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GUIDED WAVES IN A FLUID LAYER ON AN ELASTIC IRREGULAR BOTTOM

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Abstract

In this paper one considers the linearized problem to determine the movement of an ideal heavy fluid contained in an unbounded container with elastic walls. As initial data one knows the movement of both the bottom and the free surface of the fluid and also the strength of certain perturbation, strong enough to take the bottom out of its rest state.

One important point to be considered regards the influence of the bottom's geometry on the propagation of superficial waves. This problem has been already studied in other works without considering the elastic properties of the bottom and considering a cilindrical container with bounded section.

1. Introduction

The problem we are going to study corresponds to the linearization of a tsunamis wave propagation model and describes the dynamics of linear superficial waves on the sea when one considers that the perturbations altering the free surface of the fluid are of elastic origin concentrated in a certain bottom region.

We shall see that in the linearized model it is possible to note that certain wave guide superficial effects are produced by bottom irregularities of the underwater ridge type.

In the case that the bottom is a horizontal plane it is well known that the amplitude of the superficial waves decreases proportionally to R^{-1}, where R denotes the distance to the place where the initial perturbation is localized. The fundamental result of [**1**] consists in the assertion that in the presence of a rigid submarine chain of mountains, a group of superficial waves, corresponding to a non empty class of initial conditions, could appear and that they move over the chain with an amplitude decreasing order of $R^{-\alpha}$ where $\alpha = 1/2, 1/3, 1/4$, depends on the geometric form of the chain profile.

In this paper we shall study, besides the general dynamical properties, the structure of the wave spectra propagating along the submarine chain of mountains. It will be shown that if the altitude of such mountains is high enough in comparison with the altitude of the fluid layer out of the chain, then it would behave as a wave conductor in a way we are going to state precisely later. Moreover, we will show the existence of superficial waves wich propagate along the submarine chain and damp quicker than when one considers a rigid bottom.

The proof of this fact is possible due to the conjunction of the effects produced by the irregularity and the elasticity of the bottom.

Part of the results presented here are quoted without proof in [**2**].

2. Notation and Preliminaries

Let us consider in $R^3 := \{X = (x_1, x_2, x_3)\}$ the axes x_1, x_2 taken horizontally and the axis x_3 upwards in the vertical direction. We shall suppose that an ideal incompressible fluid is under the action of a homogeneeous gravitatory field occupying, in state of equilibrium, a certain region Λ of R^3 bounded by the plane $\pi := \{x_3 = 0\}$ (free surface) and by the surface $\Gamma := \{x_3 = -h(x_2)\}$ (bottom), where $h(x_2) \geq h_0 > 0$ and the function $1 - h(x_2)$ has a compact support.

The surface Γ is the border separating the volume Λ occuped by the fluid from the region $\Omega := \{x_3 < -h(x_2)\}$ occupied by the bottom in a state of static equilibrium.

When necessary, we shall distinguish the different geometrical features of the bottom acording the notation $\pi^{(h)}$, $\Lambda^{(h)}$, $\Gamma^{(h)}$, $\Omega^{(h)}$ instead of π, Λ, Γ and Ω. In the case of a plane bottom, i.e., $h =$ constant, we will use the notation $\pi^{(0)}$, $\Lambda^{(0)}$, $\Gamma^{(0)}$ and $\Omega^{(h)}$, respectively.

We will also denote by $\pi_0^{(h)}$, $\Lambda_0^{(h)}$, $\Gamma_0^{(h)}$ and $\Omega_0^{(h)}$, respectively, the transversal sections of the sets $\pi^{(h)}$, $\Lambda^{(h)}$, $\Gamma^{(h)}$ and $\Omega^{(h)}$ with respect to the plane $\{x_1 = 0\}$. Note that $\pi^{(h)}$ and $\pi_0^{(h)}$ do not depend on the election of the function h.

When it is not necessary to make any explicit reference to the bottom structure we will use simply the notation π_0, Λ_0, Γ_0 and Ω_0 for the transversal sections.

In this paper we are going to study the small oscillations of the medium formed by the fluid layer and the elastic bottom in the neighbourhood of the equilibrium state of the system, given initial conditions.

We are going to look for the pressure distribution $P(X;t)$ and for the displacement vector $\vec{U}(X;t)$ of the elastic bottom, in the form

$$P(X;t) = p^*(X) - p(X;t), \vec{U}(X;t) = \vec{u^*}(X) + \vec{u}(X;t)$$

where $p(X;t)$ and $\vec{u}(X;t)$ are small dinamical deviations of the hydrostatic pressure $p^*(X) = p_0 + \rho_0 g x_3$ and the static equilibrium state $\vec{u^*}(X)$ of the elastic medium due to the presence of the fluid layer submitted itself to the action of the gravity force.

Here p_0 denotes the exterior pressure acting on the free surface on the fluids layer and taken as constant in absence of movement, ρ_0 is the fluid's density and g is the acceleration due to gravity.

In the frame of the linear theory, the problem of the small oscillations of the just mentioned system, is described by the following Lame's equations:

$$(1) \qquad (\mathcal{L}\vec{u})_i := \sum_{k=1}^{3} \frac{\partial \sigma_{ik}(\vec{u})}{\partial x_k} = \frac{\rho}{\lambda+\mu} \frac{\partial \vec{u}}{\partial t^2}, i = 0, 1, \qquad (\Omega)$$

$$(2) \qquad \Delta p = 0, \qquad (\Lambda)$$

$$(3) \qquad (\sigma \vec{u}) \cdot \vec{n} = \frac{1}{\lambda+\mu} p\vec{n},\ \frac{\partial p}{\partial n} = \rho_0 \frac{\partial^2 u_n}{\partial t^2}, \qquad (\Gamma)$$

$$(4) \qquad 0 = g\frac{\partial p}{\partial x_3} + \frac{\partial^2 p}{\partial t^2}, \qquad (\pi)$$

and the initial conditions:

$$(5) \qquad \mathcal{G}(x_1, x_2; 0) = \mathcal{G}_0(x_1, x_2),\ \mathcal{G}_t(x1, x_2; 0) = \mathcal{G}_1(x_1, x_2),$$

$$(6) \qquad \vec{u}(x_1, x_2, x_3; 0) = \vec{u_0}(x_1, x_2, x_3),\ \vec{u_t}(x_1, x_2, x_3; 0) = \vec{u_1}(x_1, x_2, x_3)$$

where $\mathcal{G}(x_1, x_2; t)$ denotes the deviation of the free surface of the fluid with respect to the equilibrium surface coincident with the plane π. Due to the stated condition of constant pressure on the free surface of the fluid, the following relation holds $\rho_0 g \wp = -p$. In Equations (1) and (3) we use the following notations:

ρ —density of the elastic medium, $\vec{n} = (n_1, n_2, n_3)$ outer normal vector to the region Ω, u_n —normal component of the vector $\vec{u}$ over Γ, $\sigma(\vec{u}) = \frac{1}{\lambda+\mu}\tau(\vec{u})$ where $\tau(\vec{u}) = (\tau(\vec{u}))_{i,k=1}^{3}$ is the tensor of elastic tensions in Ω,

$$\tau_{ik}(\vec{u}) = \lambda \delta_{ik} \vec{u} + \mu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

λ, μ constants of elasticity, δ_{ik} —the Kronecker's delta, and

$$\mathrm{div}(\vec{u}) = \sum_{k=1}^{3} \frac{\partial u_i}{\partial x_i}.$$

In what follows we shall study oscillatory processes different to the rest state only by movements with finite energy.

3. Some functional spaces, auxiliary problems and their generalized solutions

For every non negative ℓ, we denote by $L_2^\ell(\Lambda)$ the space of generalized functions over Λ with derivatives of order ℓ belonging to the space $L_2(\Lambda)$ of square integrable functions in Λ.

We introduce in $L_2^\ell(\Lambda)$ the seminorm:

$$\|p\|^2_{L_2^\ell(\Lambda)} := \int_\Lambda \sum_{|\alpha|\le\ell} |D^\alpha p(X)|^2 dX, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^\alpha \partial x_2^\alpha \partial x_3^\alpha}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in N$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. By $\widetilde{L}_2^\ell(\Lambda)$ we shall denote the quotient $L_2^\ell(\Lambda)/P_{\ell-1}$ where $P_{\ell-1}$ is the vectorial space of the polynomials in the variables x_1, x_2, x_3 of degree less or equal to $\ell - 1$.

The seminorm defined on $L_2^\ell(\Lambda)$, restricted to the vectorial space $C_0^\infty(\Lambda \cup \Gamma)$ of the infinite differentiable complex functions whose support is a compact subset of $\Lambda \cup \Gamma$, is already a norm.

$L_{2,0}^\ell(\Lambda)$ denotes the completion of $C_0^\infty(\Lambda \cup \Gamma)$ with respect to this norm.

We shall consider moreover the Soboliev space $H_2^\ell(\Lambda)$ of the functions in $L_2(\Lambda)$ with generalized derivatives up to the order ℓ belonging to $L_2(\Lambda)$, provided with the usual norm

$$\|p\|^2_{H_2^\ell(\Lambda)} := \int_\Lambda \sum_{|\alpha|\le\ell} |D^\alpha p(X)|^2 dX$$

where $\widetilde{H}_2^\ell(\Lambda)$ denotes the closure of $C_0^\infty(\Lambda \cup \Gamma)$ in $H_2^\ell(\Lambda)$.

The space $\widetilde{H}_2^\ell(\Lambda)$ is made up of those functions from $H_2^\ell(\Lambda)$ whose derivatives of order less or equal to $\ell - 1$ have a π-trace identically equal to zero.

Obviously $\widetilde{L}_2^\ell(\Lambda)$ is a Hilbert space of generalized functions having $L_{2,0}(\Lambda)$ as a closed vector subspace. Moreover, the following immersions hold:

$$H_2^\ell(\Lambda) \subset \widetilde{L}_2^\ell(\Lambda),\ \widetilde{H}_2^\ell(\Lambda) \subset L_{2,0}^\ell(\Lambda) \subset \widetilde{L}_2^\ell(\Lambda).$$

In order to obtain other relations between such spaces and for further developments as well, it is necessary to set more conditions on the functions h determining the surface Γ.

In what follows we shall suppose that the function h is continous and continously differentiable except at most in a finite number of points where it forms no angles (inner or outer) less or equal to $\pi/2$.

Lemma 3.1. *The spaces $L^1_{2,0}(\Lambda)$ and $\widetilde{H}^1_2(\Lambda)$ coincide and their norms are equivalent.*

Proof: It is not difficult to prove this lemma just repeating the argument used in Example 3.2.2/2 from [**3**]. ■

For every positive integer $\ell > 0$, we shall denote by $H_2^{\ell-1/2}(\pi)$ and $H_2^{\ell-1/2}(\Gamma)$ the trace spaces of the functions from $H_2^\ell(\Lambda)$ to π and Γ, respectively. These trace spaces are called the Soboliev-Slobodietski spaces of order $\ell - 1/2$.

With the conditions imposed on the region Λ, the following equality holds:

$$H^1_2(\Lambda) = \{\phi \in \widetilde{L}^1_2(\Lambda) : \phi/\pi \in H_2^{1/2}(\pi)\}. \tag{7}$$

In fact, if $\phi \in \widetilde{L}^1_2(\Lambda)$ and $\phi/\pi \in H_2^{1/2}(\pi)$, then there is a function $\phi_1 \in H^1_2(\Lambda)$ satisfaying $\phi_1/\pi = \phi/\pi$. But then $\phi_0 = \phi - \phi_1 \in L^1_{2,0}(\Lambda)$ and due to Lemma 3.1 we can conclude that $\phi - \phi_1 \in H^1_2(\Lambda)$ therefore $\phi \in H^1_2(\Lambda)$.

