D be a pair of subcontinua of X. We must prove that $C \cap D$ is connected. By virtue of the above lemma we have $C = \mathrm{Lim}\,\mathcal{N}(C)$ and $D = \mathrm{Lim}\,\mathcal{N}(D)$. Each $F_a = p_a(C) \cap p_a(D)$ is connected since X_a is hereditarily unicoherent. By virtue of 2 of Lemma 2.1 each point x of $C \cap D$ is a limit of the net $\{p_a(x) : a \in A\}$. Thus, $\emptyset \neq \mathrm{Li}\,F_a \supseteq C \cap D$. On the other hand for each $y \in X \backslash C \cap D$ we have $y \notin C$ or $y \notin D$. Let $y \notin C$. By virtue of the definition of a base in X (Definition 1.9) there is a $b \in A$ such that $p_b(y)$ and $p_b(C)$ have disjoint neighborhoods U_b and V_b. From 1 of the above lemma it follows that there is a $c \geq b$ such that $p_{bd}p_d(C) \subseteq V_b$, $d \geq c$. This means that $U_b^* \cap p_d(C) = \emptyset$, $d \geq c$. We infer that $y \notin \mathrm{Ls}\,F_a$. Thus, $\mathrm{Ls}\,F_a \subseteq C \cap D$. From this and the relation $\mathrm{Li}\,F_a \supseteq C \cap D$ it follows $C \cap D = \mathrm{Li}\,F_a$. Similarly, $C \cap D = \mathrm{Ls}\,F_a$. By virtue of Lemma 3.4 $\mathrm{Ls}\,F_a$ is connected. Thus, $C \cap D$ is connected and the proof is complete. ∎

By the same method of proof as in the proof of Theorem 3.6 we have

Theorem 3.7. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of Hausdorff continua. If all the spaces* X_a *are unicoherent and if all* p_{ab} *are onto, then* $X = \lim \underline{\mathbf{X}}$ *is unicoherent.*

Remark 3.8. Without ontoness of the bonding mappings the approximate limit of unicoherent continua need not be unicoherent since this is not true for usual inverse limits [**14**, p. 228, Remark]. If $\underline{\mathbf{X}} = \{X_a, p_{ab}, A\}$

is a usual inverse system of metric locally connected unicoherent continua, then the usual limit is unicoherent (without assuming the bonding maps are onto) [**14**, p. 228, Remark]. This means that the following question is natural:

Is it true that the approximate limit of an approximate system of metric locally connected unicoherent continua and into bonding mappings is unicoherent?

Now we give an affirmative answer to the above question. Firstly, we give some necessary definitions.

Let S be the circle $|z| = 1$ in the complex plane. The space of the real numbers we denote by R.

A continuous mapping $f : X \to S$ is said to be *equivalent to 1* on a set $Y \subseteq X$, written $f \sim 1$ on Y, provided there exists a continuous mapping $\phi : Y \to R$ such that [**17**, p. 220] $f(x) = e^{i\phi(x)}$, $x \in Y$.

Two mappings f_1, $f_2 : X \to S$ will be said to be *exponentially equivalent* or simply equivalent on a set $Y \subseteq X$ provided their ratio f_1/f_2 is ~ 1 on Y [**17**, p. 225].

A space X will be said to have *property (b)* provided every mapping $f : X \to S$ is ~ 1 [**17**, p. 226].

A mapping $f : X \to S$ homotopic to the mapping $f_0 : X \to S$, $f_0(x) = 1$ for all $x \in X$, is said to be *homotopic to 1*, $f \simeq 1$.

In the sequel we need the following facts: (a) In order that a mapping $f : X \to S$ be ~ 1 it is necessary and sufficient that f be homotopic to 1 [**17**, p. 226]. (b) In order that two mappings f_1, $f_2 : X \to S$ be equivalent on X it is necessary and sufficient that they be homotopic [**17**, p. 226]. (c) Every connected space X having property (b) is unicoherent [**17**, p. 227]. (d) In order that a locally connected continuum have property (b) it is necessary and sufficient that it be unicoherent [**17**, p. 228]. (e) If X is any space and f, $g : X \to S^n$ are two maps such that for each $x \in X$, $f(x)$ and $g(x)$ are not antipodal, then $f \simeq g$. In particular, a nonsurjective $f : X \to S^n$ is always nullhomotopic [**2**, p. 316].

