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CONVEXITY THEORIES 0 FIN. FOUNDATIONS

Heinrich Kleisli and Helmut Röhrl*

Abstract

In this paper we study big convexity theories, that is convexity theories that are not necessarily bounded. As in the bounded case (see [**4**]) such a convexity theory Γ gives rise to the category $\Gamma\mathcal{C}$ of (left) Γ-convex modules. This is an equationally presentable category, and we prove that it is indeed an algebraic category over $\mathcal{S}et$. We also introduce the category $\Gamma\mathcal{A}lg$ of Γ-convex algebras and show that the category $\mathcal{F}rm$ of frames is isomorphic to the category of associative, commutative, idempotent $\mathbb{D}^U$-convex algebras satisfying additional conditions, where $\mathbb{D}$ is the two-element semiring that is not a ring. Finally a classification of the convexity theories over $\mathbb{D}$ and a description of the categories of their convex modules is given.

0. Introduction

The set theory used in this paper has as its basic concepts "sets", "classes", and "conglomerates" and is described in [**1**, p. 5–8]. The class U of all sets is called the universe. Conglomerates that can be indexed by a class are said to be legitimate and may, and indeed will, be treated as a class. Since we will be dealing with classes that are equipped with some structure it is convenient to replace the term "class" by "big set" and consequently speak of, for instance, "big group" instead of "class equipped with a group structure".

In section 1 we present the necessary background definitions. The "small" versions of these definitions appeared in [**4**]. However, here we wish to deal with the universe U and certain maps from U to semirings and similar structures. The definitions presented in this section formalize the notion of absolutely convergent series —known from classical analysis— to include infinite series with huge numbers of summands

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from a prenormed semimodule over a prenormed semiring. A prenormed semimodule possessing this type of structure is called a prenormed semimodule with U-summation.

Section 2 contains several elementary results concerning prenormed semimodules with U-summation. They deal with rearranging summands, double sums, and similar issues.

Big convexity theories are introduced in section 3. Their definition is identical with the definition of N-convexity theories in [**4**, (4.1)], —except for the size. As in [**4**] we define the notions of Γ-convex modules and their homomorphisms, leading to the category $\Gamma\mathcal{C}$ of (left) Γ-convex modules. In the case of N-convexity theories the free Γ-convex modules can be described without further preparation (cf. [**4**, proof of (4.7)]). This is not so for big convexity theories. Hence a number of computational rules for Γ-convex modules have to be proved directly, as was done in [**5**, (2.4)]. The section closes with the definition of commutative convexity theories and the notion of algebras over such convexity theories.

The main result of section 4 is the algebraicity of the category $\Gamma\mathcal{C}$. It is shown by involving a well known characterization theorem ([**3**, 3.1.13]) that, in our case, reduces the issue to the existence of free objects. If A is an infinite set and $\Gamma \mid A$ stands for the "restriction" of Γ to A then the free $\Gamma \mid A$-convex module over A carries a unique Γ-convex module structure that makes it the free Γ-convex module over A. As a consequence, $\Gamma\mathcal{C}$ is an algebraic category, and the same is true for the category $\Gamma\mathcal{A}lg_c$ of associative, commutative, and unital Γ-convex algebras where Γ is any big commutative convexity theory.

In section 5 we show that the category of frames (see [**2**, p. 39]) is isomorphic to the category of associative, commutative, idempotent $\mathbb{D}^U$-algebras satisfying additional conditions (see section 5); here $\mathbb{D}$ is the two-element semiring that is not a ring.

The last section brings an enumeration of the big convexity theories over $\mathbb{D}$ and describes the category of convex modules over those convexity theories.

1. Prenormed semirings and prenormed semimodules with U-summation

Let R be a semiring (in the sense of [**7**, section 1]) and denote by R^U the big set of all maps $U \to R$ that vanish on the complement of some subset of U. Such maps will be denoted by lower case greek letters with a lower placeholder symbol, e.g. α_* or $\alpha_\square$. We shall use freely the other definitions and notions of [**4**, 1.] pertaining to R^N and apply them to

R^U. In particular, R^U is a big hemiring (under pointwise composition) as well as a big left-R, right-R semimodule.

If M is a left-R semimodule we denote by M^U the big set of all maps $U \to M$ that vanish on the complement of some subset of U. The elements of M^U will be denoted by lower case greek letters with a lower placeholder symbol, e.g. μ_* or $\mu_\square$. Again the pertinent definitions and notions of [**4**, 1.] carry over to M^U. M^U is a big left-R^U hemimodule (under pointwise composition) as well as a left-R semimodule.

Since we are mostly dealing with left structures we will call left-R semimodules from now on R-semimodules.

Next we repeat some of the definitions of [**4**, 1.] in our current setting. The notions of positive semiring, cone semiring, prenormed semiring, and prenormed semimodule can be found in [**7**].

1.1. Definition.

Let $\mathcal{C}$ be a positive semiring. By a U-*summation* for $\mathcal{C}$ is meant a pair $(S_\mathcal{C}, \sum_\mathcal{C})$ consisting of a big twosided $\mathcal{C}$-subsemimodule $S_\mathcal{C}$ of C^U and a twosided $\mathcal{C}$-homomorphism $\sum_\mathcal{C} : S_\mathcal{C} \to \mathcal{C}$ such that

(i) $\mathcal{C}^{(U)} := \{\alpha_* \in \mathcal{C}^U : \operatorname{supp}\alpha_* \text{ is finite}\}$ is contained in $S_\mathcal{C}$ and for all $\alpha_* \in C^{(U)}$ the relation $\sum_\mathcal{C}(\alpha_*) = \sum'\{\alpha_u : u \in \operatorname{supp}\alpha_*\}$ holds, where $\sum'$ stands for the usual sum of the finitely many elements in $\{\alpha_u : u \in \operatorname{supp}\alpha_*\}$;

(ii) for all $\alpha_* \in S_\mathcal{C}$ and $\beta_* \in \mathcal{C}^U$ with $\beta_* \leq \alpha_*$, β_* is in $S_\mathcal{C}$ and $\sum_\mathcal{C}(\beta_*) \leq \sum_\mathcal{C}(\alpha_*)$;

(iii) for every $\alpha_* \in S_\mathcal{C}$ and every map $\varphi : U \to U$, $\alpha_*^{\varphi^{-1}(u)}$ is in $S_\mathcal{C}$ for all $u \in U$, and the map $\sum_\mathcal{C}(\alpha_*^{\varphi^{-1}})$ given by $U \ni u \mapsto \sum_\mathcal{C}(\alpha_*^{\varphi^{-1}(u)}) \in \mathcal{C}$ is in $S_\mathcal{C}$ and satisfies $\sum_\mathcal{C}(\sum_\mathcal{C}(\alpha_*^{\varphi^{-1}})) = \sum_\mathcal{C}(\alpha_*)$;

(iv) if α_* is in $\mathcal{C}^U$ and there exists a map $\varphi : U \to U$ such that $\alpha_*^{\varphi^{-1}(u)}$ is in $S_\mathcal{C}$ for all $u \in U$ and that $\sum_\mathcal{C}(\alpha_*^{\varphi^{-1}})$ is in $S_\mathcal{C}$, then α_* is in $S_\mathcal{C}$.

Recall (see [**4**]) that, given a possibly big subset T of U, we denote by α_*^T the map given by

$$U \ni u \mapsto \begin{cases} \alpha_u & \text{if } u \in T \\ 0 & \text{if } u \notin T \end{cases}.$$

Two comments are in place concerning (1.1). Firstly, it follows from (i) and (iii), that for any $\alpha_* \in S_\mathcal{C}$ the relation $\alpha_u \leq \sum_\mathcal{C}(\alpha_*)$ holds for all $u \in U$. As a consequence of this one obtains $S_{\mathcal{C}} \cdot S_{\mathcal{C}} \subseteq S_{\mathcal{C}}$, whence $S_{\mathcal{C}}$ is a big hemiring. Secondly, one checks easily that $\operatorname{supp} \sum_{\mathcal{C}}(\alpha_*^{\varphi^{-1}}) \subseteq \varphi(\operatorname{supp} \alpha_*)$ holds, whence for any $\alpha_* \in \mathcal{C}^U$ the map $\sum_{\mathcal{C}}(\alpha_*^{\varphi^{-1}})$ is also in $\mathcal{C}^U$.

1.2. Definition.

Let R be a prenormed semiring with prenorm $\| \ \| : R \to \mathcal{C}$ where $\mathcal{C}$ is a cone semiring with twosided U-summation $(S_{\mathcal{C}}, \sum_{\mathcal{C}})$. By a *$U$-summation* for R is meant a pair $(S_R, \sum_R)$ consisting of a big twosided R-subsemimodule S_R of R^U and a twosided R-homomorphism $\sum_R : S_R \to R$ such that

(o) $\alpha_* \in R^U$ is in S_R if and only if $\|\alpha_*\|$ is in $S_{\mathcal{C}}$;

(i) $R^{(U)} := \{\alpha_* \in R^U : \operatorname{supp} \alpha_* \text{ is finite}\}$ is contained in S_R and for all $\alpha_* \in R^{(U)}$ the relation $\sum_R(\alpha_*) = \sum' \{\alpha_u : u \in \operatorname{supp} \alpha_*\}$ holds, where $\sum'$ stands for the usual sum of the finitely many elements in $\{\alpha_u : u \in \operatorname{supp} \alpha_*\}$;

(ii) for all $\alpha_* \in S_R$, $\| \sum_R(\alpha_*)\| \leq \sum_{\mathcal{C}}(\|\alpha_*\|)$;

(iii) for every $\alpha_* \in S_R$ and every map $\varphi : U \to U$, $\alpha_*^{\varphi^{-1}(u)}$ is in S_R for all $u \in U$, and the map $\sum_R(\alpha_*^{\varphi^{-1}})$ given by $U \ni u \mapsto \sum_R(\alpha_*^{\varphi^{-1}(u)}) \in R$ is in S_R and satisfies $\sum_R(\sum_R(\alpha_*^{\varphi^{-1}})) = \sum_R(\alpha_*)$.