Just repeating the proof of Lemma 3.1, it is easy to see that if $\phi \in L^1_2(\Lambda)$ and $\phi/\pi \in L_2(\pi)$, then

$$\|\phi\|_{L_2(\Gamma)} \leq \|\phi\|_{L_2(\pi)} + \|\nabla\phi\|_{L_2(\Lambda)}. \tag{8}$$

But we know that through

$$\|\phi\| := \|\nabla\phi\|_{L_2(\Lambda)} + \|\phi\|_{L_2(\Gamma)} + \|\phi\|_{L_2(\pi)},$$

we define a norm in $H^1_2(\Lambda)$ equivalent to the usual Soboliev norm. Therefore, from (8) we can write

$$H_2^{1/2}(\Lambda) = \{\phi \in \widetilde{L}^1_2(\Lambda) : \phi/\pi \in L_2(\pi)\}. \tag{9}$$

From (9) we get immediately that

$$H_2^{1/2}(\pi) = \phi/\pi \in \widetilde{L}^1_2(\Lambda) \cap L_2(\pi). \tag{10}$$

We denote by $\vec{L}_2(\Omega)$ and $\vec{L}_2(\Gamma)$ the spaces of square integrable vectorial functions over Ω and Γ, respectively. $\vec{H}_2^\ell(\Omega)$ and $\vec{H}_2^{\ell-1/2}(\Gamma)$ denotes the Soboliev and Soboliev-Slovodietski spaces of vectorial functions.

For every pair of functions $\vec{u}$, $\vec{v} \in \vec{H}_2^1(\Omega)$ we define:

$$\begin{aligned}\varepsilon_{\lambda,\mu}(\vec{u},\vec{v}) &:= \frac{\lambda}{2(\lambda+\mu)} \operatorname{div}\vec{u} \operatorname{div}\vec{v} + \frac{\mu}{\lambda+\mu} \sum_{k=1}^{3} \frac{\partial \vec{u}_j}{\partial x_j} \frac{\partial \vec{v}_j}{\partial x_j} \\ &\quad + \frac{\mu}{2(\lambda+\mu)} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial \vec{v}_i}{\partial x_j} + \frac{\partial \vec{v}_j}{\partial x_i}\right) \\ E^*_{\lambda,\mu}[\vec{u},\vec{v}] &:= \int_\Omega \varepsilon_{\lambda,\mu}(\vec{u},\vec{v}) dX \\ E_{\lambda,\mu}[\vec{u},\vec{v}] &:= E^*_{\lambda,\mu}[\vec{u},\vec{v}] + \int_\Omega \vec{u}\vec{v} dX.\end{aligned}$$

In order to simplify notation, when not necessary we will use neither the arrow $\rightarrow$ to indicate that we are dealing with vectorial functions, nor the subindices λ, μ in the expressions above.

It is known (see [**4**]) that the expression $(E[u,u])^{1/2}$ defines a norm in $\vec{H}_2^1(\Omega)$ equivalent to the usual Soboliev norm.

The following Green formula for the Lame operator $\mathcal{L}$ defined in (1) is also well known:

$$\langle \mathcal{L}u, v\rangle_{\vec{L}_2(\Omega)} = E^*[u,v] - \langle \sigma(u)\cdot n, v\rangle_{\vec{L}_2(\Gamma)} \tag{11}$$

where $u \in \vec{H}_2^2(\Omega)$, $v \in \vec{H}_2^1(\Omega)$ and by the symbol $\langle\cdot,\cdot\rangle$ one denotes the scalar product in the mentioned spaces.

From now on we use, whenever possible, the following simplified notation for norms and scalar products:

- $|\cdot|$ for the module of a complex number.
- $|\cdot|_Q$ and $|\cdot|_{\vec{Q}}$ for the norm in spaces $L_2(Q)$ and $\vec{L}_2(Q)$, respectively, where $Q = \pi$, Λ, Γ, Ω.
- $\|\cdot\|_Q$ and $\|\cdot\|_{\vec{Q}}$ for the usual norm in the Soboliev spaces $H_2^1(Q)$ and $\vec{H}_2^1(Q)$, respectively, where $Q = \Lambda$, Ω.
- $[\cdot]_Q$ and $[\cdot]_{\vec{Q}}$ for the norm in the Soboliev-Slovodietski spaces $H_2^{1/2}(Q)$ and $\vec{H}_2^{1/2}(Q)$, respectively, where $Q = \pi$, Γ.
- $\langle\cdot,\cdot\rangle_Q$ and $\langle\cdot,\cdot\rangle_{\vec{Q}}$ for the scalar products in $L_2(Q)$ and $\vec{L}_2(Q)$, respectively, where $Q = \pi$, Λ, Γ, Ω.

Moreover, we shall use the notation $\mathcal{B}(H_1,H_2)$ for the space of bounded linear operators defined between the Banach spaces H_1 and H_2. When no doubts could arise about the nature of the spaces H_1 and H_2, we denote by $|||\cdot|||$ the norm in $\mathcal{B}(H_1,H_2)$.

Let now ϕ, ϕ_0 be scalar functions defined on π; τ, τ_0 scalar functions defined on Γ and f, F, vectorial functions defined on Γ and Ω, respectively.

Consider the following boundary problems:

$$\Delta q = 0(\Lambda), \qquad q/\pi = \phi, \qquad \left.\frac{\partial q}{\partial n}\right|_\Gamma = 0, \tag{12}$$

$$\Delta r = 0(\Lambda), \qquad r/\pi = 0, \qquad \left.\frac{\partial r}{\partial n}\right|_\Gamma = \tau \tag{13}$$

$$\Delta R = 0(\Lambda), \qquad \left.\frac{\partial R}{\partial x_3}\right|_\Gamma = 0, \qquad \left.\frac{\partial R}{\partial n}\right|_\Gamma = \tau_0, \tag{14}$$

$$\Delta S = 0(\Lambda), \qquad \left.\frac{\partial S}{\partial x_3}\right|_\pi = g^{-1/2}\phi_0, \qquad \left.\frac{\partial S}{\partial n}\right|_\Gamma = 0 \tag{15}$$

$$\mathcal{L}v + v = F(\Omega), \qquad \sigma(v)\cdot\vec{n}|_\Gamma = 0 \tag{16}$$

$$\mathcal{L}w + w = 0(\Omega), \qquad \sigma(w)\cdot\vec{n}|_\Gamma = f. \tag{17}$$

Definition 3.1. The functions $q \in \widetilde{L}_2^1(\Lambda)$, $r \in L_{2,0}^1(\Lambda)$; R, $S \in \widetilde{L}_2^1(\Lambda)$, v, $w \in \vec{H}_2^1(\Omega)$ are generalized solutions of Problems (12)-(17) if the following conditions are satisfied:

$$q/\pi = \phi, \quad \int_\Lambda \nabla q \nabla \varepsilon dX = 0, \qquad \forall \varepsilon \in L_{2,0}^1(\Lambda) \tag{12*}$$

$$\int_\Lambda \nabla r \overline{\nabla \varepsilon} dX = -\langle \tau, \varepsilon\rangle_\Gamma, \qquad \forall \varepsilon \in L_{2,0}^1(\Lambda) \tag{13*}$$

$$\int_\Lambda \nabla R \overline{\nabla \varepsilon} dX = -\langle \tau_0, \varepsilon\rangle_\Gamma, \qquad \forall \varepsilon \in \widetilde{L}_2^1(\Lambda) \tag{14*}$$

$$\int_\Lambda \nabla q \nabla \varepsilon dX = g^{1/2}\langle \phi_0, \varepsilon\rangle_\pi, \quad \forall \varepsilon \in \widetilde{L}_2^1(\Lambda) \tag{15*}$$

$$E[v,\beta] = \langle F, \beta\rangle_{\vec{\Omega}}, \qquad \forall \beta \in \vec{H}_2^1(\Omega) \tag{16*}$$

$$E[w,\beta] = \langle f, \beta\rangle_{\vec{\Gamma}}, \qquad \forall \beta \in \vec{H}_2^1(\Omega). \tag{17*}$$

With the help of variational methods and Reisz representation theorem for bounded linear functionals in Hilbert spaces, we can prove existence and unicity for generalized solutions of Problems (12)-(17).

If we denote by $\hat{}$ the Fourier transform in $L_2(R_{x_1})$ and if α is the variable in the space of transformed functions, then the boundary problem (12) can be written in the following equivalent way;

$$\frac{\partial^2 \hat{q}}{\partial x_2^2} + \frac{\partial^2 \hat{q}}{\partial x_2^3} - \alpha^2 q = 0, \quad (\Lambda_0), \tag{18}$$

$$\hat{q}/\pi_0 = \phi, \ \left.\frac{\partial \hat{q}}{\partial n_0}\right|_{\Gamma_0} = 0, \tag{19}$$

where $\vec{n}_0$ is the normal exterior to Ω_0 in Γ_0.

If we suppose that $\phi \in C_0^\infty(\pi)$, then Problem (12) has a solution in the classical sense. Using the principle of locality for elliptic boundary problems, and the fact that $\Gamma_0^{(h)}$ differs from $\Gamma_0^{(0)}$ only in a compact region, we get the following estimates (see [**5**]):

(20)

$$\max\left\{|\hat{q}(\alpha, x_2, x_3)| \left|\frac{\partial \hat{q}}{\partial x_3}(\alpha, x_2, x_3)\right|\right\} \leq Ce^{-a(\alpha^2 + x_2^2 + x_3^2)} \sup |\hat{\phi}(\alpha, x_2, 0)|$$

for every $\alpha \in R$, $(x_2, x_3) \in \Lambda_0$ where α and C are positive constants.

From (24) and Plancherel identity for the Fourier transforms, one concludes immediately that the trace of $\frac{\partial q}{\partial x_3}$ to π belongs to $L_2(\pi)$. This remark allows the definition of an operator A_0 in $L_2(\pi)$ with domain of definition $D(A_0) = C_0^\infty(\pi)$, through

$$A_0\phi = g\left.\frac{\partial q}{\partial x_3}\right|_\pi, \quad \phi \in D(A_0),$$

where q is the solution of the boundary problem (12) corresponding to ϕ.

Using now the Green formula one can see that A_0 is symmetric. Denote by A the closure of the operator A_0 in $L_2(\pi)$. The operator A is self-adjoint and positive in $L_2(\pi)$, $D(A) = H_2^1(\pi)$, $D(A^{1/2}) = H_2^{1/2}(\pi)$ and for every pair of functions ϕ_1, $\phi_2 \in D(A)$ the following relation holds:

$$\langle A\phi_1, \phi_2\rangle_\pi = g\int_\Lambda \nabla q_1 \overline{\nabla q_2} dX, \tag{21}$$

where q_i is the solution of the boundary problem (12) corresponding to ϕ_i, $i = 1, 2$.

Applying Green's formula we obtain for each $\phi \in D(A_0)$

$$|A_0\phi|_\pi^2 = \frac{g^2}{2}\int_\Lambda \frac{\partial}{\partial x_3}|\nabla q|^2 dX = \frac{g^2}{2}\int_\pi |\nabla q|^2 dS - \frac{g^2}{2}\int_\Gamma |\nabla q|^2 \cos{(n, x_3)} dS.$$

From this we have:

$$\frac{1}{2}|A_0\phi|_\pi^2 = \frac{g^2}{2}\int_\pi \left(\left|\frac{\partial\phi}{\partial x_i}\right|^2 + \left|\frac{\partial\phi}{\partial x_2}\right|^2\right) dS - \frac{g^2}{2}\int_\Gamma |\nabla q|^2 \cos(n, x_3) dS$$

since $\frac{\partial q}{\partial x_i} = \frac{\partial \phi}{\partial x_i}$, $i = 1,2$ over π and since $\Gamma = \{x_3 = -h(x_2)\}$ then $\cos(n, x_3) \geq 0$ over Γ.