Theorem 3.9. *Let* $\underline{\mathbf{X}} = \{X_a, \epsilon_a, p_{ab}, A\}$ *be an approximate inverse sequence of locally connected unicoherent metric continua. Then* $X = \lim \underline{\mathbf{X}}$ *is unicoherent.*

Proof: Let us prove that X has property (b). Let $f : X \to S$ be any mapping. By virtue of Lemma 2.5 there is a $a \in A$ such that for each $b \geq a$ there is a mapping $g : X_b \to S$ such that gp_b and f are homotopic. Since X_b has property (b), then $g \simeq 1$ and hence $f \simeq 1$. This shows that

X has property (b). By Theorem 3.1, X is a continuum. Hence by (c), X is unicoherent. ■

A Hausdorff continuum is a *tree* if each pair of points is separated by third point [**16**]. A Hausdorff continuum X is a tree iff X is locally connected and hereditarily unicoherent [**16**].

A continuum X is *smooth at a point* p [**15**] provided that for each subcontinuum K of X such that $p \in K$ and for each open set V which included K, there is an open connected set U such that $K \subseteq U \subseteq V$. Clearly, if X is smooth at a point $p \in X$, then X is locally connected at p. Moreover, X is locally connected if and only if X is smooth at each of its points [**9**, p. 84]. A continuum I is *irreducible between its points a and b* if no proper subcontinuum of I contains them. In the sequel we use the following lemma which is part of Proposition 1 [**15**].

Lemma 3.10. *Let p be a point of a Hausdorff continuum X. The following conditions are equivalent:*

(i) *X is smooth at p,*
(ii) *for each convergent net $x_a \in X$ with $\lim x_a = x$ and for each continuum $I(p,x)$ irreducible between p and x there are continua $I(p,x_a)$ each one irreducible between p and x_a such that $\operatorname{Lim} I(p,x_a) = I(p,x)$.*

Lemma 3.11. *Let $f : X \to Y$ be a monotone surjection. If X and Y are hereditarily unicoherent Hausdorff continua and if $I(a,b)$ is irreducible between a, b, then $f(I(a,b))$ is irreducible between $f(a)$ and $f(b)$, i.e., $I(f(a),f(b)) = f(I(a,b))$.*

Proof: Now, $f^{-1}(I(f(a),f(b)))$ is a continuum since f is monotone. An application of D1 shows that $f^{-1}(I(f(a),f(b))) \supseteq I(a,b)$. Thus, $f(I(a,b)) \subseteq I(f(a),f(b))$. On the other hand, $f(I(a,b)) \supseteq I(F(a),f(b))$ since $I(f(a),f(b))$ is irreducible between $f(a)$ and $f(b)$. Thus, $f(I(a,b)) = I(f(a),f(b))$ and the proof is complete. ■

The following lemma is a generalization of Lemma 2.2 of [**9**].

Lemma 3.12. *Let $\{C_n : n \in D\}$ be a net of subcontinua of a Hausdorff continuum X. If x, $y \in \operatorname{Li} C_n$ and the continuum $\operatorname{Ls} C_n$ is irreducible between x and y, then the net $\{C_n : n \in D\}$ is convergent.*

Proof: Suppose, on the contrary, that the net $\{C_n : n \in D\}$ is not convergent, i.e., there is a $c \in \operatorname{Ls} C_n \setminus \operatorname{Li} C_n$. From $c \notin \operatorname{Li} C_n$ it follows that there is a neighborhood U of c such that for each $n \in D$ there is $m \in D$, $m \geq n$, such that $C_m \cap U = \emptyset$. Let M be the set of all $m \in D$

such that $C_m \cap U = \emptyset$. The collection $\{C_m : m \in M\}$ is a net in $X \backslash U$ and a subnet of $\{C_n : n \in D\}$. This means that $L = \mathrm{Ls}\{C_m : m \in M\}$ is a nonempty subset of $X \backslash U$ and $c \in U \subseteq X \backslash L$. Since $\mathrm{Li}\{C_m : m \in M\} \supseteq \mathrm{Li}\{C_n : n \in D\}$ by Lemma 3.3 L is connected, i.e., is a subcontinuum of X. Moreover, x, $y \in L$ since $L = \mathrm{Ls}\{C_m : m \in M\} \supseteq \mathrm{Li}\{C_m : m \in M\} \supseteq \mathrm{Li}\{C_n : n \in D\}$. On the other hand, $L \subseteq \mathrm{Ls}\, C_n$. From x, $y \in L$ and from the irreducibility of $\mathrm{Ls}\, C_n$, it follows that $L = \mathrm{Ls}\, C_n$. This is impossible since $c \in \mathrm{Ls}\, C_n \backslash \mathrm{Li}\, C_n = L \backslash \mathrm{Li}\, C_n$, and $c \notin L$. The proof is complete. ■

Now, we prove the main theorem of this section.