The comments following (1.1) apply also to (1.2).

1.3. Definition.

Let R and R' be prenormed semirings with prenorms $\| \ \| : R \to \mathcal{C}$ resp. $\| \ \|' : R' \to \mathcal{C}$ where $\mathcal{C}$ is a cone semiring with twosided U-summation $(S_{\mathcal{C}}, \sum_{\mathcal{C}})$. In addition, let $(S_R, \sum_R)$ resp. $(S'_R, \sum'_R)$ be U-summations for R resp. R'. Then a map $f : R \to R'$ is called a *bounded homomorphism* of prenormed semirings with U-summation if f is a homomorphism of semirings such that

(i) $f^U(S_R) \subseteq S'_R$ and $\sum_R \left(f^U(\alpha_*)\right) = f\left(\sum_R(\alpha_*)\right)$, for all $\alpha_* \in S_R$;

(ii) there is a $c \in \mathcal{C}$ (depending on f) with
$\| \sum_{R'} \left(f^U(\alpha_*)\right) \|' \leq (\sum_{\mathcal{C}}(\|\alpha_*\|)) \cdot c,$ for all $\alpha_* \in S_R$.

A bounded homomorphism of prenormed semirings with U-summation is said to be a *contractive homomorphism* (or *contraction*) of prenormed semirings with U-summation if (1.3), (ii), holds with $c = 1$.

1.4. Definition.

Let M be a prenormed R-semimodule with prenorm $\| \,\| : M \to \mathcal{C}$ over the prenormed semiring R with prenorm $\| \,\| : R \to \mathcal{C}$ and U-summation $(S_R, \sum_R)$. By a *U-summation* for M is meant a pair $(S_M, \sum_M)$ consisting of a R-subsemimodule S_M of M^U and a R-homomorphism $\sum_M : S_M \to M$ such that

(o) $\mu_* \in M^U$ is in S_M if and only $\|\mu_*\|$ is in $S_{\mathcal{C}}$;

(i) a) $M^{(U)} := \{\mu_* \in M^U : \operatorname{supp} \mu_* \text{ is finite}\}$ is contained in S_M and for all $\mu_* \in M^{(U)}$ the relation $\sum_M(\mu_*) = \sum'\{\mu_u : u \in \operatorname{supp} \mu_*\}$ holds, where $\sum'$ stands for the usual sum of the finitely many elements in $\{\mu_u : u \in \operatorname{supp} \mu_*\}$;

(i) b) for all $\alpha_* \in S_R$ and $m \in M$, $\sum_M(\alpha_* m_*) = (\sum_R(\alpha_*)) \cdot m$;

(ii) for all $\mu_* \in S_M$, $\| \sum_M(\mu_*)\| \leq \sum_{\mathcal{C}}(\|\mu_*\|)$;

(iii) for every $\mu_* \in S_M$ and every map $\varphi : U \to U$, $\mu_*^{\varphi^{-1}(u)}$ is in S_M for all $u \in U$, and the map $\sum_M(\mu_*^{\varphi^{-1}})$ given by $U \ni u \mapsto \sum_M(\mu_*^{\varphi^{-1}(u)}) \in M$ is in S_M and satisfies $\sum_M(\sum_M(\mu_*^{\varphi^{-1}})) = \sum_M(\mu_*)$.

Again the comments following (1.1) apply here too.

We close this section with

1.5. Definition.

Let M and M' be prenormed R-semimodules with prenorms $\| \,\| : M \to \mathcal{C}$ resp. $\| \,\|' : M' \to \mathcal{C}$ where $\mathcal{C}$ is a cone semiring with twosided U-summation $(S_{\mathcal{C}}, \sum_{\mathcal{C}})$. In addition, let $(S_M, \sum_M)$ resp. $(S'_M, \sum'_M)$ be U-summations for M resp. M'. Then a map $f : M \to M'$ is called a *bounded homomorphism* of prenormed R-semimodules with U-summation if f is a homomorphism of R-semimodules such that

(i) $f^U(S_M) \subseteq S'_M$ and $\sum_M\big(f^U(\mu_*)\big) = f\,(\sum_M(\mu_*))$, for all $\mu_* \in S_M$;

(ii) there is a $c \in \mathcal{C}$ (depending on f) with
$$\| \textstyle\sum_{M'} \big(f^U(\mu_*)\big) \|' \leq (\sum_{\mathcal{C}}(\|\mu_*\|)) \cdot c, \qquad \text{for all } \mu_* \in S_M.$$

A bounded homomorphism of prenormed R-semimodules with U-summation is said to be a *contractive homomorphism* (or *contraction*) of prenormed R-semimodules with U-summation if (1.5), (ii), holds with $c = 1$.

If M is a prenormed R-semimodule with U-summation and $m \in M$, then the map $f_m : R \to M$ given by $R \ni r \mapsto rm \in M$ is a bounded homomorphism of prenormed R-semimodules with U-summation.

Let R be a fixed prenormed semiring with U-summation. Then we obtain a category ${}_R\mathcal{S}\,\mathrm{mod}_{1N}$ whose objects are the prenormed R-semimodules with U-summation and whose morphisms are the contractions of prenormed R-semimodules with U-summation, the composition being the set-theoretical one.

2. Some elementary results

2.1. Lemma.

Let M be a prenormed R-semimodule with U-summation $(S_M, \sum_M)$. Let furthermore $\mu_* \in S_M$. For $T \subseteq U$ let μ_*^T be the element of M^U given by

$$U \ni u \mapsto \begin{cases} \mu_u & \text{if } u \in T \\ 0 & \text{if } u \notin T \end{cases}.$$

Then μ_*^T is in S_M and $\| \sum_M(\mu_*^T)\| \leq \sum_{\mathcal{C}}(\|\mu_*\|)$. In particular, if $u \in U$ then $\|\mu_u\| \leq \sum_{\mathcal{C}}(\|\mu_*\|)$.

Proof:

Define a map $\varphi : U \to U$ such that for some $u \in U$, $\varphi^{-1}(u) = T$ holds. Then μ_*^T is in S_M by (1.4), (iii), and the inequality follows from (1.4), (ii). ■

2.2. Lemma.

Let M be a prenormed R-semimodule with U-summation $(S_M, \sum_M)$. Let furthermore $\mu_* \in S_M$ and $\nu_* \in M^U$ be such that for some sets $A \supseteq \operatorname{supp} \mu_*$ and $B \supseteq \operatorname{supp} \nu_*$ there is a bijection $\overline{\varphi} : A \to B$ with $\mu_u = \nu_{\overline{\varphi}(u)}$ for all $u \in \operatorname{supp} \mu_*$. Then ν_* is in S_M and $\sum_M(\nu_*) = \sum_M(\mu_*)$.

Proof:

Extend $\overline{\varphi}$ to some map $\varphi : U \to U$. One checks easily that $\nu_* = \sum_M(\mu_*^{\varphi^{-1}})$ holds. Hence (1.4), (iii), shows that $\sum_M(\nu_*) = \sum_M(\sum_M(\mu_*^{\varphi^{-1}})) = \sum_M(\mu_*)$ holds. ■

2.3. Lemma.

Let $\mathcal{C}$ be a positive semiring with U-summation $(S_{\mathcal{C}}, \sum_{\mathcal{C}})$ and let $\alpha_* \in S_{\mathcal{C}}$ satisfy the relation $\operatorname{supp} \alpha_* \subseteq N_1 \times N_2$ for two subsets N_1 and N_2 of U. Let furthermore $\varphi_i : U \to U$ be such that $\varphi_i(n_1, n_2) = n_i$ holds for all $(n_1, n_2) \in N_1 \times N_2$ and $i = 1, 2$. Suppose that $\alpha_*^{\varphi_1^{-1}(u)}$ is in $S_{\mathcal{C}}$ for all

$u \in U$ and that $\sum_{\mathcal{C}}(\alpha_*^{\varphi_1^{-1}})$ is in $S_{\mathcal{C}}$. Then $\alpha_*^{\varphi_2^{-1}(u)}$ is in $S_{\mathcal{C}}$ for all $u \in U$, $\sum_{\mathcal{C}}(\alpha_*^{\varphi_2^{-1}})$ is in $S_{\mathcal{C}}$ and $\sum_{\mathcal{C}}(\sum_{\mathcal{C}}(\alpha_*^{\varphi_2^{-1}})) = \sum_{\mathcal{C}}(\sum_{\mathcal{C}}(\alpha_*^{\varphi_1^{-1}}))$.