Therefore

$$|A_0\phi|_\pi^2 \leq g^2 \int_\pi \left(\left|\frac{\partial\phi}{\partial x_1}\right|^2 + \left|\frac{\partial\phi}{\partial x_2}\right|^2\right) dX \tag{22}$$

for every $\phi \in D(A_0)$ the inequality (22) holds for A and for every function $\phi \in H_2^1(\pi)$, just by the definition of the closure operator.

From the inequality (22) one has that

$$A \in \mathcal{B}(H_2^1(\pi), L_2(\pi)). \tag{23}$$

On the other hand,

$$|A^{1/2}\phi|_\pi = g^{1/2}|\nabla q|_\Lambda \leq g^{1/2}\sup\{|\nabla p|_\Lambda : p \in H_2^1(\Lambda),\ p/\pi = \phi\} = g^{1/2}[\phi]_\pi,$$

and from this

$$A^{1/2} \in B(H_2^{1/2}(\pi),\ L_2(\pi)).$$

Let us define now other operators that we shall use later on.

We know that the solution of Problem (12) belongs to $H_2^1(\Lambda)$ for every $\phi \in H_2^{1/2}(\pi)$, hence we can define the operator

$$B_1\phi = q/\Gamma,\ D(B_1) = H_2^{1/2}(\pi). \tag{24}$$

From (24) we get

$$[B_1\phi]_\Gamma^2 = [q]_\Gamma^2 \leq C_1\|q\|_\Lambda^2 \leq C_2\{|\phi|_\pi^2 + g|\nabla q|_\Lambda^2\} \leq C_3[\phi]_\pi^2$$

which leads us to

$$B_1 \in \mathcal{B}(H_2^{1/2}(\pi),\ H_2^{1/2}(\Gamma)). \tag{25}$$

Let us now define in $L_2(\Gamma)$ the operator

$$B\tau = r/\Gamma \tag{26}$$

where r denotes the generalized solution of (13) corresponding to τ.

From this definition it follows immediately that

$$B \in \mathcal{B}(L_2(\Gamma),\, H_2^{1/2}(\Gamma)). \tag{27}$$

If $\tau_1,\, \tau_2 \in C_0^\infty(\Gamma)$, the boundary problem (13) has a solution in the classical sense. If we apply Green's formula to the expression

$$\langle r_1, \Delta r_2\rangle_\Lambda = 0$$

where r_1 and r_2 are the solutions of (13) corresponding to τ_1 and τ_2, respectively, we obtain

$$\langle B\tau_1, \tau_2\rangle_\Gamma = -\int_\Gamma \nabla r_1 \overline{\nabla r_2} dX. \tag{28}$$

Using variational methods one proves that if $\tau \in H_2^{1/2}(\Gamma)$ then the solution r to Problem (13) belongs to $H_2^2(\Lambda)$ and therefore the traces of the derivatives $\frac{\partial r}{\partial x_3}\Big|_\pi \in L_2(\pi)$ exist.

This analysis allows us to define the operator A_1 with $D(A_1) = H_2^{1/2}(\Gamma)$ through

$$A_1(\tau) := g\, \frac{\partial r}{\partial x_3}\Big|_\pi \in L_2(\pi). \tag{29}$$

From the immersion theorems it follows immediately that

$$A_1 \in \mathcal{B}(H_2^{1/2}(\Gamma), L_2(\pi)). \tag{30}$$

We are now going to study the boundary problem (19). It is obvious that if for some function $\phi \in L_2(\pi)$ there exists a generalized solution S to Problem (15), in the sense of definition (15*), then it is unique.

We will prove that such a solution always exists for any function $\phi_0 \in D(A^{-1})$.

In fact, if $A^{-1}(\phi_0) = \phi$, then $\phi_0 = A(\phi) = g\, \frac{\partial q}{\partial x_3}\Big|_\pi$, where q is the generalized solution of Problem (12).

Obviously, the function q is also a solution of the boundary problem (15) with the right hand side $g^{-1}\phi_0$. From the uniqueness of the solution of Problem (15) it follows that $S = g^{1/2}q$ and therefore

$$S/\pi = g^{1/2}q/\pi = g^{1/2}\phi = g^{1/2}A^{-1}(\phi_0). \tag{31}$$

Then the operator A^{-1} acts according to the law:

$$A^{-1}(\phi_0) = g^{-1/2} S/\pi,\ \phi_0 \in D(A^{-1}), \tag{32}$$

where S is the generalized solution of Problem (15). On the other hand from (15*) one gets the chain of equalities:

$$\langle A^{-1}\phi_q, \phi_q\rangle_\pi = g^{-1/2}\langle S, \phi_0\rangle_\pi = \left\langle S, \frac{\partial S}{\partial x_3}\right\rangle_\pi = |\nabla S|^2_\Lambda. \tag{33}$$

Now we are going to study the boundary problem (14). Consider the operator

$$A_2(\tau_0) := A^{-1} A_1(\tau_0), \tag{34}$$

and let us show that for any $\tau_0 \in D(A_2)$ there exists a generalized solution of Problem (14) such that

$$A_2(\tau_0) = -R/\pi. \tag{35}$$

Let $\tau_0 \in D(A_2)$ and $A_2(\tau_0) = \phi$. Then $A(\phi) = g\left.\frac{\partial r}{\partial x_3}\right|_\pi$ where r is the generalized solution of Problem (13) when one takes τ_0 as a boundary condition instead of τ.

Let q be the solution of Problem (12) corresponding to $\phi = A_2(\tau_0)$, then $g\left.\frac{\partial q}{\partial x_3}\right|_\pi = A(\phi) = g\left.\frac{\partial r}{\partial x_3}\right|_\pi$. It is not difficult to verify that the function $R = r - q$ is the generalized solution of Problem (14) in the sense of Definition (14*) satisfying equality (35).

The proof of existence and uniqueness of the solution of Problems (16) and (17) for $F \in \vec{L}_2(\Omega)$, $f \in \vec{L}_2(\Gamma)$ is an immediate consequence of the Korn inequality (see [**4**]) and of the Riesz Lemma about the representation of bounded linear functionals in Hilbert spaces.

From this proof it follows the existence of bounded operators

$$P : \vec{L}_2(\Omega) \to \vec{H}^1_2(\Omega) \text{ and } Q_0 : \vec{L}_2(\Gamma) \to \vec{H}^1_2(\Omega)$$

giving generalized solutions of such problems:

$$P(F) = v, \quad Q_0(f) = w.$$

The operators P and Q are defined by the equalities

$$\langle F, \beta\rangle_{\vec{\Omega}} = E[P(F), \beta], \tag{36}$$

$$\langle f, \beta\rangle_{\vec{\Gamma}} = E[Q_0(f), \beta], \tag{37}$$

for any functions $f \in \vec{L}_2(\Gamma)$, $F \in \vec{L}_2^{(}\Omega)$, $\beta \in \vec{H}_2^1(\Omega)$.

It is well know that the closure of $D(P^{-1})$ with respect to the norm $(\langle P^{-1}v, v\rangle_{\vec{\Omega}})^{1/2}$ gives us the domain of definition of $P^{-1/2}$.

Moreover $D(P^{-1/2}) = \vec{H}_2^1(\Omega)$, and

$$E[u,v] = \langle P^{1/2}(u), P^{1/2}(v)\rangle_{\vec{\omega}};\ u, v \in \vec{H}_2^1(\Omega).$$

Therefore $|P^{-1/2}(v)|_{\vec{\Omega}}$ is a norm in $\vec{H}_2^1(\Omega)$ equivalent to the usual one.

From the last statement and (37) one gets the relation:

$$\langle P^{-1/2}Q_0(f), P^{-1/2}(\beta)\rangle_{\vec{\Omega}}; = \langle f, \beta\rangle_{\vec{\Gamma}}, \tag{38}$$

valid for any $f \in \vec{L}_2(\Gamma)$, $\beta \in \vec{H}_2^1(\Omega)$.

4. Operational statement of Cauchy problem (1)-6) and their generalized solutions

In this paragraph we are going to consider for the sake of convenience that $g = \rho_0 = 1$. We shall look for function p and u in (1)-(6) of the form $p = q + r$, $u = v + w$, where q, r, v and w are generalized solutions of the boundary problems (12), (13) and (16), (17).

Note that using the operators introduced in the previous paragraph the system (2)-(4) can be written in the equivalent operational form:

$$\sigma(u)\cdot\vec{n} = \frac{1}{\lambda+\mu}B_1(\phi)\vec{n} + \frac{1}{\lambda+\mu}\frac{\partial^2}{\partial t^2}[B(u_n)\vec{n}], \quad (\Gamma) \tag{3*}$$

$$0 = \phi + \frac{\partial^2}{\partial t^2}[A^{-1}(\phi) + A_2(u_n)], \qquad (\pi), \tag{4*}$$

where $\phi = p/\pi$ and we have just taken $\psi = \left.\frac{\partial^2 u_n}{\partial t^2}\right|_\Gamma$ in the boundary problem (13).

Now consider the system of equations

$$\mathcal{L}v + v = -\frac{\rho}{\lambda+\mu}\frac{\partial^2 u}{\partial t^2} + u(\Omega), \quad \sigma(v)\vec{n} = 0(\Gamma), \tag{39}$$

$$\mathcal{L}w + w = 0(\Omega),\ \sigma(w)\vec{n} = \frac{1}{\lambda+\mu}B_1(\varphi)\vec{n} + \frac{1}{\lambda+\mu}\frac{\partial^2}{\partial t^2}\left[B(u_n)\cdot\vec{n}\right](\Gamma) \tag{40}$$

$$u = v + w, \tag{41}$$

where $u(t)$ is a vectorial function defined on the interval $[0, +\infty[$, twice continuously differentiable, taking values in $\vec{H}_2^1(\Omega)$. For a given function $u(t)$ the generalized solutions of problems (39) and (40) are expressed by the formulas:

$$(39^*) \qquad v = \frac{\rho}{\lambda+\mu} P\left(\frac{\partial^2 u}{\partial t^2}\right) + P(u),$$

$$(40^*) \qquad w = \frac{1}{\lambda+\mu} Q_0 N B_1(\phi) + \frac{1}{\lambda+\mu}\frac{\partial^2}{\partial t^2} Q_0 N B \Gamma_n(u),$$

where N represents the operator multiplication by the normal vector $\vec{n}$, and Γ_n is the trace operator on Γ scalarly multiplied by the normal $\vec{n}$ and defined in $\vec{H}_2^1(\Omega)$. If we introduce the notations $Q := -Q_0 N B \Gamma_n, Q_1 := -Q_0 N B_1$ then from (39*) (40*) and (41) one gets the operational equation:

$$(42) \qquad u - P(u) + \frac{1}{\lambda+\mu}(\rho P + Q)\frac{d^2 u}{dt^2} + \frac{1}{\lambda+\mu} Q_1(\phi) = 0.$$

Note that the operators P and Q are bounded and self-adjoint in the space $\vec{H}_2^1(\Omega)$ with respect to the scalar product $E[\cdot,\cdot]$. Moreover the operator P is positive and Q is non negative, as follows from (37) and the relation

$$(43) \qquad E[Q(u), \beta] = -\langle B(u_n), \beta_n \rangle_\Gamma,$$

valid for any $u,\ \beta \in \vec{H}_2^1(\Omega)$.

From (43) it follows that the operator Q_1 belongs to the space $\mathcal{B}(H^{1/2}(\pi), \vec{H}_2^1(\Omega)$.

