WEIGHTED $L_p$ SPACES 
AND POINTWISE ERGODIC THEOREMS

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Abstract

In this paper we give an operator theoretic version of a recent result of F. J. Martín-Reyes and A. de la Torre concerning the problem of finding necessary and sufficient conditions for a nonsingular point transformation to satisfy the Pointwise Ergodic Theorem in $L_p$. We consider a positive conservative contraction $T$ on $L_1$ of a $\sigma$-finite measure space $(X, \mathcal{F}, \mu)$, a fixed function $e$ in $L_1$ with $e > 0$ on $X$, and two positive measurable functions $V$ and $W$ on $X$. We then characterize the pairs $(V, W)$ such that for any $f$ in $L_p(W \, d\mu)$ the averages

$$R^n_0(f, e) = \left( \sum_{k=0}^{n} T^k f \right) / \left( \sum_{k=0}^{n} T^k e \right)$$

converge almost everywhere to a function in $L_p(W \, d\mu)$. The characterizations are given for all $p$, $1 \leq p < \infty$.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space and $T$ a positive linear contraction of $L_1(\mu)$. We assume $T$ to be a conservative operator. (For the usual notation we refer the reader to Krengel’s book [2].) Thus the class

$$\mathcal{I} = \mathcal{I}(T) = \{ A \in \mathcal{F} : T^* 1_A = 1_A \}$$

(1)

of all invariant sets relative to $T$ forms a $\sigma$-field, where $1_A$ denotes the indicator function of $A$ and $T^*$ denotes the adjoint operator of $T$, acting on $L_\infty(\mu)$. Since $T$ is positive, we may extend by a canonical manner the domain of $T$ to the class $M^+(\mu)$ of all nonnegative extended real valued measurable functions on $X$. Similarly, this is done for $T^*$. Now let us fix an $e \in L_1(\mu)$ with $e > 0$ on $X$. Let $0 < V$, $W \leq \infty$ be two measurable
functions on $X$. Previously we observed in [6] that if $1 < p < \infty$ then for any $f \in L^+_p(V \, d\mu)$ the averages

\begin{equation}
R^n_0(f, e) = \left( \sum_{k=0}^{n} T^k f \right) / \left( \sum_{k=0}^{n} T^k e \right)
\end{equation}

converge to a finite limit a.e. on $X$ if and only if

\begin{equation}
E\{e^{-1}V^{1-p'}|(X, \mathcal{I}, e \, d\mu)\} < \infty \text{ a.e. on } X,
\end{equation}

where $1/p + 1/p' = 1$. In [6] we also observed implicitly (see especially p. 76–77 in [6]) that for any $f \in L^+_1(V \, d\mu)$ the averages $R^n_0(f, e)$ converge to a finite limit a.e. on $X$ if and only if there exists a function $U$, measurable with respect to $\mathcal{I}$, such that

\begin{equation}
V^{-1} \leq U < \infty \text{ a.e. on } X.
\end{equation}

In this paper we intend to study the problem of characterizing the case where the limit function $R^n_0(f, e)$ belongs to $L^+_p(W \, d\mu)$ for every $f \in L^+_p(V \, d\mu)$. This study was inspired by the work [3] of Martín-Reyes and de la Torre. See also Assani and Wós [1]. As a result, this paper may be considered to be an operator theoretic version of Martín-Reyes and de la Torre’s paper [3]. Using the result obtained we next consider multiparameter pointwise ergodic theorems for commuting positive linear contractions of $L_1(\mu)$ having a common strictly positive fixed point in $L_1(\mu)$.

2. The main result

**Theorem 1.** Let $T$ be a conservative positive linear contraction of $L_1(\mu)$. Let $V, W$ be two positive real valued measurable functions on $X$. Fix an $e \in L_1(\mu)$ with $e > 0$ on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L^+_p(V \, d\mu)$ the averages $R^n_0(f, e)$ converge a.e. to a function belonging to $L^+_p(W \, d\mu)$.