Proof:

It follows directly from (1.1), (iv), that α_* is in $S_{\mathcal{C}}$. Hence the conclusion is an immediate consequence of (1.1), (iii). ■

Due to (2.2) we may, and will, adopt the following notation. If $\mu_* \in S_M$ has the property that $\operatorname{supp}\mu_* \subseteq A$ then we write $\sum\limits_{a\in A}(\mu_a)$ instead of $\sum_M(\mu_*)$. Hence, in the situation of (2.3), we may write $\sum\limits_{n_2\in N_2}(\alpha_{n_1,n_2})$ instead of $\sum_{\mathcal{C}}(\alpha_*^{\varphi_1^{-1}(u)})$ and replace $\sum_{\mathcal{C}}(\sum_{\mathcal{C}}(\alpha_*^{\varphi_1^{-1}}))$ by $\sum\limits_{n_1\in N_1}\left(\sum\limits_{n_2\in N_2}(\alpha_{n_1,n_2})\right)$. Therefore (2.3) means that in $S_{\mathcal{C}}$ double sums may be interchanged.

2.4. Lemma.

Let R be a prenormed semiring with U-summation $(S_R, \sum_R)$. Let furthermore $\alpha_* \in S_R$, $\beta_*^\Box$ a map from U to S_R, and γ^* a map from U to R such that

$$\|\gamma^*\| \quad \text{and} \quad \textstyle\sum_{\mathcal{C}}\|\beta_*^\Box\|$$

are bounded. Then

$$\textstyle\sum_R\left(\alpha_\Box(\sum_R\beta_*^\Box\gamma^*)\right) = \sum_R\left(\sum_R(\alpha_\Box\beta_*^\Box)\gamma^*\right).$$

Proof:

Using the axiom of choice for big sets we can establish a bijection $\psi : U \to U \times U$. Denote $\psi(u)$, $u \in U$, by (u_1, u_2) and write the map $U \ni u \mapsto \alpha_{u_1}\beta_{u_2}^{u_1}\gamma^{u_2}$ as θ_*. We claim that θ_* is in S_R. In order to prove this, let $\varphi_i : U \to U$ be maps such that $\varphi_i \circ \psi^{-1}(u_1, u_2) = u_i$, $i = 1, 2$, holds. Denoting by $pr_i : U \times U \to U$ the i^{th} projection, $i = 1, 2$, we have $\varphi_i = pr_i \circ \psi$, $i = 1, 2$. Obviously, $\|\theta_*\|$ is in $(U, \mathcal{C})$. Moreover, the union $\bigcup\{\operatorname{supp}\beta_*^u : u \in \operatorname{supp}\alpha_*\}$ is a set N and we have $\theta_u = 0$ for all elements $u \notin \psi^{-1}(\operatorname{supp}\alpha_* \times N)$. This means that θ_* is in R^U, and (1.2), (o), implies $\|\theta_*\| \in C^U$. Obviously we have $\|\theta_*\|^{\varphi_i^{-1}(u)} = \|\theta_*^{\varphi_1^{-1}(u)}\|$ for all $u \in U$. Hence, for every $u \in U$, $\|\theta_*\|^{\varphi_1^{-1}(u)}$ is the map

$$U \ni v \longmapsto \begin{cases} \|\alpha_u\beta_{v_2}^u\gamma^{v_2}\| & \text{if } \psi(v) = (u, v_2), \\ 0 & \text{otherwise.} \end{cases}$$

If c is a bound for $\|\gamma^*\|$, we get

$$\|\alpha_u\beta^u_{v_2}\gamma^{v_2}\| \leq \|\alpha_u\| \cdot \|\beta^u_{v_2}\| \cdot \|\gamma^{v_2}\| \leq (\sum_{\mathcal{C}} \|\alpha_*\|) \cdot \|\beta^u_{v_2}\| \cdot c.$$

Since the U-summation for $\mathcal{C}$ is twosided, the right side of the last inequality is in $S_{\mathcal{C}}$, and by (1.1), (ii), so is the left side. Hence $\sum_{\mathcal{C}} \|\alpha_u\beta^u_*\gamma^*\|$ exists and we obtain by (1.1), (i) b) and (ii),

$$\sum_{\mathcal{C}} \|\alpha_u\beta^u_*\gamma^*\| \leq \|\alpha_u\| \cdot (\sum_{\mathcal{C}} \|\beta^u_*\|) \cdot c.$$

Since there is a bound c' for the big set of elements $\sum_{\mathcal{C}} \|\beta^u_*\|$, $u \in U$, we get

$$\sum_{\mathcal{C}} \|\alpha_u\beta^u_*\gamma^*\| \leq \|\alpha_u\| \cdot c'c,$$

whence $\|\theta_*\|^{\varphi_1^{-1}}$, that is the map $U \ni u \mapsto \sum_{\mathcal{C}} \|\alpha_u\beta^u_*\gamma^*\| \in \mathcal{C}$, satisfies $\|\theta_*\|^{\varphi_1^{-1}} \leq \|\alpha_*\| \cdot c'c$. Since the latter is in $S_{\mathcal{C}}$ it follows from (1.1), (iv), that $\|\theta_*\|$ itself is in $S_{\mathcal{C}}$. Thus (1.2), (o), implies $\theta_* \in S_R$ as was claimed. At this point we can continue as in the proof of (2.3) to arrive at our assertion. ■

3. Big convexity theories

3.1. Definition.

Let R be a prenormed semiring with prenorm $\| \; \| : R \to \mathcal{C}$ and U-summation $(S_R, \sum_R)$. By a *big convexity theory* over R is meant a big subset Γ of S_R such that

(o) $\sum_{\mathcal{C}}(\|\alpha_*\|) \leq 1$, for all $\alpha_* \in \Gamma$;

(i) the map δ^u_* given by

$$U \ni v \mapsto \delta^u_v = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

is in Γ, for all $u \in U$;

(ii) for all α_* β^u_* in Γ, $u \in U$, the map $\langle \alpha_\square, \beta^\square_* \rangle$ given by $U \ni u \mapsto \sum_R\{\alpha_v\beta^v_u : v \in U\} \in R$ is in Γ.

A comment is in order concerning (3.1), (ii). Denote by $\alpha_\square\beta^\square_u$ the map $U \ni v \mapsto \alpha_v\beta^v_u$. Then $\operatorname{supp} \alpha_\square\beta^\square_u \subseteq \operatorname{supp} \alpha_*$. Hence $\alpha_\square\beta^\square_u$ is in R^U. Moreover, $\|\alpha_\square\beta^\square_u\| \leq \|\alpha_*\|$ as $\|\beta^v_u\| \leq \sum_{\mathcal{C}}(\|\beta^v_*\|) \leq 1$. By (1.1), (ii), and (1.2), (o), $\alpha_\square\beta^\square_u$ is in S_R, implying that $\langle \alpha_\square, \beta^\square_* \rangle$ is defined. In addition, since $\|\alpha_u\| \, \|\beta^u_*\|$ is the map $U \ni v \mapsto \|\alpha_u\| \, \|\beta^u_v\|$ we have $\|\alpha_u\| \, \|\beta^u_*\| \leq \|\beta^u_*\|$ by (2.1) and (3.1), (o). Due to (1.1), (ii), $\|\alpha_u\| \, \|\beta^u_*\|$ is in $S_{\mathcal{C}}$ and $\sum_{\mathcal{C}}(\|\alpha_u\| \, \|\beta^u_*\|) \leq \|\alpha_u\| \sum_{\mathcal{C}}(\|\beta^u_*\|) \leq \|\alpha_u\|$. Therefore the map $U \ni u \mapsto \sum_{\mathcal{C}}(\|\alpha_u\| \, \|\beta^u_*\|)$ is $\leq \|\alpha_*\|$. Since $\|\alpha_*\|$ is in $S_{\mathcal{C}}$, this also holds for the last map and $\sum\limits_u \big(\sum_{\mathcal{C}}(\|\alpha_u\|\,\|\beta_*^u\|)\big) \leq \sum_{\mathcal{C}}(\|\alpha_*\|) \leq 1$. In terms of the notation following (2.3) this means that $\sum\limits_u(\sum\limits_v(\|\alpha_u\|\,\|\beta_v^u\|)) \leq 1$ holds. Hence (2.3) shows that $\sum\limits_v \left(\sum\limits_u(\|\alpha_u\|\,\|\beta_v^u\|) \right) \leq 1$ is satisfied. The latter, however, is $\sum_{\mathcal{C}}(\|\langle \alpha_\square, \beta_*^\square \rangle\|) \leq 1$. In other words, the map defined in (3.1), (ii), satisfies without additional hypotheses the condition (3.1), (o).

Let Γ be a big convexity theory over R such that for some set T the condition $\operatorname{card}(\operatorname{supp}\alpha_*) \leq \operatorname{card}(T)$ is satisfied for all $\alpha_* \in \Gamma$. Then $\operatorname{card}(T)$ is called a *bound* for Γ and Γ is said to be *bounded*.

3.2. Lemma.

Any big convexity theory is the union of bounded big convexity subtheories.

Proof:

Let T be any infinite set. If Γ is a big convexity theory, put $\Gamma^T := \{\alpha_* \in \Gamma : \operatorname{card}(\operatorname{supp}\alpha_*) \leq \operatorname{card}(T)\}$. Since $\operatorname{card}(T \times T) = \operatorname{card}(T)$, it is easy to check that Γ^T is a big convexity theory with bound $\operatorname{card}(T)$. Clearly, $\Gamma = \bigcup\{\Gamma^T : T \in \mathcal{P}_{\inf}(U)\}$ where $\mathcal{P}_{\inf}(U)$ is the totality of all infinite subsets of U. ■

3.3. Definition.

Let Γ be a big convexity theory over the prenormed semiring R. By a *left* Γ-*convex module* is meant a set X, non-empty whenever $0_* \in \Gamma$, together with a map

$$\Gamma \times (U, X) \ni (\alpha_*, x^*) \longmapsto \langle \alpha_*, x^* \rangle \in X$$

such that

(i) $\langle \delta_*^u, x^* \rangle = x^u,$ for all $u \in U$, $x^* \in (U, X)$;

(ii) $\big\langle \alpha_\square, \langle \beta_*^\square, x^* \rangle \big\rangle = \big\langle \langle \alpha_\square, \beta_*^\square \rangle, x^* \big\rangle,$ for all $\alpha_* \in \Gamma$, $\beta_*^\square \in (U, \Gamma)$, $x^* \in (U, X)$.