Obviously the study of the solutions of the system (1)-(4) can be reduced to the study of the matrix operational equation

$$(44) \quad \left[\begin{pmatrix} \frac{1}{\lambda+\mu}(\rho P + Q) & 0 \\ A_2 \Gamma_n & A^{-1} \end{pmatrix} \frac{d^2}{dt^2} + \begin{pmatrix} I - P & \frac{1}{\lambda+\mu} Q_1 \\ 0 & I \end{pmatrix}\right] \begin{pmatrix} u \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where the matrix operator act on the space $\vec{H}_2^1(\Lambda) \times L_2(\pi)$. If in (44) we change the variable $(\lambda+\mu)^{1/2}\varphi_0 = \varphi$ and afterwards we apply the operator

$$\begin{pmatrix} I & -(\lambda+\mu)^{-1} Q_1 \\ 0 & (\lambda+\mu)^{-1/2} I \end{pmatrix}$$

to the left of (44) we obtain the equivalent operational equation:

$$(45) \qquad \left(M_1 \frac{d^2}{dt^2} + M_2\right) V = 0,$$

where

$$M_1 = \begin{pmatrix} (\lambda+\mu)^{-1}(\rho P + Q - Q_1 A_2 \Gamma_n) & -(\lambda+\mu)^{-1/2} Q_1 A^{-1} \\ (\lambda+\mu)^{-1/2} A_2 \Gamma_n & A^{-1} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} I - P & 0 \\ 0 & I \end{pmatrix}$$

$$V = \begin{pmatrix} u \\ \phi_0 \end{pmatrix}$$

and the scalar function $\mathcal{G} = (\lambda+\mu)^{1/2}\phi_0$ gives the vertical displacement of the free surface of the fluid layer.

For every $u \in D(Q_1 A_2 \Gamma_n)$ the following equality holds:

$$E[Q_1 A_2 \Gamma_n(u), u] = |\nabla r|^2_\Lambda - |\nabla R|^2 \Lambda.$$

From this equality we can prove that $\widetilde{Q} := Q - Q_1 A_2 \Gamma_n$ is positive and selfadjoint in $\widetilde{H}^1_2(\Omega)$ and for every $u \in D(\widetilde{Q})$ it holds:

$$E[\widetilde{Q}(u), u] = |R|^2_\Lambda. \tag{46}$$

In the two previous equalities, r and R are the generalized solutions of the boundary problems (13) and (14) corresponding to $\tau = u_n/\Gamma$ and $\tau_0 = -u_n/\Gamma$.

If now we decompose the matrix M_1 defined in (45) in the form

$$M_1 = M_{10} + M_{11}, \tag{47}$$

where

$$M_{10} = \begin{pmatrix} (\lambda+\mu)^{-1}\rho P & 0 \\ 0 & A^{-1} \end{pmatrix}$$

$$M_{11} = \begin{pmatrix} (\lambda+\mu)^{-1}(Q - Q_1 A_2 \Gamma_n) & -(\lambda+\mu)^{1/2} Q_1 A^{-1} \\ (\lambda+\mu)^{-1/2} A_2 \Gamma_n & 0 \end{pmatrix}$$

then, taking into account (46) and the fact that $A_2 \Gamma_n(u) = R/\pi$, where R is the generalized solution of the problem (14) for $\tau = -u_n/\Gamma$, we obtain that the operators M_1, M_2 are selfadjoint in $\vec{H}^1_2(\Omega) \times L_2(\pi)$.

Moreover, M_1 and M_2 are positive and M_2 is bounded. More exactly the following equalities hold.

$$\langle M_{11}(V), V\rangle_{\vec{\Omega}\times\pi} = |\nabla(\lambda+\mu)^{-1/2}R + S|^2_\Lambda - |\nabla S|^2_\Lambda \tag{48}$$

$$\langle M_1(V), V\rangle_{\vec{\Omega}\times\pi} = \rho(\lambda+\mu)^{-1}|u|^2_{\vec{\Omega}} + |\nabla(\lambda+\mu)^{-1/2}R + S|^2_\Lambda. \tag{49}$$

Remark. It is not difficult to see that under the assumption $\frac{\lambda}{\mu}$ = const., the operators P and Q_0 do not depend on λ nor on μ. So that, if we put $(\lambda+\mu)^{-1} = \epsilon^2$, the condition give us that the operator M is an analytic operator function of the variable ϵ in a neighbourhood of $\epsilon = 0$ in Kato's sense ([**7**, XII.2]). In fact, from (49) we have that $M_1(\varepsilon)$ is a family of type B. So the problem (45) becomes the problem $\left(A^{-1}\frac{d^2}{dt^2} + I\right)$ as $\epsilon \to 0$, which corresponds to the hard bottom case wich has been studied in [**1**].

Consider Cauchy's problem for the operational equation (45) under the initial conditions

$$V|_{t=0} = \begin{pmatrix} u_0 \\ (\lambda+\mu)^{-1/2}\wp_0 \end{pmatrix}, \ V_t|_{t=0} = \begin{pmatrix} u_1 \\ -(\lambda+\mu)^{-1/2}\wp_1 \end{pmatrix} \tag{50}$$

where u_0, u_1, $\wp_0$ and $\wp_1$ appear in the initial conditions (5) and (6).

It is known from the general theory of Cauchy's problem solubility that the problems (45) and (50) could not have a solution in $\vec{H}^1_2(\Omega)\times L_2(\pi)$.

Let us make a construction which describes the space where the solution could be found.

Denote by H the Hilbert space obtained by completing the pre-Hilbert space $\vec{H}^1_2(\Omega)\times L_2(\pi)$ with the inner product

$$\langle U, V\rangle = \langle M_2 U, V\rangle_{\vec{\Omega}\times\pi} = E[(I-P)u, v] + \langle \phi_0, \phi_1\rangle_\pi \tag{51}$$

where

$$U = \begin{pmatrix} u \\ \phi_0 \end{pmatrix}, \quad V = \begin{pmatrix} v \\ \phi_1 \end{pmatrix}.$$

Obviously

$$\langle U, V\rangle = E^*[u, v] + \langle \phi_0, \phi_1\rangle_\pi.$$

Consider the vector space $\vec{L}^1_2(\Omega)$ of the generalized vector functions on Ω wich fulfill the following property: the first derivatives satisfy $E^*[u,u] \leq +\infty$. Every function in $\vec{L}^1_2(\Omega)$ also belongs to the space

$\vec{L}^1_{2,\text{loc}}(\Omega)$ of the vector functions on Ω which are locally square integrable. The mapping $(E^*[u,u])^{1/2}$ defines a seminorm in $\vec{L}^1_2(\Omega)$.

We denote by $\vec{\tilde{L}}^1{}_2(\Omega)$ the quotient space given in $\vec{L}^1_2(\Omega)$ by the subspace of constant functions.

It is not difficult to verify the equality

$$H = \vec{\tilde{L}}^1_2(\Omega) \times L_2(\pi). \tag{52}$$

Furthermore, from Korn's inequality (cf. [**4**]) we have that

$$\vec{H}^1_2(\Omega) = \vec{\tilde{L}}^1_2(\Omega) \cap \vec{L}_2(\Omega). \tag{53}$$

By a direct check it can be seen that the operator $M_2^{-1}M_1$ is symmetric in H.

Applying Lemma 3 of [**9**] to operators M_1 and M_2 we obtain that the closure $\overline{M_1^{-1}M_2}$ of the operator $M_1^{-1}M_2$ in the space $H = \vec{\tilde{L}}^1_2(\Omega) \times L_2(\pi)$ is selfadjoint and positive.

Definition 4.1. The pair (u,p) is a generalized solution of Cauchy's problem (1)-(6) if $V = (u, \phi_0)$ is a solution of Cauchy's problem for the operational equation

$$\frac{d^2V}{dt^2} + \overline{M_1^{-1}M_2}V = 0 \tag{54}$$

with the initial conditions

$$V|_{t=0} = \begin{pmatrix} u_0 \\ (\lambda+\mu)^{-1/2}\wp_0 \end{pmatrix},\ V|_{t=0} = \begin{pmatrix} u_1 \\ -(\lambda+\mu)^{-1/2}\wp_1 \end{pmatrix}. \tag{55}$$

Remark. Notice that from the solution V of Cauchy's problem (54)-(55), the hydrodynamic pressure p in the problems (1)-(6) can be reconstructed. In fact, when V is obtained, the auxiliary boundary problems (12) and (13) with the conditions

$$q/\pi = \phi_0,\ \left.\frac{\partial r}{\partial n}\right|_\Gamma = \frac{\partial^2 u_n}{\partial t^2}$$

are solved.

As a result we obtain the functions q and r, so $p = q + r$.

From the general theory of solubility of Cauchy's problem for abstract hyperbolic equations we conclude (cf. [**7**]) that, if we suppose in (55) that $V|_{t=0} \in D(\overline{M_1^{-1}M_2})$ and $V_t|_{t=0} \in D(\overline{[M_1^{-1}M_2]}^{1/2}) \cap R(\overline{[M_1^{-1}M_2]}^{1/2})$, then the problem (54) and (55) has unique solution $V(t)$, such that $V(t) \in D(\overline{M_1^{-1}M_2})$ has two continuous derivatives and satisfies equation (54) for every $t \in [0,T]$. Besides, the function $V'(t)$ has an image in $[D(\overline{M_1^{-1}M_2}]1/2$ and the function $\overline{[M_1^{-1}M_2]}^{1/2}V'(t)$ is continuous on $[0,T]$.

The solution $V(t)$ of the problem (54)-(55) depends continuously on the initial data and it is expressed by means of the formula

$$\begin{aligned} (56) \quad V(t) = (\cos t\overline{[M_1^{-1}M_2]}^{1/2})\, V|_{t=0} \\ + (\sin t\overline{[M_1^{-1}M_2]}^{1/2})\overline{[M_1^{-1}M_2]}^{-1/2}\, V_t|_{t=0}\,. \end{aligned}$$

5. The spectral problem of the normal oscillations of the system (1)-(4)

Let us write $\rho_0 = g = 1$ as in the last paragraph. The solutions of problem (1)-(4) will be found in the form of travelling waves of the type

$$u(X;t) = e^{i(wt-\alpha x_1)}v(x_2,x_3);\ p(X;t) = e^{i(wt-\alpha x_1)}q(x_2,x_3);\ \alpha, w \in R.$$

Replacing these expressions in (1)-(4), we set down the following spectral problem in the parameters α and w:

$$(57) \qquad [\alpha^2 A - \alpha B + C](c) = \rho w^2 v, \qquad (\Omega_0)$$

$$(58) \qquad \Delta q - \alpha^2 q = 0, \qquad (\Lambda_0)$$

$$(59) \qquad [M + i\alpha N]\,(v) = q\vec{n},\ \frac{\partial q}{\partial n} = -w^2 v_n, \quad (\Gamma_0)$$

$$(60) \qquad \frac{\partial q}{\partial x_3} - w^2 q = 0, \qquad (\pi_0)$$

where $\vec{n}$ denotes the unitary normal exterior vector to Γ_0, Δ represents the Laplace operator for to the variables x_2 and x_3, and the coefficients of (57) and (59) are matrix differential expressions:

$$A := \begin{pmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$$B := i(\lambda + \mu) \begin{pmatrix} 0 & D_2 & D_3 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \end{pmatrix}$$

$$C := - \begin{pmatrix} \mu\Delta & 0 & 0 \\ 0 & \mu\Delta + (\lambda+\mu)D_2^2 & (\lambda+\mu)D_2D_3 \\ 0 & (\lambda+\mu)D_2D_3 & \mu\Delta + (\lambda+\mu)D_3^2 \end{pmatrix}$$

$$N := \begin{pmatrix} 0 & \mu n_2 & \mu n_3 \\ \lambda n_2 & 0 & 0 \\ \lambda n_3 & 0 & 0 \end{pmatrix}$$

$$M := \begin{pmatrix} \mu(n_2D_2+n_3D_3) & 0 & 0 \\ 0 & (\lambda+2\mu)n_2D_2+\mu n_3D_3 & \lambda n_2D_3+n_3D_2 \\ 0 & \lambda n_3D_2+\mu n_2D_3 & \mu n_2D_2+(\lambda+2\mu)n_3D_3 \end{pmatrix}$$

$$\Delta := D_2^2 D_3^2;\ D_k := \frac{\partial}{\partial x_k};\ k = 2, 3;\ \vec{n} = (n_2, n_3).$$

Let α be a fixed real number. We shall study the weak solutions $q \in H_2^1(\Lambda_0)$, $v \in \vec{H}_2^1(\Lambda)$ of the spectral problems (57)-(60) associated to w.