(b) $E\{e^{-1}W|(X, \mathcal{I}, e \, d\mu)\}^{1/p} \cdot E\{e^{-1}V^{1-p'}|(X, \mathcal{I}, e \, d\mu)\}^{1/p'} \leq C \text{ a.e. on } X$, where $C$ is a positive constant.

(c) For any $f \in L^+_p(W^{1-p'} \, d\mu)$ the averages $R^n_0(f, e)$ converge a.e. to a function belonging to $L^+_p(V^{1-p'} \, d\mu)$.

If $p = 1$, then (a) is equivalent to

(d) $E\{e^{-1}W|(X, \mathcal{I}, e \, d\mu)\} \leq CV \text{ a.e. on } X.$
Proof: Let $1 < p < \infty$.

(a) $\Rightarrow$ (b). By (a) the limit function

$$R_0^\infty(f,e) = \lim_{n} R_0^n(f,e)$$

is finite a.e. on $X$. Thus by (3) we have

$$E\{e^{-1}V^{1-p'}|(X,I,e d\mu)\} < \infty \ \text{a.e. on} \ X.$$  

Choose $X_n \in I$, $n = 1, 2, \ldots$, so that

$$X_n \uparrow X \ \text{and} \ \int_{X_n} V^{1-p'} \ d\mu < \infty.$$  

Since $V^{(1-p')p} \cdot V = V^{1-p'}$, it follows that

$$V^{1-p'} \in L^+_p(X_n, V d\mu).$$

On the other hand, since $R_0^\infty(\cdot,e)$ is a positive linear operator from $L_p(V d\mu)$ into $L_p(W d\mu)$ by (a), it is bounded, i.e., there exists a constant $K > 0$ such that

$$\int |R_0^\infty(f,e)|^pW d\mu \leq K^p \int |f|^pV d\mu \quad (f \in L_p(V d\mu)).$$

Therefore, for any $A \in I$ with $A \subset X_n$, (7) yields

$$\int_{A} R_0^\infty(V^{1-p'}, e)^pW d\mu \leq K^p \int_{A} V^{1-p'} d\mu < \infty.$$  

Since $R_0^\infty(V^{1-p'}, e) = E\{e^{-1}V^{1-p'}|(X,I,e d\mu)\}$ a.e. on $X$ (cf. p. 73 in [6]), these inequalities imply

$$R_0^\infty(V^{1-p'}, e)^pE\{e^{-1}W|(X,I,e d\mu)\} \leq K^p E\{e^{-1}V^{1-p'}|(X,I,e d\mu)\}$$

$$< \infty \ \text{a.e. on} \ X;$$

and (b) follows.

(b) $\Rightarrow$ (a). Since $e^{-1}f = (e^{-1/p}fV^{1/p})(e^{-1/p'}V^{-1/p})$, the Hölder inequality for the conditional expectation operator and (b) imply that if $f \in L^+_p(V d\mu)$ then

$$R_0^\infty(f,e) = E\{e^{-1}f|(X,I,e d\mu)\} \leq E\{e^{-1/p}fV|(X,I,e d\mu)\}^{1/p} \cdot E\{e^{-1}V^{1-p'}|(X,I,e d\mu)\}^{1/p'} \leq CE\{e^{-1/p}fV|(X,I,e d\mu)\}^{1/p} \cdot E\{e^{-1}W|(X,I,e d\mu)\}^{-1/p}$$
a.e. on $X$; and thus
\[
\int R^\infty_0 (f,e)^p W \, d\mu \leq C_p \int \frac{E\{e^{-1} f^p V|(X,I,e \, d\mu)\}}{E\{e^{-1} W|(X,I,e \, d\mu)\}} W \, d\mu \\
= C_p \int E\{e^{-1} f^p V|(X,I,e \, d\mu)\} e \, d\mu \\
= C_p \int f^p V \, d\mu < \infty,
\]
which proves (a).

(b) $\iff$ (c). Direct from (a) $\iff$ (b).