Here, (U, X) stands for the conglomerate of all maps $U \to X$ and, similarly, (U, Γ) for the conglomerate of all maps $U \to \Gamma$.

Since we will only deal with left Γ-convex modules, we shall drop the epithet "left" from now on.

3.4. Definition.

Let Γ be a big convexity theory. Then a map $f: X \to X'$ of Γ-convex modules is said to be a *homomorphism of* Γ*-convex modules* if

$$f(\langle \alpha_*, x^* \rangle) = \langle \alpha_*, f^U(x^*) \rangle, \quad \text{for all} \quad \alpha_* \in \Gamma,\, x^* \in (U, X),$$

where $f^U(x^*)$ is the map $U \ni u \mapsto f(x^u) \in X'$.

Given a big convexity theory Γ we denote by $\Gamma\mathcal{C}$ the category of Γ-convex modules and their homomorphisms, with composition the set-theoretical one.

Next we need some computational rules concerning Γ-convex modules. Since Γ is a big convexity theory, free Γ-convex modules may not be available and thus we can not argue that the computational rules for all Γ-convex modules are just those for free Γ-convex modules.

We begin with a useful notation. If $\psi : U \to U$ is any map and if x^* is in (U, X), then we denote the map $U \xrightarrow{\psi} U \xrightarrow{x^*} X$ by $x^{\psi(*)}$.

3.5. Lemma.

Let Γ be a big convexity theory and let X be a Γ-convex module. Let furthermore α_* be in Γ and let $x^*, y^* \in (U, X)$ satisfy $x^u = y^u$ for all $u \in \operatorname{supp} \alpha_*$. Then $\langle \alpha_*, x^* \rangle = \langle \alpha_*, y^* \rangle$.

Proof: (see [**5**, (2.4), (iii)]).

Write U as the disjoint union of U' and U'' such that there are bijections $\varphi' : U' \to U$ and $\varphi'' : U'' \to U$. Define $z^* \in (U, X)$ by

$$z^u := \begin{cases} x^{\varphi'(u)} & \text{if } u \in U', \\ y^{\varphi''(u)} & \text{if } u \in U''. \end{cases}$$

Then, denoting $U \xrightarrow{\varphi'^{-1}} U' \hookrightarrow U$ again by φ'^{-1},

$$\langle \alpha_*, x^* \rangle = \langle \alpha_*, z^{\varphi'^{-1}(*)} \rangle = \big\langle \alpha_*, \langle \delta_\square^{\varphi_1'^{-1}(*)}, z^\square \rangle \big\rangle = \big\langle \langle \alpha_*, \delta^{\varphi'^{-1}(*)} \rangle, z^\square \big\rangle,$$

where $\beta_\square := \langle \alpha_*, \delta_\square^{\varphi'^{-1}(*)} \rangle$ satisfies

$$\beta_v = \begin{cases} \alpha_{\varphi'(v)} & \text{if } v \in \varphi'^{-1}(\operatorname{supp} \alpha_*) \\ 0 & \text{otherwise.} \end{cases}$$

Define the map $\psi : U \to U$ by

$$\psi(u) := \begin{cases} \varphi'^{-1}(u) & \text{if } u \in \operatorname{supp} \alpha_* \\ \varphi''^{-1}(u) & \text{otherwise.} \end{cases}$$

Then

$$\langle \alpha_*, y^* \rangle = \langle \alpha_*, z^{\psi(*)} \rangle = \big\langle \alpha_*, \langle \delta_\Box^{\psi(*)}, z^\Box \rangle \big\rangle = \big\langle \langle \alpha_*, \delta_\Box^{\psi(*)} \rangle, z^\Box \big\rangle,$$

where $\gamma_\Box := \langle \alpha_*, \delta_\Box^{\psi(*)} \rangle$ satisfies

$$\gamma_v = \begin{cases} \alpha_{\psi^{-1}(v)} & \text{if } v \in \psi(\operatorname{supp} \alpha_*) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\psi(\operatorname{supp} \alpha_*) = \varphi'^{-1}(\operatorname{supp} \alpha_*)$, we obtain $\beta_* = \gamma_*$ and hence $\langle \alpha_*, x^* \rangle = \langle \alpha_*, y^* \rangle$. ■

3.6. Lemma.

Let Γ be a big convexity theory and let X be a Γ-convex module. If α_* is in Γ, x^* is in (U, X), and $\varphi : U \to U$ is a bijection, then $\alpha_{\phi(*)}$ is in Γ and $\langle \alpha_{\varphi(*)}, x^{\varphi(*)} \rangle = \langle \alpha_*, x^* \rangle$.

Proof: (see [**5**, (2.4), (iv)]).

Since $\alpha_{\varphi(*)} = \langle \alpha_\Box, \delta_*^{\varphi^{-1}(\Box)} \rangle$, (3.1), (i) and (ii), imply that $\alpha_{\varphi(*)}$ is in Γ. Hence

$$\langle \alpha_{\varphi(*)}, x^{\varphi(*)} \rangle = \big\langle \langle \alpha_\Box, \delta_*^{\varphi^{-1}(\Box)} \rangle, x^{\varphi(*)} \big\rangle = \big\langle \alpha_\Box, \langle \delta_*^{\varphi^{-1}(\Box)}, x^{\varphi(*)} \rangle \big\rangle = \langle \alpha_\Box, x^\Box \rangle. ■$$

While (3.3), (ii), means that the “sums” $\langle \alpha_*, x^* \rangle$ in a Γ-convex module are associative and distributive, (3.6) says that they are also commutative.

3.7. Lemma.

Let Γ be a big convexity theory and let X be a Γ-convex module. Let furthermore $\alpha_*, \beta_* \in \Gamma$ and $x^*, y^* \in (U, X)$ satisfy the following conditions

(i) *for some sets $A \supseteq \operatorname{supp} \alpha_*$ and $B \supseteq \operatorname{supp} \beta_*$ there is a bijection $\varphi : A \to B$ such that*
$\alpha_u = b_{\varphi(u)}$ *for all $u \in A$;*

(ii) $x^u = y^{\varphi(u)}$ *for all $u \in A$.*

Then $\langle \alpha_*, x^* \rangle = \langle \beta_*, y^* \rangle$.

Proof:

This is an immediate consequence of (3.5) and (3.6). ■

3.8. Lemma.

Let X be a Γ-convex module, $\alpha_* \in \Gamma$, and $x^* \in (U, X)$. Denote the map $U \xrightarrow{x^*} X \hookrightarrow U$ by φ and let $\iota^* \in (U, X)$ be any map with $\iota^* \mid X = \mathrm{id}_X$. Then $\langle \alpha_*, x^* \rangle = \langle \sum_R (\alpha_*^{\varphi^{-1}}), \iota^* \rangle$.

Proof:

Let $\beta_*^{\square} \in (U, \Gamma)$ be the map $U \ni u \mapsto \delta_*^{x^u} \in \Gamma$. Then $x^* = \langle \beta_{\square}^*, \iota^{\square} \rangle$ and hence

$$\langle \alpha_*, x^* \rangle = \big\langle \alpha_*, \langle \beta_{\square}^*, \iota^{\square} \rangle \big\rangle = \big\langle \langle \alpha_{\square}, \beta_*^{\square} \rangle, \iota^* \big\rangle = \langle \textstyle\sum_R (\alpha_*^{\varphi^{-1}}), \iota^* \rangle$$

since $\sum_R (\alpha_*^{\varphi^{-1}}) = \langle \alpha_{\square}, \beta_*^{\square} \rangle$ as a simple computation shows. ∎

Let X be a Γ-convex module and let $x^{\square,*}$ be any map $U \times U \to U$. If α_* is in Γ, we denote by $\langle \alpha_*, x^{\square,*} \rangle$ the map $U \ni u \mapsto \langle \alpha_*, x^{u,*} \rangle \in X$ and by $\langle \alpha_{\square}, x^{\square,*} \rangle$ the map $U \ni x \mapsto \langle \alpha_{\square}, x^{\square,u} \rangle \in X$. With this notation we have

3.9. Lemma.

Let $\alpha_*, \beta_* \in \Gamma$ satisfy $\alpha_u \beta_v = \beta_v \alpha_u$ for all $u, v, \in U$. Then for any map $x^{\square,*}$ from $U \times U$ to some Γ-convex module X, $\big\langle \alpha_{\square}, \langle \beta_*, x^{\square,*} \rangle \big\rangle = \big\langle \beta_*, \langle \alpha_{\square}, x^{\square,*} \rangle \big\rangle$.

Proof:

Identical with the proof of (2.4), (ix), in [**5**, p. 968]. ∎

A big convexity theory Γ is said to be a *convexity theory with zero* if $0_* \in \Gamma$ holds.