We shall define the bilinear functional E_0 for vector functions u and v smooth enough on $\overline{\Omega_0}$

$$E_0[u,v] := \int_{\Omega_0} \varepsilon_0(u,v)dX + \langle u, v\rangle_{\vec{\Omega}_0}$$

where

$$\begin{aligned} \varepsilon_0(u,v) &:= \lambda \operatorname{div}' u \overline{\operatorname{div}' v} + \mu(D_2u_1\overline{D_2v_1} + D_3u_1\overline{D_3v_1}) \\ &\quad + \frac{\mu}{2} \sum_{i,k=2}^{3} (D_ku_i + D_iu_k)(\overline{D_kv_i + D_iv_k}), \\ \operatorname{div}' u &:= D_2u_2 + D_3u_3,\ u = (u_1, u_2, u_3). \end{aligned}$$

Henceforth, the so-called second Korn's inequality (cf. [**4**])

$$\|v\|^2_{\vec{H}_2^1(\Omega_0)} \leq CE_0[v,v],\ v \in H_2^1(\Omega_0),\ C > 0$$

tells us that the expression $(E_0[v,v])^{1/2}$ defines an equivalent norm to the usual Sobolev norm on the space $\vec{H}_2^1(\Omega_0)$. Let us introduce the auxiliary problems

$$\Delta q_1 - \alpha^2 q_1 = 0(\Lambda_0), \quad q_1/\pi_0 = \phi, \quad \left.\frac{\partial q}{\partial n}\right|_{\Gamma_0} = 0, \tag{61}$$

$$\Delta q_2 - \alpha^2 q_2 = 0(\Lambda_0), \quad q_2/\pi_0, \quad \left.\frac{\partial q_2}{\partial n}\right|_{\Gamma_0} = \tau, \tag{62}$$

$$Cv_1 + v_1 = f(\Omega_0), \qquad Mv_1|_{\Gamma_0} = 0, \tag{63}$$

$$Cv_2 + v_2 = 0(\Omega_0), \qquad Mv_2|_{\Gamma_0} = \tau, \tag{64}$$

$$\begin{aligned}
&q_1 \in H_2^1(\Lambda_0),\\
&q_2 \in H_2^1(\Lambda_0) = \{q \in H_2^1(\Lambda_0);\ q/\pi_0 = 0\},\\
&v_i \in \vec{H}_2^1(\Omega_0),\ i = 1, 2, \dots ,
\end{aligned}$$

The weak solutions are defined in the following manner

$$q_1/\pi_0 = \phi,\ \langle \nabla q_1, \nabla r\rangle_{\Lambda_0} + \alpha^2 \langle q_1, r\rangle_{\Lambda_0} = 0, \qquad \forall r \in \widetilde{H}_2^1(\Lambda_0) \tag{65}$$

$$\langle \nabla q_2, \nabla r\rangle_{\Lambda_0} + \alpha^2 \langle q_2, r\rangle_{\Lambda_0} = \langle \psi, r\rangle_{\Gamma_0},\ \ \forall r \in \widetilde{H}_2^1(\Lambda_0) \tag{66}$$

$$E_0[v_1, w] = \langle f, w\rangle_{\vec{\Omega_0}},\ \forall w \in \vec{H}_2^1(\Omega_0) \tag{67}$$

$$E_0[v_2, w] = \langle \tau, w\rangle_{\vec{\Gamma_0}},\ \forall w \in \vec{H}_2^1(\Omega_0). \tag{68}$$

Hence forward we shall denote by $V_\alpha[q,r]$ the bilinear functional $\langle \nabla q, \nabla r\rangle_{\Lambda_0} + \alpha^2 \langle q, r\rangle_{\Lambda_0}$. The definitions (65)-(68) correspond in a natural way to Green's formulas for Δ and C:

$$\begin{aligned}
\langle \Delta q, r\rangle_{\Lambda_0} &= \left\langle \frac{\partial q}{\partial x_3}, r\right\rangle_{\pi_0} - \left\langle \frac{\partial q}{\partial n}, r\right\rangle_{\Gamma_0} - \langle \nabla q, \nabla r\rangle_{\Lambda_0},\\
\langle Cv + v, w\rangle_{\vec{\Omega}_0} &= E_0[v,w] - \langle M(v)\cdot \vec{n}, w\rangle_{\vec{\Gamma}_0},
\end{aligned}$$

which are valid for scalar complex functions q, r and vector functions v, w smooth enough, defined on $\overline{\Lambda}_0$ and $\overline{\Omega}_0$, respectively.

Let us observe that (61) is the Euler equation for the functional $V_\alpha[q_1, q_1]$. Thus, from the general theory of variational methods for the solution of elliptic boundary problems, we deduce the existence and uniqueness of a weak solution of the problem (61) for every $\phi \in H_2^{1/2}(\pi_0)$. Such solutions are a minimum for the functional V_α over the class $H_2^1(\Lambda_0)$, whose trace in π_0 is exactly ϕ. Existence and unicity of weak solutions of the problems (62)-(64), when the functions $\phi \in L_2(\Gamma_0)$, $f \in \vec{L}_2(\Omega_0)$, $r \in \vec{L}_2(\Gamma_0)$, are possible to prove using the Riesz theorem for Hilbert spaces.

We shall find the weak solutions (q, v) of the problems (57)-(60) in the form $q = q_1 + q_2$, $v = v_1 + v_2$, where q_i, v_i, $i = 1, 2$ are weak solutions of the auxiliary problems (61)-(64) when the right side function ϕ, ψ, f and τ are conveniently chosen.

Now let us define some operators related to the operational statement of the spectral problem (57)-(60).

If the function ϕ, given by the problem (61), is in the space $C_0^\infty(\pi_0)$, then $q_1 \in H_2^2(\Lambda_0)$ and hence we can define an operator according to the formula

$$A_\alpha^0(\phi) := \left.\frac{\partial q_1}{\partial x_3}\right|_{\pi_0}$$

where $C_0^\infty(\pi_0)$ is the domain of definition.

Integrating by parts the expression $\langle \Delta q_1 - \alpha^2 q_1, q_1\rangle_{\Lambda_0}$ we deduce the relation

$$\langle A_\alpha^0 \phi, \phi\rangle_{\pi_0} = V_\alpha[q_1, q_1],\ \forall \phi \in C_0^\infty(\pi_0) \tag{69}$$

so we deduce that the operator A_α^0 is symmetric and positive for every $\alpha \in R$.

Let us denote the closure of the operator A_α^0 by means of A_α. In the same way as for the operator A from (21), we can prove that A_α is a self-adjoint and positive operator in the space $L_2(\pi_0)$, whose domain of definition is $H_2^1(\pi_0)$ and $D(A_\alpha^{1/2}) = H_2^{1/2}(\pi_0)$.

Relation (69) also takes place for the operator A_α and for every $\phi \in D(A_\alpha)$. These arguments ensure that the operator A_α^{-1} is bounded when $\alpha \in R/\{0\}$.

Also define the operator B_α in $L_2^1(\Gamma_0)$, according to the formula $B_\alpha(\psi) = q_2/\Gamma_0$. Obviously, the operator B_α is bounded and negative in $L_2(\Gamma_0)$ and also satisfies the relation

$$\langle B_\alpha \psi, \psi\rangle_{\Gamma_0} = -V_\alpha[q_2, q_2], \forall \psi \in L_2(\Gamma_0). \tag{70}$$

By using immersion theorems it is easy to see that the operator $B_{1,\alpha}(\phi) = q_1/\Gamma_0$, is bounded between the space $H_2^{1/2}(\pi_0)$ and $H_2^{1/2}(\Gamma_0)$.

Consider the operator $A_\alpha^{(1)}$ whose definition domain is $H_2^{1/2}(\Gamma_0)$ and ranges over $L_2(\pi_0)$ given by

$$A_\alpha^{(1)}(\psi) := \left.\frac{\partial q_2}{\partial x_3}\right|_{\pi_0}.$$

Doing again the same analysis which we did for the operator defined in (34), we can verify that the operator $A_{1,\alpha}^{(1)} := A_\alpha^{-1}A_\alpha^{(1)}$ satisfies $A_{1,\alpha}(\theta) = -q_3/\pi_0$ where q_3 is the weak solution of the boundary problem

$$\Delta q_3 - \alpha^2 q_3 = 0(\Lambda_0),\ \left.\frac{\partial q_3}{\partial x_3}\right|_{\pi_0} = 0,\ \left.\frac{\partial q_3}{\partial n}\right|_{\Gamma_0} = \theta \tag{71}$$

in the sense that

$$\langle\theta, r\rangle_{\Gamma_0} = -V_\alpha[q_3, r],\ \forall\tau \in H_2^1(\Lambda_0).$$

Finally, similar to those auxiliary problems (16) and (17), the weak solutions of problems (63) and (64) define bounded operators $P_0 : \vec{L}_2(\Omega) \to \vec{H}_2^1(\Omega_0)$ and $Q^0 : \vec{L}_2(\Gamma_0) \to \vec{H}_2^1(\Omega_0)$ such that $P_0 f := v_1, Q^0\tau := v_2$.

The operator P_0 is still self-adjoint and positive in $\vec{L}_2(\Omega_0)$ when we consider the inner product $E_0[\cdot,\cdot]$. Furthermore, $D(P_0^{-1/2}) = \vec{H}_2^1(\Omega_0)$ and for every u, $v \in \vec{H}_2^1(\Omega_0)$ the relation

$$E_0[u, v] = \langle P_0^{-1/2}u, P_0^{-1/2}v\rangle_{\vec{\Omega}_0} \tag{72}$$

holds.

Let (q, v) be the weak solution of the problem (57)-(60). Represent the function q in the form $q = q_1+q_2$ where q_1 and q_2 are weak solutions of the auxiliary problems (61) and (62), corresponding to $\phi = q/\pi_0 \in H_2^1(\pi_0)$ and $\psi = -w^2 v_n$. Then

$$\left.\frac{\partial q}{\partial n}\right|_{\Gamma_0} = \left.\frac{\partial q_2}{\partial n}\right|_{\Gamma_0},\ q/\Gamma_0 = q_1/\Gamma_0 + q_2/\Gamma_0 = B_{1,\alpha}(\phi) - w^2 B_\alpha(v_n)$$
$$\left.\frac{\partial q}{\partial x_3}\right|_{\pi_0} = \left.\frac{\partial q_1}{\partial x_3}\right|_{\pi_0} + \left.\frac{\partial q_2}{\partial x_3}\right|_{\pi_0} = A_\alpha(\phi) - w^2 A_\alpha^{(1)}(v_n).$$

From the above equalities we deduce that the equations (58)-(60) can be written in the equivalent operational form

$$[M + i\alpha N](v) = B_{1,\alpha}(\phi)\vec{n} - w^2 B_\alpha(v_n)\vec{n}, \tag{73}$$
$$\phi - w^2[A_\alpha^{-1}(\phi) - w^2 A_{1,\alpha}(v_n)] = 0. \tag{74}$$

If in the problems (63) and (64) we take

$$f = [-\alpha^2 A + \alpha B + I + \rho\omega^2 I](v)$$
$$\tau = B_{1,\alpha}(\phi)\vec{n} - w^2 B_\alpha(v_n)\vec{n} - i\alpha N(v),$$

where (q, v) is the weak solution of (57)-(60), we obtain the following relation for the function v:

$$\begin{aligned} v - P_0 f - Q^0\tau = v - P_0[I - \alpha^2 A + \alpha(B - iP_0^{-1}Q^0 NT) \\ + \rho\omega^2 I](v) - Q^0[(\cdot\vec{n})B_{1,\alpha}(\phi) - \omega^2(\cdot\vec{n})B_\alpha T_n(v)] = 0. \end{aligned} \tag{75}$$

In this relation T is the trace operator on Γ_0 defined in $\vec{H}_2^1(\Omega_0)$, T_n is the trace operator multiplied by the normal vector $\vec{n}$ and $(\cdot\vec{n})$ is the scalar product operator by the normal vector.