Let $p = 1$.

(a) $\iff$ (d). For any $f \in L^+_1(V \, d\mu)$ we obtain
\[
\int R^\infty_0 (f,e) W \, d\mu = \int E\{e^{-1} f|(X,I,e \, d\mu)\} W \, d\mu \\
= \int E\{e^{-1} f|(X,I,e \, d\mu)\} E\{e^{-1} W|(X,I,e \, d\mu)\} e \, d\mu \\
= \int f E\{e^{-1} W|(X,I,e \, d\mu)\} \, d\mu \\
= \int f V(\{e^{-1} W|(X,I,e \, d\mu)\} \cdot V^{-1}) \, d\mu.
\]
Hence, by (8) with $p = 1$, (a) is equivalent to
\[(a') \int f V(\{e^{-1} W|(X,I,e \, d\mu)\} \cdot V^{-1}) \, d\mu \leq K \int f V \, d\mu \text{ for every } f \in L^+_1(V \, d\mu);
\]
and (a') is clearly equivalent to (d). The proof is complete.

**Corollary 1.** In addition to the hypotheses of Theorem 1, if we assume that $T$ is ergodic, i.e., that $I$ is trivial, then the following are equivalent, for every $1 \leq p < \infty$:

(a) For any $f \in L^+_p(V \, d\mu)$ the averages $R^\infty_0 (f,e)$ converge a.e. to a function belonging to $L^+_p(W \, d\mu)$, 
(b) $W \in L_1(\mu)$ and $V^{-1} \in L^p(V \, d\mu)$, where $p' = \infty$ when $p = 1$.

3. Applications

Let $d \geq 1$ be an integer and $T_1, \ldots, T_d$ be commuting positive linear contractions of $L_1(\mu)$. In this section we assume that there exists an $e \in L_1(\mu)$ with $e > 0$ on $X$ such that
\[(10) \quad T_i e = e \quad (1 \leq i \leq d).
\]
Thus each $T_i$ is a conservative operator and satisfies the mean ergodic theorem in $L_1(\mu)$. And by an induction argument we see that for any $f \in L_1(\mu)$ the averages

$$A_n(T_1, \ldots, T_d)f = A_n(T_1) \ldots A_n(T_d)f$$

converge in $L_1$-norm, where $A_n(T_i) = \frac{1}{n} \sum_{k=0}^{n-1} T_i^k$. By Theorem 1 of [5], for any $f \in L_1(\mu)$ the averages $A_n(T_1, \ldots, T_d)f$ converge a.e. on $X$. Let us denote the limit function by $A(T_1, \ldots, T_d)f$; thus

$$A(T_1, \ldots, T_d)f = \lim_n A_n(T_1, \ldots, T_d)f$$

a.e. on $X$.

If we let

$$T = \frac{1}{d} \sum_{i=1}^d T_i$$

then $T$ also satisfies the mean ergodic theorem in $L_1(\mu)$; and we get the direct decomposition

$$L_1(\mu) = \{ f \in L_1(\mu) : Tf = f \} \oplus \{ g - Tg : g \in L_1(\mu) \}.$$

Since $Tf = f$ if and only if $T_i f = f$ for each $1 \leq i \leq d$ by the Brunel-Falkowitz lemma (cf. p. 82 in [2]) and

$$\lim_n \|A_n(T_1, \ldots, T_d)(g - Tg)\|_1 = 0$$

by the equation $g - Tg = \frac{1}{d} \sum_{i=1}^d (g - T_i g)$, it follows that for any $f \in L_1(\mu)$ the limit function $A(T_1, \ldots, T_d)f$ coincides a.e. with the limit function

$$A(T)f = \lim_n A_n(T)f.$$

Further, since $\mathcal{I}(T) = \bigcap_{i=1}^d \mathcal{I}(T_i)$ (in the sequel $\mathcal{I}$ will denote this $\sigma$-field), it follows that for any $f \in L_1^+(\mu)$