Theorem 3.13. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of trees and monotone onto bonding mappings. Then* $X = \lim \underline{\mathbf{X}}$ *is a tree.*

Proof: The proof is broken into severals steps.

Step 1. The limit X is a continuum and the projections are onto.

See Theorem 3.1 and [**12**, (4.5) Corollary].

Step 2. By virtue of Theorem 3.6 X is hereditarily unicoherent.

Step 3. The limit X is locally connected.

We shall use Lemma 3.10 to prove that X is smooth at each point $y \in X$. Let $\{x^\mu : \mu \in M\}$ be a net which converges to a point $x \in X$. The irreducible subcontinua $I(y, x)$ and $I(y, x^\mu)$, $\mu \in M$, needed in Lemma 3.10, are unique since X is a Hausdorff hereditarily unicoherent continuum. For each $a \in A$ we have also uniquely determined subcontinua $I(y_a, x_a)$, $I(y_a, x^\mu_a)$, $\mu \in M$, irreducible between $y_a = p_a(y)$ and $x^\mu_a = p_a(x^\mu)$ since X_a is hereditarily unicoherent. It is obvious that each net $\{x^\mu_a : \mu \in M\}$ converges to x_a. Moreover, from the smoothness of X_a (X_a is locally connected) and Lemma 3.10 it follows that the net $\{I(y_a, x^\mu_a) : \mu \in M\}$ of subcontinua converges to $I(y_a, x_a)$. By virtue of Lemma 3.10 we must prove $I(y, x) = \mathrm{Lim}\{I(y, x^\mu) : \mu \in M\}$ (see Step 3.4). We start with auxiliary Steps 3.1-3.3.

Step 3.1. $\mathrm{Ls}\{I(y_a, x^\mu_a) : a \in A\} = K^\mu = I(y, x^\mu)$, $\mu \in M$.

By virtue of Lemma 3.4 each net $\{I(y_a, x^\mu_a) : a \in A\}$ has a nonempty and connected $\mathrm{Ls}\{I(y_a, x^\mu_a) : a \in A\} = K^\mu$. Clearly, $K^\mu \supseteq I(y, x^\mu)$ since $I(y, x^\mu)$ is the unique subcontinuum irreducible between y, x^μ and $\{y, x^\mu\} \subseteq K^\mu$. By virtue of Lemma 3.5 we have $I(y, x^\mu) =$

$\mathrm{Lim}\{p_a(I(y,x^\mu)) : a \in A\}$. Since each $p_a(I(y,x^\mu))$ contains $I(y_a,x_a^\mu)$, we infer that $K^\mu \subseteq I(y,x^\mu)$. Finally, we have $K^\mu = I(y,x^\mu)$.

Step 3.2. For each $a \in A$ and each $\mu \in M$ we have $p_a(K^\mu) = I(y_a,x_a^\mu)$.

Clearly, $p_a(K^\mu) \supseteq I(y_a,x_a^\mu)$. Suppose that there is an $a \in A$ and a point $z_a \in p_a(K^\mu)\backslash I(y_a,x_a^\mu)$. This means that there are disjoint open sets U_a and V_a such that $z_a \in V_a$ and $I(y_a,x_a^\mu) \subseteq U_a$. From the local connectedness of X_a it follows that there is an open and connected set W_a such that $I(y_a,x_a^\mu) \subseteq ClW_a \subseteq U_a$. From the definition of thread it follows that there is a $b \in A$ such that $p_{ac}(y_c)$ and $p_{ac}(x_c^\mu)$ are in W_a for each $c \geq b$. This means that $p_{ac}(I(y_c,x_c^\mu)) \subseteq ClW_a$ since $p_{ac}(I(y_c,x_c^\mu))$ is irreducible between $p_{ac}(x_c)$ and $p_{ac}(x_c^\mu)$ (see Lemma 3.11). It follows that U_a^* is a neighborhood of a point $z \in K$, $p_a(z) = z_a$, such that $U_a^* \cap I(y_c,x_c^\mu) = \emptyset$. This means that $z \notin \mathrm{Ls}\{I(y_a,x_a^\mu) : a \in A\} = K^\mu$. This is impossible since $z \in K^\mu$. By Theorem 1.10 it follows that $K^\mu = \cap\{p_a^{-1}(I(y_a,x_a^\mu)) : a \in A\}$. Similarly, we have $K = \cap\{p_a^{-1}(I(y_a,x_a)) : a \in A\}$, where $K = \mathrm{Ls}\{I(y_a,x_a) : a \in A\}$.

Step 3.3. $\mathrm{Ls}\{K^\mu : \mu \in M\} = \mathrm{Ls}\{I(y,x^\mu) : \mu \in M\} = I(y,x)$.