If in (75) we make the substitution $u = P_0^{-1/2}(v) \in \vec{L}_2(\Omega_0)$ and we operate at the left side by means of $P_0^{-1/2}$, we obtain the equations

$$L(\alpha)u := [\alpha^2 A_0 - \alpha B_0 + i - P_0](u) = \omega^2 K_\alpha(u) - R_\alpha(\phi), \tag{76}$$

being

$$A_0 = P_0^{1/2} A P_0^{1/2}, B_0 = P_0^{1/2} B P_0^{1/2} - iP_0^{-1/2} Q^0 NT P_0^{-1/2}$$
$$K_\alpha = \rho P_0 - P_0^{-1/2} Q^0(\cdot\vec{n}) B_\alpha T_n P_0^{-1/2}, R_\alpha = -P_0^{-1/2} Q^0(\cdot\vec{n}) B_{1,\alpha}.$$

Writing the equation (74) by using the new variables we obtain:

$$\phi = \omega^2[A_\alpha^{-1}(\phi) + S_\alpha(u)] \tag{77}$$

where

$$S_\alpha = A_{\alpha,1} T_n P_0^{1/2}.$$

Thus, we have posed the spectral problem (76) and (77) related to the spectral parameter ω^2, which can be expressed in the matrix form

$$\begin{pmatrix} L(\alpha) & R(\alpha) \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ \phi \end{pmatrix} = \omega^2 \begin{pmatrix} K_\alpha & 0 \\ S_\alpha & A_\alpha^{-1} \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix}. \tag{78}$$

Applying the results of the lemma in [**10**] we can deduce that the operators B_0 and K_α given in (76), are self-adjoint in $\vec{L}_2(\Omega_0)$. Besides K_α is positive and the operator R_α is $-A_\alpha S_\alpha$.

It is not difficult to verify that for $\alpha \in R$, $|\alpha|$ sufficiently large, the operator $L(\alpha)$ is positive ($L(\alpha) \gg 0$). Thus when α is chosen as above there exists the operator $L(\alpha)^{1/2}$ and it is invertible.

If in (78) we make the change of variables $L(\alpha)^{1/2}u = w$ and we operate on the left side by means of the invertible matrix operator

$$\begin{pmatrix} L(\alpha)^{1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} K_\alpha^{-1} & 0 \\ -A_\alpha S_\alpha K_\alpha^{-1} & A_\alpha \end{pmatrix}$$

we obtain a spectral problem equivalent to (78) for the operator $\mathcal{M}(\alpha)$ in the space $\vec{L}_2(\Omega_0) \times L_2(\pi_0)$:

$$\mathcal{M}(\alpha) \begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \omega^2 \begin{pmatrix} \omega \\ \varphi \end{pmatrix} \tag{79}$$

where

$$\mathcal{M}(\alpha) := \begin{pmatrix} L(\alpha)^{1/2} & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} K_\alpha^{-1} & K_\alpha^{-1} R_\alpha \\ -A_\alpha S_\alpha K_\alpha^{-1} & A_\alpha - A_\alpha S_\alpha K_\alpha^{-1} R_\alpha \end{pmatrix} \begin{pmatrix} L(\alpha)^{1/2} & 0 \\ 0 & I \end{pmatrix}.$$

Notice that the operator $\mathcal{M}(\alpha)$ is self-adjoint.

If we write (74) in the form

$$\begin{pmatrix} \omega \\ \varphi \end{pmatrix} = \omega^2 \mathcal{M}(\alpha)^{-1} \begin{pmatrix} \omega \\ \varphi \end{pmatrix}$$

make the substitution $w = L(\alpha)^{1/2}u$, $\varphi = A_\alpha^{1/2}(\psi)$ and we operate on the left side by means of

$$\begin{pmatrix} L(\alpha)^{1/2} & 0 \\ 0 & A_\alpha^{1/2} \end{pmatrix}$$

we obtain the following equivalent spectral problem,

$$\begin{pmatrix} L(\alpha) & 0 \\ 0 & A_\alpha \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix} = \begin{pmatrix} K_\alpha - R_\alpha S_\alpha & -R_\alpha A_\alpha^{1/2} \\ A_\alpha^{1/2} S_\alpha & I \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix} \tag{80}$$

where both matrix operators are self-adjoint in $\vec{L}_2(\Omega_0) \times L_2(\pi_0)$.

6. Spectra of the operator $\mathcal{M}(\alpha)$ and existence of wave guides in the problem (1)-(4)

Let us denote by $\mathcal{M}^{(h)}(\alpha)$ and $\mathcal{M}^{(0)}(\alpha)$ the operator $\mathcal{M}(\alpha)$ in the cases of irregular and plane bottom respectively. A simple application of perturbation theory proves that for every $\alpha \in R$, the limit spectra $\sigma_\pi(\mathcal{M}^{(h)}(\alpha))$ and $\sigma_\pi(\mathcal{M}^{(0)}(\alpha))$ of the operators $\mathcal{M}^{(h)}(\alpha)$ and $\mathcal{M}^{(0)}(\alpha)$ coincide.

Theorem 6.1. *When $\alpha \in R$ has a sufficiently large modulus the following inclusions hold:*

$$\sigma_\pi(\mathcal{M}^h(\alpha)) \supset \sigma_\pi(A^h_\alpha) \supset \sigma(\mathcal{L}^h_\alpha). \tag{81}$$

Besides,

$$\inf \sigma(\mathcal{M}^{(h)}(\alpha) \leq \inf \sigma(A^{(h)}_\alpha). \tag{82}$$

Proof: Observe that the equality

$$\inf \sigma(\mathcal{M}(\alpha)) = \inf \frac{\langle L(\alpha)u, u\rangle_{vec\Omega_0} + \langle A_\alpha \psi, \psi\rangle_{\pi_0}}{\rho\langle P_0 u, u\rangle_{\vec{\Omega}_0} + \langle K_{0,\alpha} u, u\rangle_{\vec{\Omega}_0} + \mid W_\alpha u + \psi \mid^2_{\pi_0}}, \tag{83}$$

takes place for an arbitrary operator $\mathcal{M}^{(h)}(\alpha)$. So the first inclusion in (81) can be deduced from the equality of the limit spectra of the operators $\mathcal{M}^{(h)}(\alpha)$ and $\mathcal{M}^{(0)}(\alpha)$ by assuming $h = 0$ in (83) and bearing in mind that $K_{0,\alpha} \geq 0$ and the infimum of the right side of (83) is smaller than the infimum ranging over all the functions of the type $(0, \psi)^t$.

The above reasoning for an arbitrary h shows the validity of the inequality (82). Observe that for $\epsilon = 1/\alpha$, $1/\alpha^2 L(\alpha)$ can be written in the form $\mathcal{A}_0 - \epsilon\mathcal{B}_0 + \epsilon^2(I - P_0) = \mathcal{M}(\epsilon)$.

For $\epsilon = 0$ we easily obtain $M(0) > \mu/2P_0$. The expression of $L(\alpha)$ is given in terms of bounded operators, so the inequality $M(\epsilon) > \mu/2P_0$ is valid for ϵ sufficiently small.

From the above statements we deduce

$$L(\alpha) \geq (\mu/2)\alpha^2 P_0, \quad \alpha \in \Re, \quad \mid \alpha \mid \gg 0.$$

Such real numbers α satisfy

$$\inf \sigma(\mathcal{L}^{(h)}_\alpha) = \inf_{u \in \vec{H}^1_2(\Omega^{(h)}_0)} \frac{\langle L(\alpha)u, u\rangle_{\vec{\Omega}^{(h)}_0}}{\rho\langle P_0 u, u\rangle_{\vec{\Omega}^{(h)}_0}} \geq \frac{\mu}{2\rho}\alpha^2.$$

Then the second inclusion in (81) is obtained from the equality

$$\sigma_\pi(A^{(h)}_\alpha) = [\mid \alpha \mid an \mid \alpha \mid, +\infty[. \quad \blacksquare$$

It is known (see [**1**]) that when the bottom is hard the condition $\min h(x_2) < 1$ is sufficient for the existence of traveling waves which are propagated troughout the underwater ridge, that is, for the existence of eigenvalues $\omega_k(\alpha)$ of the operator $A^{(h)}_\alpha$ outside its limit spectrun. If the elastic properties of the bottom are considered, it is necesary to introduce stronger constraints to the bottom geometry to obtain similar results.

Theorem 6.2. *Suppose that the function h satisfies the required conditions of Section 1 and further* $\min h(x_2) < 1/2$. *Then for every α whose modulus is sufficiently large, the operator $\mathcal{M}^{(h)}(\alpha)$ has at least an isolated eigenvalue $\omega(\alpha)$.*

Proof: It is sufficient to prove that

$$\inf \sigma(\mathcal{M}^{(h)}(\alpha)) < \inf \sigma_\pi(\mathcal{M}^{(h)}(\alpha)). \tag{84}$$

Suppose for convenience that $\alpha \gg 0$.

We can deduce (84) from (82) provided we can prove that for $\alpha \gg 0$ the following inequality holds

$$\alpha \tanh \alpha - \inf \sigma(A_\alpha^{(h)}) > \alpha \tanh \alpha - \inf \sigma_\pi(\mathcal{M}^{(h)}(\alpha)). \tag{85}$$

It is known that the following equality is valid

$$\inf \sigma(A_\alpha^{(h)}) = \inf \left\{ V_\alpha[q_1] : q_1 \in H_2^1(\Lambda_0^{(h)}),\, q_1/\pi_0 = \varphi, |\, \varphi\,|_{\pi_0} < 1 \right\}. \tag{86}$$

Consider $x_2 = x$, $x_3 = z$ and define the function $q(x,z) = \sqrt{a} e^{-a|x|} g(z)$ where

$$g(z) = \frac{1}{\cosh \alpha} \begin{cases} \cosh \alpha(z+1), & -1 < z < 0 \\ 1, & z < -1. \end{cases}$$

Denote by $m = \min h(x)$, $M = \max h(x)$, and let $\epsilon > 0$ such that $m + \epsilon < 1/2$.

Then we have the estimate

$$\begin{aligned} V_\alpha[q] \le \alpha \tanh \alpha + \frac{a^2}{2} \left(\frac{\tanh \alpha}{\alpha} + \frac{1}{\cosh^2 \alpha} \right) \\ - a \left[\frac{k_1 \alpha}{w \cosh^2 \alpha} \sinh 2\alpha(1 - m - \epsilon) - k_2 \frac{\alpha^2 + 1}{\cosh^2 \alpha} \right] \end{aligned} \tag{87}$$

where the constant k_1, only depends on ϵ and k_2 depends on ϵ and M, moreover $k_2 = 0$ if $M = 1$. The term between brackets in (87) can be made positive for α sufficiently large. For such α we have the estimate

$$V(\alpha)[q] \le \alpha \tanh \alpha - c(\alpha), \tag{88}$$

where $c(\alpha) > 0$ and

$$1/2 c(\alpha) k_1^2 \alpha^{-3} e^{4(m+\epsilon)\alpha} \to 1 (\alpha \to \infty). \tag{89}$$

Now from (86) and (88) we obtain the estimate

$$\alpha \tanh \alpha - \inf \sigma(A_\alpha^{(h)}) \geq c(\alpha). \tag{90}$$

Let us get an upper estimate for the right term in (85) .