$$A(T_1, \ldots, T_d)f = \lim_n A_n(T)f$$

$$\frac{1}{n} \sum_{k=0}^{n-1} T^k f \right/ \left( \frac{1}{n} \sum_{k=0}^{n-1} T^k e \right)$$

$$= eE\{ e^{-1} f | (X, \mathcal{I}, e d\mu) \} \text{ a.e. on } X.$$

Hence, by an approximation argument, for any $f \in M^+(\mu)$ the limit $A(T_1, \ldots, T_d)f = \lim_n A_n(T_1, \ldots, T_d)f$ exists a.e. on $X$ and satisfies (15).

We are now in position to state the first application of Theorem 1.
Theorem 2. Let $T_1, \ldots , T_d$ be commuting positive linear contractions of $L^1(\mu)$ such that $T_i e = e$ ($1 \leq i \leq d$) for some $e \in L^1(\mu)$ with $e > 0$ on $X$. Let $0 < V$, $W < \infty$ be two measurable functions on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L_p^+(V \,d\mu)$ the limit function $A(T_1, \ldots , T_d)f$ belongs to $L_p^+(W \,d\mu)$.
(b) $E\{e^{p-1} V | (X, I, e \,d\mu)e\}^{1/p} E\{e^{-1} W^{1-p'} | (X, I, e \,d\mu)e\}^{1/p'} \leq C$ a.e. on $X$.
(c) For any $f \in L_p^+(e^{-p'} W^{1-p'} \,d\mu)$ the limit function $A(T_1, \ldots , T_d)f$ belongs to $L_p^+(e^{-p'} V^{1-p'} \,d\mu)$.

If $p = 1$, then (a) is equivalent to
(d) $E\{W | (X, I, e \,d\mu)e\} \leq CV$ a.e. on $X$.

Consequently, in case $I$ is trivial, (a) is equivalent, for every $1 \leq p < \infty$, to
(e) $e^p W \in L_1(\mu)$ and $V^{-1} \in L_p^+(V \,d\mu)$, where $p' = \infty$ when $p = 1$.

Proof: For any $f \in L_p^+(V \,d\mu)$ we have, by (15), $A(T_1, \ldots , T_d)f = e R_0^\infty (f, e)$. Thus (a) is equivalent to

(a') For any $f \in L_p^+(V \,d\mu)$ the limit function $R_0^\infty (f, e)$ (relative to $T$) belongs to $L_p^+(e^p W \,d\mu)$.

Therefore, by Theorem 1, we see (a) $\Leftrightarrow$ (b) when $1 < p < \infty$, and (a) $\Leftrightarrow$ (d) when $p = 1$. When $1 < p < \infty$, (b) $\Leftrightarrow$ (c) follows from the equivalence (a) $\Leftrightarrow$ (b). This completes the proof. $\blacksquare$

Corollary 2. Let $T_1, \ldots , T_d$ and $e$ be the same as in Theorem 2. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L_p^+(\mu)$ the limit function $A(T_1, \ldots , T_d)f$ belongs to $L_p^+(\mu)$.
(b) $E\{e^{p-1} | (X, I, e \,d\mu)e\}^{1/p} E\{e^{-1} | (X, I, e \,d\mu)e\}^{1/p'} \leq C$ a.e. on $X$.

Consequently, in case $I$ is trivial, (a) is equivalent, for every $1 < p < \infty$, to
(c) $\mu(X) < \infty$ and $e \in L_p(\mu)$.

Remark. We note that (a) of Corollary 2 always holds when $p = 1$.