3.10. Lemma.

Let Γ be a big convexity theory with zero and let X be any Γ-convex module. Then $\langle 0_*, x^* \rangle$ is independent of the choice of $x^* \in (U, X)$.

Proof:

This is an immediate consequence of (3.5). ∎

For a big convexity theory Γ with zero and a Γ-convex module X the element described in (3.10) is denoted by 0_X.

3.11. Lemma.

Let Γ be a big convexity theory with zero, let X be a Γ-convex module, and denote the constant map $U \to X$ with value 0_X by 0_X^*. Then $\langle \alpha_*, 0_X^* \rangle = 0_X$ for all $\alpha_* \in \Gamma$.

Proof:

See the proof of (2.4), (vi), in [**5**, p. 967]. ■

3.12. Lemma.

Let Γ be a big convexity theory with zero, let X be a Γ-convex module, and let x^* be an element of (U, X), Suppose that $\alpha_*, \beta_* \in \Gamma$ satisfy $\alpha_u = \beta_u$ for all $u \in U$ with $x^u \neq 0_X$. Then $\langle \alpha_*, x^* \rangle = \langle \beta_*, x^* \rangle$.

Proof:

Identical with the proof of (2.4), (vii), in [**5**, p. 967]. ■

3.13. Lemma.

Let Γ be any big convexity theory with zero, $\alpha_* \in \Gamma$, and $T \subseteq U$. Let furthermore X be any Γ-convex module and $x^* \in (U, X)$. Then, for any $u \neq v$ in U, with $\delta_*^u + \delta_*^v \in \Gamma$

$$\langle \alpha_*, x^* \rangle = \langle \delta_*^u + \delta_*^v, z^* \rangle$$

where $z^* \in (U, X)$ is any map satisfying $z^u = \langle \alpha_*^T, x^* \rangle$ and $z^v = \langle \alpha_*^{U \setminus T}, x^* \rangle$.

Proof:

Since $\langle \delta_*^u + \delta_*^v, z^* \rangle$ does not depend on $z^* \mid (U \setminus \{u, v\})$ due to (3.5), we may assume $z^w = 0_X$ for all $w \in U \setminus \{u, v\}$. Define $\beta_*^\square$ by

$$\beta_*^t := \begin{cases} \alpha_*^T, & \text{for all } t = u \\ \alpha_*^{U \setminus T}, & \text{for all } t = v \ , \\ 0_*, & \text{otherwise} \end{cases} \quad t \in U.$$

Then $\alpha_* = \langle \delta_\square^u + \delta_\square^v, \beta_*^\square \rangle$ and $z^\square = \langle \beta_*^\square, x^* \rangle$. Hence

$$\langle \alpha_*, x^* \rangle = \big\langle \langle \delta_\square^u + \delta_\square^v, \beta_*^\square \rangle, x^* \big\rangle = \big\langle \delta_\square^u + \delta_\square^v, \langle \beta_*^\square, x^* \rangle \big\rangle = \langle \delta_\square^u + \delta_\square^v, z^\square \rangle. \quad ■$$

3.14. Lemma.

Let Γ be any big convexity theory with zero and let $\alpha_* \in \Gamma$. Let furthermore X be any Γ-convex module, $y \in X$, and denote by $c_y^* \in (U,X)$ the constant map with value y. Then, for any $u \neq v$ in U, with $\delta_*^u + \delta_*^v \in \Gamma$

$$\langle \alpha_*, c_y^* \rangle = \langle (\textstyle\sum_R \{\alpha_w : w \in U\})\delta_*^u + \delta_*^v, z^* \rangle$$

where $z^* \in (U,X)$ is any map satisfying $z^u = y$ and $z^v = 0_X$.

Proof:

Define $\beta_*^\square \in (U,\Gamma)$ by $\beta_*^t := \delta_*^u$, for all $t \in U$. Then $\langle \alpha_\square, \beta_*^\square \rangle = (\sum_R \{\alpha_w : w \in U\}) \cdot \delta_*^u$ and $\langle \beta_*^\square, c_y^* \rangle = c_y^\square$. Hence

$$\begin{aligned}
\langle \alpha_*, c_y^* \rangle &= \big\langle \alpha_\square, \langle \beta_*^\square, c_y^* \rangle \big\rangle \\
&= \big\langle \langle \alpha_\square, \beta_*^\square \rangle, c_y^* \big\rangle \\
&= \langle (\textstyle\sum_R \{\alpha_w : w \in U\})\delta_*^u, c_y^* \rangle \\
&= \langle (\textstyle\sum_R \{\alpha_w : w \in U\})\delta_*^u + \delta_*^v, z^* \rangle. \ \blacksquare
\end{aligned}$$

3.15. Definition.

A big convexity theory Γ over R is called *commutative* if, for all $\alpha_*, \beta_* \in \Gamma$ and all $u, v \in U$, $\alpha_u \beta_v = \beta_v \alpha_u$ holds.

If Γ is a big commutative convexity theory and if X and Y are Γ-convex modules, then the tensorproduct $X \bigotimes_\Gamma Y$ in $\Gamma\mathcal{C}$, defined by the standard universal property (cf. [**5**, section 5]), exists.

3.16. Definition.

Let Γ be a big commutative convexity theory. Then a Γ-*convex algebra* is a pair (X, μ) consisting of a Γ-convex module X and a homomorphism $\mu : X \bigotimes_\Gamma X \to X$ of Γ-convex modules. A *homomorphism* $f : (X, \mu) \to (X', \mu')$ of Γ-*convex algebras* is a homomorphism $f : X \to X'$ of Γ-convex modules satisfying $f \circ \mu = \mu' \circ (f \bigotimes f)$.

If (X, μ) is a Γ-convex algebra and x', x'' are in X, then $\mu(x' \bigotimes x'')$ is usually denoted by $x'x''$ or $x' \cdot x''$. Such an algebra is, in general, not required to satisfy any laws involving products except those resulting from the universal property of the tensorproduct:

$$\begin{aligned}
\langle \alpha_*, x^* \rangle y &= \langle \alpha_*, x^* y \rangle, \quad \text{for all} \quad \alpha_* \in \Gamma,\ x^* \in (U,X),\ y \in X, \\
x \langle \beta_*, y^* \rangle &= \langle \beta_*, x y^* \rangle, \quad \text{for all} \quad x \in X,\ \beta_* \in \Gamma,\ y^* \in (U,X),
\end{aligned}$$

where x^*y is the map $U \ni u \mapsto x^u y \in X$ and xy^* is defined similarly.

Given a big commutative convexity theory Γ we denote by $\Gamma\mathcal{A}lg$ the category of Γ-convex algebras and their homomorphisms, with composition the set-theoretical one.

We shall also consider full subcategories of $\Gamma\mathcal{A}lg$ whose objects satisfy additional sets of relations such as the category of Γ-convex associative algebras. For $\Gamma = \Omega_{\mathbb{C}}$ and $\Gamma = \Omega_{\mathbb{R}}$, Γ-convex associative algebras were discussed in [**6**]. Later on we will have to deal with the category $\mathbb{D}^U\mathcal{A}lg_F$ of associative, commutative, idempotent algebras satisfying additional conditions (see section 5).

3.17. Proposition.

Let Γ *be a bounded big convexity theory with bound* $\operatorname{card}(N)$*. Then the categories* $\Gamma\mathcal{C}$ *and* $(\Gamma|N)\mathcal{C}$ *are canonically isomorphic.*

Proof:

Let X be a Γ-convex module with composition $\Gamma \times (U,X) \ni (\alpha_*, x^*) \mapsto \langle \alpha_*, x^* \rangle \in X$. If β_* is in $\Gamma|N$ then there is a unique $\alpha_* \in \Gamma$ with $\alpha_* = \alpha_*^N$ and $\beta_* = \alpha_*|N$. Let $y^* \in X^N$ and extend y^* to some $\bar{y}^* \in (U,X)$. Put $\langle \beta_*, y^* \rangle' := \langle \alpha_*, \bar{y}^* \rangle$. Due to (3.5), $\langle \beta_*, y^* \rangle'$ is independent of the extension $\bar{y}^*$ of y^*. One checks easily that the composition $(\Gamma|N) \times X^N \ni (\beta_*, y^*) \mapsto \langle \beta_*, y^* \rangle' \in X$ makes X a $\Gamma|N$-convex module X'. A simple computation shows that for every map $f : X \to Y$ between Γ-convex modules, $f \in \Gamma\mathcal{C}(X,Y)$ implies $f \in (\Gamma|N)\mathcal{C}(X',Y')$. Conversely assume that the map $f : X' \to Y'$ satisfies $f \in (\Gamma|N)\mathcal{C}(X',Y')$. If α_* is in Γ then there is an injective map $\varphi' : \operatorname{supp} \alpha_* \to N$. Extend φ' to a bijection $\varphi^{-1} : U \to U$. Then $\alpha_{\varphi(*)}$ is in Γ and satisfies $\alpha_{\varphi(*)} = \alpha_{\varphi(*)}^N$, whence $\beta_* := \alpha_{\varphi(*)}|N$ is in $\Gamma|N$. Let $x^* \in (U,X)$. Then by (3.6)

$$\begin{aligned} f(\langle \alpha_*, x^* \rangle) &= f(\langle \alpha_{\varphi(*)}, x^{\varphi(*)} \rangle) \\ &= f(\langle b_*, x^{\varphi(*)}|N \rangle') \\ &= \left\langle \beta_*, f^N(x^{\varphi(*)}|N) \right\rangle' \\ &= \langle \alpha_{\varphi(*)}, f^U(x^{\varphi(*)}) \rangle \\ &= \langle \alpha_*, f^U(x^*) \rangle. \end{aligned}$$