It is obvious that $\mathrm{Ls}\{I(y,x^\mu) : \mu \in M\} \supseteq I(y,x)$ since $\mathrm{Ls}\{I(y,x^\mu) : \mu \in M\}$ contains x and y and $I(y,x)$ is irreducible between x and y. Now we prove that $\mathrm{Ls}\{I(y,x^\mu) : \mu \in M\} \subseteq I(y,x)$. Let z be any point in $X - I(y,x)$. By virtue of the definition of a base in X, there is an $a \in A$ such that $p_a(z) = z_a \notin p_a(I(y,x)) =$ (by Steps 3.1 and 3.2) $I(y_a,x_a)$. This means that there is a neighborhood U_a of z_a and a neighborhood V_a of $p_a(I(y,x))$ such that $U_a \cap V_a = \emptyset$. By Step 3.2 $p_a(I(y,x)) = I(y_a,x_a)$. Since $I(y_a,x_a) = \mathrm{Lim}\{I(y_a,x_a^\mu) : \mu \in M\}$ we infer that there is a $\mu_0 \in M$ such that, for each $\mu \geq \mu_0$, U_a and $I(y_a,x_a^\mu)$ are disjoint. From 3.2 it follows that $p_a^{-1}(U_a)$ and $I(y,x^\mu)$ are disjoint. Since $p_a^{-1}(U_a)$ is a neighborhood of z, we infer that $z \notin \mathrm{Ls}\{I(y,x^\mu) : \mu \in M\}$. Thus, $\mathrm{Ls}\{I(y,x^\mu) : \mu \in M\} = I(y,x)$ and 3.3 is proved.

Step 3.4. $I(y,x) = \mathrm{Lim}\{I(y,x^\mu) : \mu \in M\}$.

Apply Step 3.3 and Lemma 3.12.

By virtue of Lemma 3.10 and Step 3.4 it follows that X is smooth at y. We infer that X is smooth in any of its point y. This means that X is locally connected. The proof of Theorem 3.13 is complete. ■

A Hausdorff continuum X with precisely two nonseparating points is called a *generalized arc*. A continuum X is said to be an *arc* if X is a metrizable generalized arc. A tree X is a generalized arc if and only if X is atriodic.

Theorem 3.14. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of generalized arcs. Then* $X = \lim \underline{\mathbf{X}}$ *is atriodic.*

Proof: Suppose that T is a subcontinuum of X which is a triod. This means that T is the sum of three generalized arcs C_x, C_y, and C_z, such that the common part of each two of them is the common part of all three of them and is a point. Let $x \in C_x$- $(C_y \cup C_z)$, $y \in C_y$- $(C_x \cup C_z)$, $z \in C_z$- $(C_x \cup C_y)$ and $t = C_x \cap C_y \cap C_z$. By virtue of the definition of a basis in X, there exist $a \in A$ and open sets V_x, V_y, V_z of X_a which are pairwise mutually exclusive and which contain x_a, y_a, z_a, respectively, so that

$$p_a^{-1}(V_x) \bigcap C_y = \emptyset = p_a^{-1}(V_x) \bigcap C_z,$$
$$p_a^{-1}(V_y) \bigcap C_x = \emptyset = p_a^{-1}(V_y) \bigcap C_z,$$
$$p_a^{-1}(V_z) \bigcap C_y = \emptyset = p_a^{-1}(V_z) \bigcap C_x.$$

Now, one of x_a, y_a or z_a lies between t_a and one of x_a, y_a or z_a. Suppose that $t_a \prec x_a \prec y_a$. Then $p_a(C_y)$ intersects t_a and y_a and hence x_a, but $p_a(C_y)$ does not intersect V_x. This is a contradiction. So X contains no triod. ■

Theorem 3.15. *Let* $\underline{\mathbf{X}} = \{X_a, \mathcal{U}_a, p_{ab}, A\}$ *be an approximate inverse system of generalized arcs with limit* X. *If the bonding mappings are monotone and onto, then* X *is a generalized arc.*

Proof: By virtue of Theorem 3.13 X is a tree. From 3.14 it follows that X is atriodic. Thus X is a generalized arc. ■

Corollary 3.16. *Let* $\underline{\mathbf{X}} = \{X_n, \epsilon_n, p_{mn}, N\}$ *be an approximate inverse sequence of arcs and monotone onto mappings. Then* $X = \lim \underline{\mathbf{X}}$ *is an arc.*

Proof: Now, from 3.15, it follows that X is a generalized arc. Moreover, X is a metrizable generalized arc. Thus, X is an arc. ■

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