We next consider the adjoint operators $T_1^*, \ldots , T_d^*$. Since $f(T_i^*f) \,d\mu = \int f(T_i e) \,d\mu = \int f \,e \,d\mu$ for $f \in L_2^+(\mu)$, $T_1^*, \ldots , T_d^*$ can be regarded as commuting positive linear contractions of $L_1(e \,d\mu)$. Since

$$(16) \quad T_i^* 1 = 1 \in L_1(e \,d\mu) \quad (1 \leq i \leq d),$$
if we replace the measure $\mu$ and the function $e$ by $e\,d\mu$ and 1, respectively, then the above-given argument shows that for any $f \in M^+(\mu) = M^+(e\,d\mu)$ the limit

$$A(T_1^*, \ldots, T_d^*)f = \lim_{n} A_n(T_1^*, \ldots, T_d^*)f$$

exists a.e. on $X$; further, since $I = \bigcap_{i=1}^{d} I(T_i) = \bigcap_{i=1}^{d} I(T_i^*)$, it follows that

$$A(T_1^*, \ldots, T_d^*)f = \lim_{n} A_n(T^*)f = E\{f|(X,I,e\,d\mu)\} \text{ a.e. on } X,$$

where $T^* = \frac{1}{d} \sum_{i=1}^{d} T_i^*$.

**Theorem 3.** Let $T_1, \ldots, T_d$ and $e$ be the same as in Theorem 2. Let $0 < V, W < \infty$ be two measurable functions on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L^p_{\mu}(V\,d\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^p_{\mu}(W\,d\mu)$.

(b) $E\{e^{-1}W|(X,I,e\,d\mu)\}^{1/p}E\{e^{-V}1^{1-p'}|(X,I,e\,d\mu)\}^{1/p'} \leq C$ a.e. on $X$.

(c) For any $f \in L^p_{\mu}(e^pW^{-1-p'}\,d\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^p_{\mu}(e^pV^{1-p'}\,d\mu)$.

(d) For any $f \in L^p_{\mu}(W^{-1-p'}\,d\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^p_{\mu}(V^{1-p'}\,d\mu)$.

(e) For any $f \in L^p_{\mu}(e^{-p}V\,d\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^p_{\mu}(e^{-p}W\,d\mu)$.

If $p = 1$, then (a) is equivalent to

(f) $E\{e^{-1}W|(X,I,e\,d\mu)\} \leq C(e^{-1}V)$ a.e. on $X$.

Consequently, in case $I$ is trivial, (a) is equivalent, for every $1 \leq p < \infty$, to

(g) $W \in L_1(\mu)$ and $eV^{-1} \in L_p(V\,d\mu)$, where $p' = \infty$ when $p = 1$.

**Proof:** Since $L^p_{\mu}(V\,d\mu) = L^p_{\mu}(e^{-1}V\,d\mu)$ and $L^p_{\mu}(W\,d\mu) = L^p_{\mu}(e^{-1}W\,d\mu)$, if we apply Theorem 2 to commuting positive linear contractions $T_1^*, \ldots, T_d^*$ of $L_1(e\,d\mu)$, then (16) yields (a) $\iff$ (f) when $p = 1$, and (a) $\iff$ (b) $\iff$ (c) when $1 < p < \infty$. If we write (b) as

$$E\{e^{-1}(e^{-p}W)|(X,I,e\,d\mu)\}^{1/p}E\{e^{-V}1^{1-p'}|(X,I,e\,d\mu)\}^{1/p'} \leq C$$

a.e. on $X$,
and apply Theorem 2 to commuting positive linear contractions $T_1, \ldots, T_d$ of $L_1(\mu)$, then we obtain (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (d) when $1 < p < \infty$. The proof is complete. 

**Corollary 3 (cf. [3] and [4]).** Let $T_1, \ldots, T_d$ and $e$ be the same as in Theorem 2. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L^+_p(\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^+_p(\mu)$.

(b) $E\{e^{-1}(X,\mathcal{I},e\,d\mu)\}^{1/p}E\{e^{p'-1}(X,\mathcal{I},e\,d\mu)\}^{1/p'} \leq C \text{ a.e. on } X$.

(c) For any $f \in L^+_p(\mu)$ the limit function $A(T_1, \ldots, T_2)f$ belongs to $L^+_p(\mu)$.

If $p = 1$, then (a) is equivalent to

d) $E\{e^{-1}(X,\mathcal{I},e\,d\mu)\} \leq C e^{-1} \text{ a.e. on } X$.

Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 \leq p < \infty$, to

e) $\mu(X) < \infty$ and $e \in L^{p'}(\mu)$, where $p' = \infty$ when $p = 1$.

**Corollary 4.** Suppose $(X,\mathcal{F},\mu)$ is a finite measure space. Let $T_1, \ldots, T_d$ be commuting positive linear contractions of $L_1(\mu)$, and assume that $\mu$ is invariant under $T_1, \ldots, T_d$, i.e., that $T_i 1 = 1 \in L_1(\mu)$ $(1 \leq i \leq d)$. Let $0 < V, W < \infty$ be two measurable functions on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L^+_p(V\,d\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^+_p(W\,d\mu)$.

(b) $E\{W|(X,\mathcal{I},\mu)\}^{1/p}E\{V^{1-p'}|(X,\mathcal{I},\mu)\}^{1/p'} \leq C \text{ a.e. on } X$.

(c) For any $f \in L^+_p(W^{1-p'}\,d\mu)$ the limit function $A(T_1^*, \ldots, T_d^*)f$ belongs to $L^+_p(V^{1-p'}\,d\mu)$.

If $p = 1$, then (a) is equivalent to

d) $E\{W|(X,\mathcal{I},\mu)\} \leq CV \text{ a.e. on } X$.

Consequently, in case $\mathcal{I}$ is trivial, (a) is equivalent, for every $1 \leq p < \infty$, to

e) $W \in L_1(\mu)$ and $V^{-1} \in L^{p'}(V\,d\mu)$, where $p' = \infty$ when $p = 1$.

**Remark.** Under the hypotheses of Corollary 4, it follows (see (15) and (18)) that for any $f \in M^+(\mu)$

$$A(T_1, \ldots, T_d)f = A(T_1^*, \ldots, T_d^*)f = E\{f|(X,\mathcal{I},\mu)\} \text{ a.e. on } X,$$

so that the function $A(T_1^*, \ldots, T_d^*)f$ can be replaced by the function $A(T_1, \ldots, T_d)f$ in Corollary 4, without any influence.
4. Concluding remarks

Throughout this section, \((X, \mathcal{F}, \mu)\) is a \(\sigma\)-finite measure space, and \(T_1, \ldots, T_d\) are commuting positive linear contractions of \(L_1(\mu)\) such that \(T_ie = e\) \((1 \leq i \leq d)\) for some \(e \in L_1(\mu)\) with \(e > 0\) on \(X\). Here we briefly discuss the problem of characterizing a positive measurable function \(V\) on \(X\) such that if \(f \in L_1^+(Vd\mu)\) then the limit function \(A(T_1, \ldots, T_d)f\) (or \(A(T_1^*, \ldots, T_d^*)f\)) is finite a.e. on \(X\). As in the preceding section, we will denote \(\mathcal{I} = \bigcap_{i=1}^d \mathcal{I}(T_i)\). The results may be stated as follows. (For a related result we refer the reader to [7].)

**Theorem 4.** Let \(0 < V \leq \infty\) be a measurable function on \(X\). If \(1 < p < \infty\) and \(1/p + 1/p' = 1\), then the following are equivalent:

(a) For any \(f \in L_1^+(Vd\mu)\) the limit function \(A(T_1, \ldots, T_d)f\) is finite a.e. on \(X\).

(b) \(E\{e^{-1}V^{1-p'}|(X, \mathcal{I}, \mu d\mu)\} < \infty\) a.e. on \(X\).

If \(p = 1\), then (a) is equivalent to

(c) \(V^{-1} \leq U < \infty\) a.e. on \(X\) for some \(U\), measurable with respect to \(\mathcal{I}\).

**Proof:** By virtue of (15) and the result mentioned in Introduction (see especially (3) and (4)), Theorem 4 follows immediately.