Hence $\Gamma\mathcal{C}(X,Y) = (\Gamma|N)\mathcal{C}(X',Y')$. Next let Y be a $\Gamma|N$-convex module with composition $(\Gamma|N) \times Y^N \ni (\beta_*, y^*) \mapsto \langle \beta_*, y^* \rangle' \in Y$. If $\alpha_* \in \Gamma$, choose φ' as above and extend it to a bijection $\varphi^{-1} : U \to U$. Let $y^* \in (U,Y)$ and define

$$\langle \alpha_*, y^* \rangle := \langle \alpha_{\varphi(*)}|N, y^{\varphi(*)}|N \rangle'. \tag{3.17.1}$$

(3.6) implies that $\langle \alpha_*, y^* \rangle$ is independent of the choice of φ. It follows from (2.2) that for $\alpha_* \in \Gamma$ and $\beta_*^{\square} \in \Gamma^N$ the relation $\langle \alpha_{\varphi(\square)}, \beta_*^{\varphi(\square)} \rangle = \langle \alpha_\square, \beta_*^{\square} \rangle$ holds, where φ is any bijection from U to U. Since Γ is bounded by $\operatorname{card}(N)$ and since N is an infinite set, we have for $A := \operatorname{supp} \alpha_* \cup \bigcup\{\operatorname{supp} \beta_*^u : u \in \operatorname{supp} \alpha_*\}$ the relation $\operatorname{card} A \preceq \operatorname{card}(N)$. In order to get a better visual display we write in (3.17.1) $\alpha_{\varphi|N(*)}$ instead of $\alpha_{\varphi(*)}|N$ and $x^{\varphi|N(*)}$ instead of $x^{\varphi(*)}|N$. Then we have

$$
\begin{aligned}
\big\langle \langle \alpha_\square, \beta_*^{\square} \rangle, y^* \big\rangle &= \big\langle \langle \alpha_{\psi(\square)}, \beta_*^{\psi(\square)} \rangle, y^* \big\rangle \\
&= \big\langle \langle \alpha_{\psi(\square)}, \beta_{\psi|N(*)}^{\psi(\square)} \rangle, y^{\psi|N(*)} \big\rangle' \\
&= \big\langle \langle \alpha_{\psi|N(\square)}, \beta_{\psi|N(*)}^{\psi|N(\square)} \rangle, y^{\psi|N(*)} \big\rangle' \\
&= \big\langle \alpha_{\psi|N(\square)}, \langle \beta_{\psi|N(*)}^{\psi|N(\square)}, y^{\psi|N(*)} \rangle' \big\rangle' \\
&= \big\langle \langle \alpha_\square, \langle \beta_{\psi|N(*)}^{\square}, y^{\psi|N(*)} \rangle' \big\rangle \\
&= \big\langle \alpha_\square, \langle \beta_*^{\square}, y^* \rangle \big\rangle,
\end{aligned}
$$

which is (3.3), (ii). Since the verification of (3.3), (i), for the composition (3.17.1) is straight forward we obtain that Y equipped with the composition (3.17.1) is a Γ-convex module $Y^\sim$. One checks easily that for any Γ-convex module X the relation $(X')^\sim = X$ and for any $\Gamma|N$-convex module Y the relation $(Y^\sim)' = Y$ is satisfied. Hence the asserted isomorphy is satisfied. ■

4. The category $\Gamma\mathcal{C}$ is algebraic

4.1. Theorem.

Let Γ be any big convexity theory. Then $\Gamma\mathcal{C}$ is an algebraic category.

Proof:

In [**3**, Chap. 3, (1.13)], algebraic categories are characterized as equationally presentable categories satisfying three conditions. The last two of these conditions are trivially satisfied by $\Gamma\mathcal{C}$ as $\Gamma\mathcal{C}$ has separators (formed as in $\mathcal{S}et$) and the underlying-set functor $\Gamma\mathcal{C} \to \mathcal{S}et$ creates quotients of congruences. Hence it remains to be shown that $\Gamma\mathcal{C}$ has arbitrary free objects. Let A be any set and consider the set $B := \{\xi_* \in \Gamma : \operatorname{supp} \xi_* \subseteq A\}$. If α_* is in Γ and $\xi_*^{\square} \in (U, B)$ then $\langle \alpha_\square, \xi_*^{\square} \rangle$ is in Γ and $\operatorname{supp}\langle \alpha_\square, \xi_*^{\square} \rangle \subseteq A$ holds, whence $\langle \alpha_\square, \xi_*^{\square} \rangle$ is an element of B. Evidently, this makes B a Γ-convex module $\mathcal{F}(A)$. Let $\delta^* : A \to \mathcal{F}(A)$ be the map $A \ni a \mapsto \delta_*^a \in \mathcal{F}(A)$. Now let φ be any map from A to some Γ-convex module X and denote by $\varphi^* \in (U, X)$ the map given by

$$\varphi^*(u) := \begin{cases} \varphi(u), & \text{if } u \in A \\ \text{anything}, & \text{if } u \notin A. \end{cases}$$

Define $h : \mathcal{F}(A) \to X$ by $h(\xi_*) := \langle \xi_*, \varphi^* \rangle$ for all $\xi_* \in \mathcal{F}(A)$. Due to (3.5), h is well defined. Obviously, $h \circ \delta^* = \varphi$. Moreover if α_* is in Γ and $\xi_*^\square$ is in $(U, \mathcal{F}(A))$ then

$$h(\langle \alpha_\square, \xi_*^\square \rangle) = \big\langle \langle \alpha_\square, \xi_*^\square \rangle, \varphi^* \big\rangle = \big\langle \alpha_\square, \langle \xi_*^\square, \varphi^* \rangle \big\rangle = \langle \alpha^\square, h^U(\xi_*^\square) \rangle,$$

showing that h is a homomorphism of Γ-convex modules. The uniqueness of h (subject to $h \circ \delta^* = \varphi$) follows from the fact that $\delta^*(A)$ is a system of generators of the Γ-convex module $\mathcal{F}(A)$. ■

Let Γ be any big convexity theory over the commutative prenormed semiring R and let X be a Γ-convex module. Let furthermore N be a set and $\kappa : X^N \to X$ be a map (equivalently: a composition of arity N). Then κ is called a Γ-*multi-homomorphism* if for every $\bar{n} \in N$ and every $y^* \in X^{N \setminus \{\bar{n}\}}$ the map

$$X \xrightarrow{\psi_{y^*}} X^N \xrightarrow{\kappa} X$$

is a homomorphism of Γ-convex modules, where ψ_{y^*} is given by

$$\psi_{y^*}(x)(n) := \begin{cases} x, & \text{if } n = \bar{n} \\ y^n, & \text{if } n \neq \bar{n}. \end{cases}$$

Let $\mathcal{T}$ be an algebraic theory that is given by a set $\{\kappa_i : i \in I\}$ of compositions and a set $\{\rho_j : j \in J\}$ of relations. Denote the arity of κ_i by k_i, $i \in I$. Then the category $\Gamma\mathcal{T}$ of Γ-convex $\mathcal{T}$-algebras has as its objects the tuples $\{X, \kappa_{i,X} : i \in I\}$ where X is any Γ-convex module and $\kappa_{i,X} : X^{k_i} \to X$, $i \in I$, is a map such that

(i) every $\kappa_{i,X}$, $i \in I$, is a Γ-multi-homomorphism;

(ii) every ρ_j, $j \in J$, is satisfied on X.

Furthermore, $\Gamma\mathcal{T}$ has as its morphisms $f : X \to Y$ precisely those maps that are homomorphisms of Γ-convex modules and are compatible with the compositions $\kappa_{i,X}$ and $\kappa_{i,Y}$, $i \in I$.

In order to prove that $\Gamma\mathcal{T}$ is an algebraic category we need few preliminary statements.

4.2. Lemma.

Let Γ be any big convexity theory over the prenormed semiring R. Let furthermore $\alpha_*^{(1)}, \dots, \alpha_*^{(k)}$ be in Γ and denote by $\beta_* \in (U, R)$ the map

$$\beta_u := \begin{cases} \alpha_{u_1}^{(1)} \cdot \ldots \cdot \alpha_{u_k}^{(k)}, & \textit{if } u = (u_1, \ldots, u_k) \in \operatorname{supp} \alpha_*^{(1)} \times \cdots \times \operatorname{supp} \alpha_*^{(k)} \\ 0, & \textit{otherwise.} \end{cases}$$

Then β_* is in Γ.

Proof:
It suffices to consider the case $k = 2$. For $u \in \operatorname{supp} \alpha_*^{(1)}$ define β_*^u by

$$\beta_v^u := \begin{cases} \alpha_w^{(2)}, & \text{if } v = (u, w) \text{ and } w \in U \\ 0, & \text{otherwise.} \end{cases}$$

By (3.6) we have $\beta_*^u \in \Gamma$ for all $u \in \operatorname{supp} \alpha_*^{(1)}$. Choose $\beta_*^u \in \Gamma$ arbitrarily for all $u \notin \operatorname{supp} \alpha_*^{(1)}$. Then $\beta_* = \langle \alpha_\Box, \beta_*^\Box \rangle$ and hence $\beta_* \in \Gamma$. ■

4.3. Lemma.

Let Γ be any big convexity theory over the prenormed semiring R with U-summation $(S_R, \sum_R)$. Let furthermore $\alpha_* \in \Gamma$ and let $\varphi : U \to U$ be any map. Then $\sum_R(\alpha_*^{\varphi-1})$ is in Γ.