Obviously, for $h \equiv 1$ we have

$$\sigma(\mathcal{M}^{(0)}(\alpha)) = \sigma_\pi(\mathcal{M}^{(0)}(\alpha)). \tag{91}$$

From (91) and the equality $\sigma_\pi(\mathcal{M}^{(h)}(\alpha)) = \sigma_\pi(\mathcal{M}^{(0)}(\alpha))$ we have

$$\sigma(\mathcal{M}^{(0)}(\alpha)) = \sigma_\pi(\mathcal{M}^{(h)}(\alpha)). \tag{92}$$

Relation (92) allows us to calculate $\inf \sigma_\pi(M^{(h)}(\alpha))$ according to the equality (83) with index (0). Thus from (83) the inequality

$$\begin{aligned} &\inf \sigma_\pi(\mathcal{M}^{(h)}(\alpha)) \\ &\geq \inf \frac{\langle L(\alpha)u, u\rangle_{\vec{\Omega}_0^{(0)}} + \langle A_\alpha^{(0)}\psi, \psi\rangle_{\pi_0^{(0)}}}{\rho\langle P_0 u, u\rangle_{\vec{\Omega}_0^{(0)}} + \langle K_{0,\alpha}u, u\rangle_{\vec{\Omega}_0^{(0)}} + (\mid W_\alpha(u) \mid_{\pi_0^{(0)}} + \mid \psi \mid_{\pi_0^{(0)}})^2} \end{aligned} \tag{93}$$

is obtained where the operators in the right hand side are with index (0). Now let us use the inequality

$$2|W_\alpha u|_{\pi^{(0)}} |\psi|_{\pi^{(0)}} \leq \frac{|W_\alpha u|^2_{\pi^{(0)}}}{\epsilon^2(\alpha)} + |\psi|^2_{\pi^{(0)}} \epsilon^2(\alpha) \tag{94}$$

where the number $\epsilon(\alpha)$ will be chosen below.

From (93) by using (94) we obtain:

$$\begin{aligned} &\inf \sigma_\pi(\mathcal{M}^{(h)}(\alpha)) \geq (1+\epsilon^2(\alpha))^{-1} \\ &\inf \frac{\langle L(\alpha)u, u\rangle_{\vec{\Omega}_0^{(0)}} + \langle A_\alpha^{(0)}\psi, \psi\rangle_{\pi_0^{(0)}}}{\rho\langle P_0 u, u\rangle_{\vec{\Omega}_0^{(0)}} + \langle K_{0,\alpha}u, u\rangle_{\vec{\Omega}_0^{(0)}} + \mid \psi \mid_{\pi_0^{(0)}} + (1+\epsilon^{-2}(\alpha)) \mid W_\alpha(u) \mid^2_{\pi_0^{(0)}}}. \end{aligned} \tag{95}$$

Now we observe that the infimum taken on the right side of (95) is exactly the infimum of the spectrum of the following problem:

$$\begin{aligned} &\begin{pmatrix} L(\alpha) & 0 \\ 0 & A_\alpha^{(0)} \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix} \\ &\qquad = \omega^2 \begin{pmatrix} \rho P_0 + K_{0,\alpha} + (1+\epsilon^{-2}(\alpha))W_\alpha^* W_\alpha & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ \psi \end{pmatrix} \end{aligned} \tag{96}$$

where ω^2 is the spectral parameter.

But the spectrum σ_0 of problem (96) is the union of the spectra of the problems given below

$$L(\alpha)u = \omega^2[\rho P_0 + K_{0,\alpha} + (1+\epsilon^{-2}(\alpha))W_\alpha^* W_\alpha]u, \tag{97}$$

$$A_\alpha^{(0)}\psi = \omega^2\psi. \tag{98}$$

From the definition of the operators $K_{0,\alpha}$ and W_α when the bottom is flat ($h=1$), the following estimates are obtained

$$\langle K_{0,\alpha}u, u\rangle_{\vec{\Omega}_0^{(0)}} \le \frac{k_3}{\alpha}|u|^2_{\vec{\Omega}_0^{(0)}},\ |W_\alpha u|_{\pi^{(0)}} \ge k_1\frac{e^{-2\alpha}}{\alpha}|u|^2_{\vec{\Omega}_0^{(0)}}. \tag{99}$$

Let us put $\epsilon(\alpha) = e^{-2\alpha}$. Then from (82), (99) and the fact that the infimum of the spectrum σ_1 of problem (97) coincides with the expression

$$\inf \frac{\langle L(\alpha)u, u\rangle_{\vec{\Omega}_0^{(0)}}}{\rho\langle P_0 u, u\rangle_{\vec{\Omega}_0^{(0)}} + \langle K_{0,\alpha}u, u\rangle_{\vec{\Omega}_0^{(0)}} + (\mid W_\alpha u\mid^2_{\pi_0^{(0)}}(1+\epsilon^{-2}(\alpha))}$$

we obtain the estimates

$$\inf \sigma_1 \ge k_4\alpha^2,\ \alpha > 0. \tag{100}$$

On the other hand, it is known (see [**1**]), that

$$\sigma(A_\alpha^{(0)}) = \sigma_\pi(A_\alpha^{(0)}) = \sigma_\pi(A_\alpha^{(h)}) = [\alpha \tanh\alpha, +\infty[. \tag{101}$$

Thus, from (100) and (101) we deduce that if $\alpha \gg 0$ then

$$\sigma_0 = [\alpha\tanh\alpha, +\infty[. \tag{102}$$

From the first inclusion given by (81) and from (102) and (95) the estimates given below are obtained

$$(1+\epsilon^{-2\alpha})^{-1}\alpha\tanh\alpha \le \inf\sigma_\pi(\mathcal{M}^h(\alpha)) \le \alpha\tanh\alpha. \tag{103}$$

From (103) we obtain the inequality

$$\alpha\tanh\alpha - \inf\sigma_\pi(\mathcal{M}^h(\alpha)) \le \alpha\tanh\alpha[1-(1+\epsilon^{-2\alpha})^{-1}] \le \alpha e^{-2\alpha}. \tag{104}$$

From (84) and taking into account that $m+\epsilon < 1/2$, we have

$$\lim_{\alpha\to+\infty}\frac{\alpha e^{-2\alpha}}{c(\alpha)} = 0.$$

Thus, from (90) and (94) we deduce (85) for $\alpha \gg 0$, so that the theorem is proved. ■

Remark. Analyzing the specific form of the operators considered in the definition of $M^{(h)}(\alpha)$, we deduce that not every eigenfunction of the operator $M^{(h)}(\alpha)$ has the type $(0,\varphi)^t$. In fact, from this assumption and from (77), every isolated eigenvalues of $M^{(h)}(\alpha)$ should also be an isolated eigenvalue of $A_\alpha^{(h)}$. Bu it is easy to verify that it is not valid in general, at least for specific profile of the bottom.

7. Asymptotic solution of the Cauchy problem (1)-(6)

Let us make a change of variable $q = i\omega\phi$ in the problem (57)-(60) and number the isolated eigenvalues of the operator $\mathcal{M}^{(h)}(\alpha)$ so obtained by taking into account their multiplicities:

$$\omega_k(\alpha),\ k = 1, \dots, n(\alpha) \leq \infty,\ \alpha \in R,\ |\alpha| \gg 0.$$

If we consider the complex conjugate of the problem (57)-(60), it is easy to see that the following relations are satisfied

$$\begin{aligned} \omega_k(-\alpha) &= -\omega_k(\alpha) \\ \overline{v_k(\alpha, x_2, x_3)} &= v_k(-\alpha, x_2, x_3) \\ \overline{\phi_k(\alpha, x_2, x_3)} &= \phi_k(-\alpha, x_2, x_3) \end{aligned} \tag{105}$$

for every real α, $|\alpha| \gg 0$.

It is not difficult to prove that the operator $[\mathcal{M}^{(h)}(\alpha)]^{-1}$ can be expressed in the following way

$$[\mathcal{M}^{(h)}(\alpha)]^{-1} = \begin{pmatrix} L(\alpha)^{-1/2} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} K_\alpha - R_\alpha S_\alpha & -R_\alpha A_\alpha^{-1} \\ S_\alpha & A_\alpha^{-1} \end{pmatrix} \begin{pmatrix} L(\alpha)^{-1/2} & 0 \\ 0 & I \end{pmatrix}.$$

Let us represent the operator $[\mathcal{M}^{(h)}(\alpha)]^{-1}$ in the form

$$[\mathcal{M}^{(h)}(\alpha)]^{-1} = \mathcal{R}(\alpha) + \mathcal{H}(\alpha)$$

where

$$\mathcal{H}(\alpha) = \begin{pmatrix} 0 & -L(\alpha)^{-1/2} R_\alpha A_\alpha^{-1} \\ S_\alpha L(\alpha)^{-1/2} & 0 \end{pmatrix}.$$

Let $a = \inf \sigma_\pi([\mathcal{M}^{(h)}(\alpha)]^{-1})$ and $b > a$.

For every self-adjoint operator C defined in a Hilbert space H and for $\gamma < \beta$ arbitrary numbers, let us define $\pi[C](\gamma, \beta) := \dim E_C(\gamma, \beta)H$, where E_C is the spectral measure of the operator C.

Then from Lemma 9.4.1 [**8**] we obtain the inequality

$$\pi[\mathcal{R}(\alpha)](a - \|\mathcal{H}(\alpha)\|,\ b + \|\mathcal{H}(\alpha)\|) \geq \pi\left[[\mathcal{M}^{(h)}(\alpha)]^{-1}\right](a, b) \tag{106}$$

where $\|\mathcal{H}(\alpha)\|$ denotes the norm of the operator $\mathcal{H}(\alpha)$.

We have seen in the proof of Theorem 6.2 that for every real α whose modulus is sufficiently large the equality $\sigma(\mathcal{R}(\alpha)) = \sigma(A_\alpha^{-1})$ takes place.

Indeed we can find a number $k(\alpha)$, in such a way that the following estimates are valid

$$\begin{aligned} \frac{1}{a} = \inf \sigma_\pi(\mathcal{M}^{(h)}(\alpha)) \leq k(\alpha) < |\alpha| \tanh |\alpha|, \\ \|\mathcal{H}(\alpha)\| \leq \frac{1}{k(\alpha)} - \frac{1}{|\alpha| \tanh |\alpha|} \end{aligned} \tag{107}$$

for every real α whose modulus is sufficiently large.

From (107) we deduce that the part of the spectrum of the operator $\mathcal{R}(\alpha)$ which lies on the interval $(a - \|\mathcal{H}(\alpha)\|, b + \|\mathcal{H}(\alpha)\|)$ consists of a finite number of eigenvalues with finite multiplicity.

From the last statements we deduce that the left side in (106) is bounded. Therefore, the spectral multiplicity of the operator $[\mathcal{M}^h(\alpha)]^{-1}$ in the interval (a, b) is finite.

It is obvious that $\mathcal{M}^h(\alpha)$ is an analytic family related to α in Kato's sense (see [**6**, p. 24]), and hence it is possible to use the Kato-Rellix's Theorem from the regular theory of perturbations (see Theorem 12-13 of [**6**]). Combining all results above mentioned, we obtain the following theorem:

Theorem 7.1. *Let us suppose that a function h fulfills the conditions of Theorem 6.2. Then every function $\omega_k(\alpha)$ admits an analytical continuation to some subset of the real axis of the type $I_k \cup \{-I_k\}$ where $I_k =]\alpha_k, \beta_k[$. The total multiplicity of the spectrum of the operator $\mathcal{M}^{(h)}(\alpha)$ does not depend on $\alpha : n(\alpha) = r$. The functions $\omega_k(\alpha)$ are real and odd in $I_k \cup \{-I_k\}$. For each $\alpha \in I_k \cup \{-I_k\}$ eigenfunctions $(v_k(\alpha), -i\omega_k(\alpha)\phi_k(\alpha))^t$, $k = 1 \dots r$ of the problem (57)-(60) can be found which depend analytically on $\alpha \in I_k \cup -I_k$ and determine an orthonormal system in $\vec{H}_2^1(\Omega_0) \times H_2^1(\Lambda_0)$.*

Let us consider the study of the influence of the irregularity and elasticity of the bottom on the asymptotic form of the amplitude of the superficial waves in the problem (1)-(6) as $| x_1 | \sim t \to \infty$.