**Corollary 5.** If \(1 < p < \infty\), then the following are equivalent:

(a) For any \(f \in L_1^+(\mu)\) the limit function \(A(T_1, \ldots, T_d)f\) is finite a.e. on \(X\).

(b) There exist \(X_n \in \mathcal{I}, n = 1, 2, \ldots\), such that \(X_n \uparrow X\) and \(\mu(X_n) < \infty\).

(c) For any \(f \in \bigcup_{1 \leq r \leq \infty} L_1^+(\mu)\) the limit function \(A(T_1, \ldots, T_d)f\) is finite a.e. on \(X\).

**Proof:** Since the implications (b) \(\Rightarrow\) (c) \(\Rightarrow\) (a) are obvious, we only prove (a) \(\Leftrightarrow\) (b). To do this we apply Theorem 4 with \(V = 1\) on \(X\) and see that (a) is equivalent to

\[E\{e^{-1}|(X, \mathcal{I}, \mu d\mu)\} < \infty\] a.e. on \(X\),

which is clearly equivalent to (b). The proof is complete.
Theorem 5. Let $0 < V \leq \infty$ be a measurable function on $X$. If $1 < p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L^+_p(V\,d\mu)$ the limit function $A(T^*_1, \ldots, T^*_d)f$ is finite a.e. on $X$.
(b) $E\{ (e^{-1}V)^{1-p'} \mid (X, \mathcal{I}, e\,d\mu) \} < \infty$ a.e. on $X$.

If $p = 1$, then (a) is equivalent to
(c) $eV^{-1} \leq U < \infty$ a.e. on $X$ for some $U$, measurable with respect to $\mathcal{I}$.

Proof: By the proof of Theorem 4, we see that in this case it is enough to use (18) instead of (15), completing the proof. ■

Corollary 6. If $1 \leq p < \infty$ and $1/p + 1/p' = 1$, then the following are equivalent:

(a) For any $f \in L^+_{p'}(\mu)$ the limit function $A(T^*_1, \ldots, T^*_d)f$ is finite a.e. on $X$.
(b) There exists an $\tilde{e} \in L_1(\mu) \cap L_{p'}(\mu)$ such that $\tilde{e} > 0$ on $X$ and $T^*_i \tilde{e} = \tilde{e}$ $(1 \leq i \leq d)$.
(c) For any $f \in \bigcup_{p \leq r \leq \infty} L^+_{r'}(\mu)$ the limit function $A(T^*_1, \ldots, T^*_d)f$ is finite a.e. on $X$.

Proof: (a) $\Rightarrow$ (b). By Theorem 5 with $V = 1$ on $X$, (a) implies the existence of $X_n \in \mathcal{I}$, $n = 1, 2, \ldots$, such that

$$X_n \uparrow X \text{ and } e \cdot 1_{X_n} \in L_{p'}(\mu).$$

Thus choosing a suitable sequence $d_n$, $n = 1, 2, \ldots$, of positive real numbers we have

$$\tilde{e} = \sum_{n=1}^{\infty} d_n (e \cdot 1_{X_n}) \in L_1(\mu) \cap L_{p'}(\mu)$$

and

$$T^*_i \tilde{e} = \tilde{e} \text{ for all } 1 \leq i \leq d.$$

(b) $\Rightarrow$ (c). Since $T^*_1, \ldots, T^*_d$ are commuting positive linear contractions of $L_1(\tilde{e}\,d\mu)$ with $T^*_i1 = 1 \in L_1(\tilde{e}\,d\mu)$ $(1 \leq i \leq d)$, it is enough to show that

$$f \in L_1(\tilde{e}\,d\mu) \text{ for every } f \in \bigcup_{p \leq r \leq \infty} L^+_{r'}(\mu).$$

And this follows, as $\tilde{e} \in L_1(\mu) \cap L_{p'}(\mu)$ implies

$$\tilde{e} \in \bigcap_{1 \leq r' \leq p'} L_{r'}(\mu).$$
by the Hölder inequality.
(c) ⇒ (a). Trivial. The proof is complete. ■

References


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