Proof:
Put $\beta_*^u := \delta_*^{\varphi(u)}$, $u \in U$. Then $\sum_R(\alpha_*^{\varphi^{-1}}) = \langle \alpha_\Box, \beta_*^\Box \rangle$. ■

4.4. Theorem.

Let Γ be any big convexity theory over the commutative prenormed semiring R. Let furthermore $\mathcal{T}$ be any algebraic theory that is given by a set $\{\kappa_i : i \in I\}$ of finitary compositions of arity ≥ 1 and a set $\{\rho_j : j \in J\}$ of relations. Then $\Gamma\mathcal{T}$ is an algebraic category.

Proof:
Let A be any set and denote $B := \mathcal{F}_{\mathcal{T}'}(A)$ the free $\mathcal{T}'$-object on A, where $\mathcal{T}'$ is the algebraic theory with compositions $\{\kappa_i : i \in I\}$ and no relations. B can be thought of as the free "$\mathcal{T}'$-multi-magma" on A. Let $\mathcal{F}(A)$ be the free Γ-convex module on the set B and denote by δ^* the canonical map from B to $\mathcal{F}(A)$. If $i_A^* \in (U, \mathcal{F}(A))$ is any map with $i_A^* \mid B = \delta^*$ then for every $x \in \mathcal{F}(A)$ there is a unique $\alpha_*^x \in \Gamma$ with $(\alpha_*^x)^B = \alpha_*^x$ and $x = \langle \alpha_*^x, i_A^* \rangle$. Let k_i be the (finite) arity of κ_i,

$i \in I$, and denote the corresponding composition $B^{k_i} \to B$ by κ_i'. Given $\bar{x} := (x_1, \dots, x_{k_i}) \in \mathcal{F}(A)^{k_i}$, $i \in I$, denote by $\beta_*^{\bar{x}}$ the map

$$U \ni u \longmapsto \sum_R \{\alpha_{b_1}^{x_1} \cdot \dots \cdot \alpha_{b_{k_i}}^{x_{k_i}} : b_1, \dots, b_{k_i} \in B \text{ and } \kappa_i'(b_1, \dots, b_{k_i}) = u\}.$$

Due to (4.2) and (4.3), $\beta_*^{\bar{x}}$ is well defined and belongs to Γ. Now put

$$\kappa_i''(x_1, \dots, x_{k_i}) := \langle \beta_*^{\bar{x}}, i_A^* \rangle, \quad \bar{x} \in \mathcal{F}(A)^{k_i}.$$

We claim that κ_i'' is a Γ-multi-homomorphism. Let k_i be ≥ 2. For simplicity we check this only for the first component. Put $y := (x_2, \dots, x_{k_i})$. For $\gamma_* \in \Gamma$ and $x^* \in (U, \mathcal{F}(A))$ we have

$$\langle \gamma_*, x^* \rangle = \big\langle \gamma_\square, \langle \alpha_*^{x^\square}, i_A^* \rangle \big\rangle = \big\langle \langle \gamma_\square, \alpha_*^{x^\square} \rangle, i_A^* \big\rangle$$

and hence $\alpha_*^{\langle \gamma_\square, x^\square \rangle} = \langle \gamma_\square, \alpha_*^{x^\square} \rangle$. Therefore

$$\kappa_i''(\langle \gamma_*, x^* \rangle, y) = \langle \bar{\beta}_*, i_A^* \rangle$$

where

$$\bar{\beta}_u = \sum_R \{\langle \gamma_\square, \alpha_{b_1}^{x^\square} \rangle \cdot \alpha_{b_2}^{x_2} \cdot \dots \cdot \alpha_{b_{k_i}}^{x_{k_i}} : b_1, \dots, b_{k_i} \in B \\ \text{and } \kappa_i'(b_1, \dots, b_{k_i}) = u\}, \quad u \in U.$$

On the other hand,

$$\langle \gamma_\square, \kappa_i''(x^\square, y) \rangle = \big\langle \gamma_\square, \langle \beta_*^{(x^\square, y)}, i_A^* \rangle \big\rangle = \big\langle \langle \gamma_\square, \beta_*^{(x^\square, y)} \rangle, i_A^* \big\rangle$$

and $\bar{\bar{\beta}}_* := \langle \gamma_\square, \beta_*^{(x^\square, y)} \rangle$ satisfies

$$\bar{\bar{\beta}}_u = \sum_R \{\gamma_v \cdot \alpha_{b_1}^{x^v} \cdot \alpha_{b_2}^{x_2} \cdot \dots \cdot \alpha_{b_{k_i}}^{x_{k_i}} : v \in U, b_1, \dots, b_{k_i} \in B, \\ \text{and } \kappa_i'(b_1, \dots, b_{k_i}) = u\}, \quad u \in U.$$

Since $\bar{\beta}_u = \bar{\bar{\beta}}_u$ due to (4.2) and (4.3), it is clear that κ_i'' is a Γ-multi-homomorphism. In case $k_i = 1$ a similar, but simpler, argument can be used to prove that κ_i'' is a Γ-multi-homomorphism. Now let X be any $\Gamma\mathcal{T}$-object and let $\varphi : A \to X$ be any map. Since X is also a $\mathcal{T}'$-object there is a map, indeed a $\mathcal{T}'$-homomorphism, $\varphi' : B \to X$ with $\varphi = \varphi' \circ \delta'^*$ where $\delta'^* : A \to B$ is the canonical map. Since $\mathcal{F}(A)$ is the free Γ-convex module on B there is a homomorphism of Γ-convex modules $f : \mathcal{F}(A) \to X$ with $\varphi' = f \circ \delta^*$. Since $f \circ i_A^* \mid B = \varphi'$ and since φ' is a $\mathcal{T}'$-homomorphism we have for every $(x_1, \dots, x_{k_i}) \in \mathcal{F}(A)^{k_i}$

$$\begin{aligned} f\big(\kappa_i''(x_1, \dots, x_{k_i})\big) &= f(\langle \beta_*^{\bar{x}}, i_A^* \rangle) \\ &= \langle \beta_*^{\bar{x}}, f^U(i_A^*) \rangle \\ &= \langle \beta_*^{\bar{x}}, f \circ i_A^* \rangle \\ &= \kappa_i^X\big(f(x_1), \dots, f(x_{k_i})\big), \end{aligned}$$

where κ_i^X is the composition in X that corresponds to κ_i. This means that f is a $\mathcal{T}'$-homomorphism. Since X is a $\Gamma\mathcal{T}$-object, f factors through $g : \mathcal{F}(A) \to \mathcal{F}(A)/\sim$ where g is the quotient map with respect to the smallest Γ-congruence relation $\sim$ that is compatible with the set $\{\rho_j : j \in J\}$ of relations of $\mathcal{T}$. If $f = \bar{f} \circ g$ is this factorization then

$$\varphi = \varphi' \circ \delta'^* = f \circ \delta^* \circ \delta'^* = \bar{f} \circ g \circ \delta^* \circ \delta'^*,$$

and this factorization determines $\bar{f}$ uniquely in terms of φ. Hence $\mathcal{F}(A)/\sim$ is the free $\Gamma\mathcal{T}$-object on A with $g \circ \delta^* \circ \delta'^*$ the canonical map $A \to \mathcal{F}(A)/\sim$. Since $\Gamma\mathcal{C}$ has separators, which are formed as in $\mathcal{S}et$ and since the underlying-set functor $\Gamma\mathcal{C} \to \mathcal{S}et$ creates quotients of congruences, [**3**, Chap. 3, (1.13)], implies that $\Gamma\mathcal{T}$ is an algebraic category. ■

Suppose that on each Γ-convex module X a map $\lambda^X : X^\ell \to X$ is given. Then $\lambda := \{\lambda^X : X \in 0b\Gamma\mathcal{C}\}$ is called an *ℓ-ary Γ-composition* if for every homomorphism $f : X \to Y$ of Γ-convex modules $f \circ \lambda^X = \lambda^Y \circ f^\ell$ is satisfied. The primary example is obtained as follows. Let α_* be given and denote card(supp α_*) by ℓ. For each $\varphi \in X^\ell$ choose an $x_\varphi^* \in (U, X)$ with $x_\varphi^* | \operatorname{supp} \alpha_* = \varphi$ and put $\lambda^X(\varphi) = \langle \alpha_*, x_\varphi^* \rangle, \varphi \in X^\ell$. Then (3.5) implies that λ is an ℓ-ary Γ-composition.

4.5. Addendum.

Let Γ be any big convexity theory over the commutative prenormed semiring R. Let furthermore $\mathcal{T}$ be any algebraic theory as specified in (4.4). Given any set $\{\lambda_k : k \in K\}$ of finitary Γ-compositions, denote by $\mathcal{T}^*$ the set of compositions $\{\kappa_i : i \in I\}$ together with the class of relations $\{\rho_j : j \in J\} \cup \{\sigma_\ell : \ell \in L\}$, where each σ_ℓ involves the compositions $\{\kappa_i : i \in I\}$ and the Γ-compositions $\{\lambda_k : k \in K\}$. Then $\Gamma\mathcal{T}^*$ is an algebraic category.