From now on, initial data u_0, u_1, $\mathcal{G}_0$, $\mathcal{G}_1$ will satisfy the condition $u_0 = v_0 + w_0$ where v_0 and w_0 are generalized solutions of the Cauchy problems (16) and (17); besides $u_1 \in \vec{H}_2^1(\Omega)$, $\mathcal{G}_0 \in H_2^1(\pi)$, $\mathcal{G}_1 \in L_2(\pi)$.

Under such conditions the problem (1)-(6) has only one solution in the space $\vec{\widetilde{L}}{}_2^1(\Omega) \times L_2(\pi)$. Let us denote this solution by $(u, \mathcal{G})^t$, and let us consider the vector functions defined by means of the expression

$$e^{i\omega_k(\alpha)t}(v_k(\alpha, x_2, x_3),\ -i\omega_k(\alpha)\phi_k(\alpha, x_2, x_3))^t;\ k = 1, \dots, r \tag{108}$$

for $\alpha \in I_k \cup \{-I_k\}$ and equal to zero otherwise, where v_k and ϕ_k were described in Theorem 7.1. These functions are solutions of the system which are obtained from (1)-(4), after mapping by means of the Fourier transform with respect to x_1. Let us denote such system by (1')-(4'). Because the derivatives with respect to t in (1)-(4) are of second order, we conclude that those functions which can be obtained from (108) substituting $e^{i\omega_k(\alpha)t}$ by $\sin\omega_k(\alpha)t$ and $\cos\omega_k(\alpha)t$, respectively are also solutions of the problem (1')-(4').

Any linear combination of such solutions of type

$$\left(b_k(\alpha)\cos\omega_k(\alpha)t + a_k(\alpha)\frac{\sin\omega_k(\alpha)t}{\omega_k(\alpha)}\right)\begin{pmatrix} v_k(\alpha,x_2,x_3) \\ -i\omega_k(\alpha)\phi_k(\alpha,x_2,x_3)\end{pmatrix} \tag{109}$$

are also solutions of the system (1')-(4').

Let us denote by $U_k(\alpha, x_2, x_3; t)$ the solutions of the type (109) and by $\widetilde{U}(\alpha, x_2, x_3; t)$ their inverse Fourier transform with respect to α.

Obviously, the functions $\widetilde{U}_k$, $k = 1, \dots, r$ are solutions of the problem (1)-(4), and from Theorem 7.1 we deduce that they are orthogonal to each other.

Let us find the solution of the Cauchy's problem (1)-(6) in the form

$$U(x_1, x_2, x_3; t) = \sum_{k=1}^{r} \widetilde{U}_k + \widetilde{U}_0 \tag{110}$$

where the function $\widetilde{U}_0$ is otrhogonal to the sum $\sum_{k=1}^{r} \widetilde{U}_k$. In the expansion (110), the function $\widetilde{U}_0$ corresponds to the continuous spectrum of the operator $\mathcal{M}^{(h)}(\alpha)$ and the sum corresponds to the discrete spectrum.

Let us find the coefficients $a_k(\alpha)$ and $b_k(\alpha)$ in the expression of U_k, which depend on the initial conditions (5) and (6).

Let us observe that from the orthogonality of the functions $(v_k(\alpha, x_2, x_3), -i\omega_k(\alpha)\phi_k(\alpha, x_2, x_3))^t$ in the space $\vec{H}_2^1(\Omega_0) \times H_2^1(\Lambda_0)$ and from relation (69) for the operator A_α we deduce that the functions

$$(v_k(\alpha, x_2, x_3),\ -i\omega_k(\alpha)\phi_k(\alpha, x_2, 0))^t$$

form and orthonormal system in $\vec{H}_2^1(\Omega_0) \times L_2(\pi_0)$. From this we have:

$$\begin{aligned}\sqrt{2\pi}b_k(\alpha) = &\int_{-\infty}^{\infty} e^{-i\alpha x_1} E_0[u_0(x_1,\cdot,\cdot), v_k(\alpha,\cdot,\cdot)]dx_1 \\ &+ i\omega_k(\alpha)\int\int_{\Re^2} e^{-i\alpha x_1}\mathcal{G}_0(x_1,x_2)\overline{\phi_k(\alpha,x_2,0)}dx_1dx_2.\end{aligned} \tag{111}$$

It is not difficult to show that $a_k(\alpha)$ can be obtained from (111) substituting u_0 by u_1, and $\mathcal{G}_0$ by $\mathcal{G}_1$.

Because the functions $\omega_k(\alpha)$ in relations (131) are odd, we deduce immediately that

$$a_k(\alpha) = a_k(-\alpha),\, b_k(\alpha) = b_k(-\alpha). \tag{112}$$

If we take into account the fact that the functions $\widetilde{U}_k$ are Fourier transforms of functions which vanish outside the sets $[-\beta_k, -\alpha_k] \cup [\alpha_k, \beta_k]$, we obtain an expression for $\widetilde{U}_k$ as the sum of two integrals: $\int_{\alpha_k}^{\beta_k}$ and $\int_{-\beta_k}^{-\alpha_k}$. If in the second integral we make the change of variables $\alpha \to -\alpha$, then from (112) we obtain an integral which is the complex conjugate of the first one. In this way we obtain

(113)
$$\widetilde{U}_k(x_1, x_2, x_3; t) = \frac{1}{\sqrt{2\pi}} \operatorname{Re} \int_{\alpha_k}^{\beta_k} \left\{ \left(b_k(\alpha) - i\frac{a_k(\alpha)}{\omega_k(\alpha)} \right) e^{i(\omega_k(\alpha)t + \alpha x_1)} \right.$$
$$\left. + \left(b_k(\alpha) + i\frac{a_k}{\omega_k(\alpha)} \right) e^{-i(\omega_k(\alpha)t - \alpha x_1)} \right\} \begin{pmatrix} v_k(\alpha, x_2, x_3) \\ -i\omega_k(\alpha)\phi_k(\alpha, x_2, x_3) \end{pmatrix} d\alpha.$$

Thus we have proved the following result.

Theorem 7.2. *The solutions $U_k(x_1, x_2, x_3; t)$ of the Cauchy problem (1)-(6) admits an orthogonal expansion of the type (110). The functions $\widetilde{U}_k$ are given by the expression (113) where the coefficients $b_k(\alpha)$ and $a_k(\alpha)$ are defined by means of relation (111) where it is necessary to change u_0 to u_1, and $\mathcal{G}_0$ to $\mathcal{G}_1$ when the coefficient is $a_k(\alpha)$.*

Thus we see that the initial data can be written as a sum of terms orthogonal to each other, every one of them affects the corresponding component of the solution in (110). So, for specific choices of the initial conditions some terms of the sum in (110) are zero.

Each function $\widetilde{U}_k$ in (110) describes a group of waves which are moving along the x_1-axis and they are located inside a band.

Under the assumption that $\omega_k'(\alpha) \neq 0$, $\alpha \in (\alpha_k, \beta_k)$ the rate of decreasing of the corresponding component of the displacement vector $u_k(x_1, x_2, x_3; t)$ and that of the amplitude of the superficial waves $\mathcal{G}_k(x_1, x_2, t)$ as $|x_1| \sim t \to \infty$ is determined by the real zeros of the function ω_k'' according to the principle of stationary phase.

Similarly in the case when the bottom is hard (see [**1**]) each zero of the function ω_k'' of order m in the interior of (α_k, β_k) incorporates the order $t^{-1/m+2}$ to the asymptotic behaviour.

Let us consider an example. Suppose to simplify that the initial conditions u_1 and $\mathcal{G}_1$ in (5) and (6) are zero. Then from (113) and from the relation $p = \mathcal{G}$ on π, we have:

(114)

$$\mathcal{G}_k(x_1, x_2; t) = \sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left\{\int_{\alpha_k}^{\beta_k} \phi_k(\alpha, x_2, 0)\omega_k(\alpha)b_k(\alpha)e^{i\alpha x_1}\cos\omega_k(\alpha)t d\alpha\right\}.$$

Let us study the asymptotic behaviour of the function $\mathcal{G}_k(x_1, x_2 : t)$ as $|x_1| \sim t \to \infty$, in the particular case given by the condition $w_k'(\alpha) \neq 0$ in (α_k, β_k) and $w_k''(\alpha)$ has one and only one zero of second order at some point $\alpha_0 \in R$:

$$w_k''(\alpha_0) = 0,\ w_k''(\alpha_0) = 0,\ w_k^{IV}(\alpha_0) > 0. \tag{115}$$

Applying the principle of stationary phase we obtain the next theorem.

Theorem 7.3. *Consider the Cauchy problem (1)-(6) with initical conditions u_0, $\mathcal{G}_0$, $u_1 = 0$, $\mathcal{G}_1 = 0$. Suppose that the function h satisfies the required condition in Theorem 6.2 and that the function $w_k(\alpha)$ does not have stationary points and satisfies conditions (115) at a unique point $\alpha_0 \in R$. Then the component $\mathcal{G}_k(x_1, x_2 : t)$ of the amplitude of the superficial waves, produced by the initial conditions $(u_0, \mathcal{G}_0)$, has the asymptotic expressions:*

$$\mathcal{G}_k(x_1, x_2 : t) = \alpha_0 v_k(\kappa_k t)^{-1/4}\,\mathrm{Im}\Bigg\{\phi_k(\alpha_0, x_2, 0)b_k(\alpha_0)$$
$$\times\left[e^{i\alpha_0(v_k t + x_1)}F_+\left(\frac{x_1 + c_k t}{(\kappa_k t)^{1/4}}\right) + e^{-i\alpha_0(v_k t - x_1)}F_-\left(\frac{x_1 - c_k t}{(\kappa_k t)^{1/4}}\right)\right]\Bigg\} + o(t^{1/4}),$$

as $|\, x_1 \,| \sim t \to \infty$, where the function

$$F \pm (z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(sz \pm s^4/4)}ds,$$

are entire and even, and satisfy the relations

$$\overline{F_-(z)} = F_+(\overline{z}),$$
$$F_-(z) = \frac{1-i}{\sqrt{24}}|z|^{-1/3}e^{-i\frac{3}{4}|z|^{4/3}} + 0(|z|^{-1}),$$

and besides $\kappa_k = \omega^{IV}(\alpha_0)/6$, $v_k = \frac{\omega_k(\alpha_0)}{\alpha_0}$ and $c_k = w_k'(\alpha_0)$.

Remark. Suppose that the profile h is such that the component v_k of the eigenvector $(v_k, i\omega_k(\alpha)\phi_k)$ is not zero in the holomorphic interval given in Theorem 7.1. According to the remark posed at the end of Section 6 we know that such profile exists. An asymptotic formula for $\mathcal{G}_k(x_1, x_2; t)$ is obtained in [**1**] when the depth is hard under the same conditions on $\omega_k(\alpha)$ required in Theorem 7.3. Comparing both asymptotic expressions we arrive at the conclusion that for every initial condition $\mathcal{G}_0$ in the free surface we can find and initial perturbation u_0 of the elastic bottom, so the component $\mathcal{G}_k(x_1, x_2; t)$ of the amplitude of the superficial waves decrease more rapidly than that when the bottom is hard. In fact, from $v_k(\alpha, x_2, x_3) \neq 0$ and from (111) we deduce that there exist initial conditions such that $b(\alpha) = 0$. Then, from Theorem 7.3 we have that $\mathcal{G}_k(x_1, x_2 : t) = o(t^{-1/4})$ as $|x_1| \sim t \to \infty$. Thus we see that considering the elastic properties of the bottom in the Cauchy-Poisson problem leads to some weakening of the wave conductor effect which produces an underwater ridge.

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