Proof:

Same as for (4.4), with the appropriate change in the congruence relation $\sim$. ■

5. A simple example: Frames

Let $\mathbb{D}$ denote the semiring consisting of the two element 0 and 1, with 0 the neutral element for addition and 1 the neutral element for multiplication, satisfying $1 + 1 = 1$. There is only one other semiring having precisely two elements, namely the field F_2.

The ring $\mathbb{D}$ is a positive semiring (indeed a cone semiring) with U-summation $(\mathbb{D}^U, \sum)$, where $\sum(\alpha_*) := \max\{\alpha_u : u \in U\}$. It is also a prenormed semiring with prenorm $\mathrm{id}_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ and U-summation $(\mathbb{D}^U, \sum)$. Furthermore $\mathbb{D}^U$ is a big hemiring as well as a big commutative convexity theory.

Given a $\mathbb{D}^U$-convex module X we obtain, for any $\alpha_* \in \mathbb{D}^U$ and any $x^* \in (U, X)$, the element $\langle \alpha_*, x^* \rangle \in X$. Due to (3.5), $\langle \alpha_*, x^* \rangle$ depends only on the restriction $x^* \mid \operatorname{supp} \alpha_*$. Hence $\langle \alpha_*, x^* \rangle$ may be written as a formal sum $\sum\{x^u : u \in \operatorname{supp} \alpha_*\}$. Conversely, if $\xi : I \to X$ is any family of elements of X and $\alpha_* \in \mathbb{D}^U$ is given by

$$\alpha_u := \begin{cases} 1 & \text{if } u \in I \\ 0 & \text{otherwise} \end{cases}, \quad u \in U,$$

while x^* is any extension of ξ, then the formal sum associated with $\langle \alpha_*, x^* \rangle$ is $\sum\{x^u : u \in \operatorname{supp} \alpha_*\} = \sum\{x^i : i \in I\} = \sum\{\xi(i) : i \in I\}$. In other words we have in X the (formal) sums of arbitrary set-indexed families of elements of X. These sums are associative and distributive by (3.3), (ii), and commutative due to (3.6).

5.1. Lemma.

Every $\mathbb{D}^U$-convex module X admits the structure of a semimodule over the semiring $\mathbb{D}$.

Proof:

Since $\mathbb{D}^U$ is a convexity theory with zero, (3.10) shows that X has a distinguished element $0_X := \langle 0_*, x^* \rangle$. Let x', x'' be in X. Given any two distinct elements u and v of U and any $x^* \in (U, X)$ with $x^u = x'$ and $x^v = x''$, $\langle \delta^u_* + \delta^v_*, x^* \rangle$ is independent of $x^* \mid (U \setminus \{u, v\})$ by (3.5) and is independent of the choice of u, v by (3.6). Hence we define

$$x' + x'' := \langle \delta^u_* + \delta^v_*, x^* \rangle.$$

(3.6) shows that $x' + x'' = x'' + x'$ holds. It follows from (3.12) that $x + 0_X = x$ is satisfied for all $x \in X$. Next let u, v, w be mutually distinct elements of U and let $x^* \in (U,X)$ satisfy $x^u = x'$, $x^v = x''$, $x^w = x'''$. Furthermore let

$$\beta_*^t := \begin{cases} \delta_*^u + \delta_*^v, & \text{for } t = u \\ \delta_*^w, & \text{for } t = v \\ \text{anything}, & \text{otherwise} \end{cases}, \quad t \in U.$$

Then $\beta_u^\square = \beta_v^\square = \delta_u^\square$ and $\beta_w^\square = \delta_v^\square$. Hence $\langle \delta_\square^u + \delta_\square^v, \beta_*^\square \rangle = \delta_*^u + \delta_*^v + \delta_*^w$. Since $\langle \beta_*^u, x^* \rangle = x' + x''$ and $\langle \beta_*^v, x^* \rangle = x'''$, we have

$$(x'+x'')+x''' = \big\langle \delta_\square^u + \delta_\square^v, \langle \beta_*^\square, x^* \rangle \big\rangle = \big\langle \langle \delta_\square^u + \delta_\square^v, \beta_*^\square \rangle, x^* \big\rangle = \langle \delta_*^u + \delta_*^v + \delta_*^w, x^* \rangle.$$

However, if

$$\gamma_*^t := \begin{cases} \delta_*^u, & \text{for } t = u \\ \delta_*^v + \delta_*^w, & \text{for } t = v \\ \text{anything}, & \text{otherwise} \end{cases}, \quad t \in U,$$

then $\gamma_u^\square = \delta_u^\square$ and $\gamma_v^\square = \gamma_w^\square = \delta_v^\square$. Hence $\langle \delta_\square^u + \delta_\square^v, \gamma_*^\square \rangle = \delta_*^u + \delta_*^v + \delta_*^w$. Since $\langle \gamma_*^u, x^* \rangle = x'$ and $\langle \gamma_*^v, x^* \rangle = x'' + x'''$, we have

$$x' + (x'' + x''') = \big\langle \delta_\square^u + \delta_\square^v, \langle \gamma_*^\square, x^* \rangle \big\rangle = \langle \delta_*^u + \delta_*^v + \delta_*^w, x^* \rangle$$

and therefore $(x' + x'') + x''' = x' + (x'' + x''')$. Finally, let $x^* \in (U,X)$ satisfy $x^u = x^v = x$ and define $\alpha_*^\square$ by

$$\alpha_*^t := \begin{cases} \delta_*^u, & \text{for } t = u \text{ and } t = v \\ \text{anything}, & \text{otherwise} \end{cases}, \quad t \in U.$$

Then

$$\langle \alpha_*^t, x^* \rangle = x, \quad \text{for} \quad t = u \text{ and } t = v.$$

Hence $x + x = \big\langle \delta_\square^u + \delta_\square^v, \langle \alpha_*^\square, x^* \rangle \big\rangle$. Since $\langle \delta_\square^u + \delta_\square^v, \alpha_*^\square \rangle = \delta_*^u$, we obtain

$$x + x = \big\langle \delta_\square^u + \delta_\square^v, \langle \alpha_*^\square, x^* \rangle \big\rangle = \big\langle \langle \delta_\square^u + \delta_\square^v, \alpha_*^\square \rangle, x^* \big\rangle = \langle \delta_*^u, x^* \rangle = x.$$

This implies that X is a unital semimodule over the semiring $\mathbb{D}$. ■

Lemma 5.1 allows us to define a full subcategory $\mathbb{D}^U \mathcal{A}lg_F$ of the category $\mathbb{D}^U \mathcal{A}lg$ of $\mathbb{D}^U$-convex algebras and their homomorphisms which will be shown to be isomorphic to the category $\mathcal{F}rm$ of frames (see [**2**, p. 39]). The objects of $\mathbb{D}^U \mathcal{A}lg_F$ are the associative, commutative and idempotent $\mathbb{D}^U$-convex algebras X that satisfy the *Absorption Law*

$$xy + y = y, \quad \text{for all} \quad x, y \in X.$$

The category $\mathbb{D}^U \mathcal{A}lg_F$ is a category of algebras in the sense of Addendum (4.5) and therefore an algebraic category.

5.2. Theorem.

The categories $\mathcal{F}rm$ and $\mathbb{D}^U\mathcal{A}lg_F$ are isomorphic as concrete categories (over Set).

Proof:

Let X be a frame, that is a partially ordered set (with order relation "$\leq$") that is complete and satisfies the infinite distributive law. The $\mathbb{D}^U$-convex module structure on X is given by

$$(*) \quad \langle \alpha_*, x^* \rangle := \bigvee \{x^v : \alpha_v = 1\}, \quad \text{for all} \quad \alpha_* \in \mathbb{D}^U \text{ and } x^* \in (U, X).$$

We have

$$\langle \delta_*^u, x^* \rangle = \bigvee \{x^v : \delta_v^u = 1\} = x^u, \quad \text{for all} \quad u \in U, x^* \in (U, X),$$

which is (3.3), (i). Moreover, for $\alpha_* \in \mathbb{D}^U$, $\beta_*^\square \in (U, \mathbb{D}^U)$ and $x^* \in (U, X)$,

$$\begin{aligned}\big\langle \alpha_\square, \langle \beta_*^\square, x^* \rangle \big\rangle &= \bigvee \{\bigvee \{x^v : \beta_v^u = 1\} : \alpha_u = 1\} \\ &= \bigvee \{x^v : \max\{\alpha_u \beta_v^u : u \in U\} = 1\} \\ &= \big\langle \langle \alpha_\square, \beta_*^\square \rangle, x^* \big\rangle,\end{aligned}$$

which is (3.3), (ii). Next we define a multiplication on X by putting $x' \cdot x'' := x' \wedge x''$ for all $x', x'' \in X$. Then by the infinite distributive law

$$x' \cdot \langle \alpha_*, x^* \rangle = x' \wedge \bigvee \{x^v : \alpha_v = 1\} = \bigvee \{x' \wedge x^v : \alpha_v = 1\} = \langle \alpha_*, x' \cdot x^* \rangle,$$

where $x' \cdot x^*$ is the map $U \ni u \mapsto x' \cdot x^u \in X$. This shows that X equipped with this structure is an associative, commutative, idempotent $\mathbb{D}^U$-convex algebra $A(X)$ satisfying the Absorption Law. Indeed it follows from $(*)$ that the sum $x' + x''$ in the $\mathbb{D}^U$-convex module X is given by the join $x' \vee x''$. Moreover, a frame morphism $f : X \to Y$ becomes a homomorphism of $\mathbb{D}^U$-convex algebras $A(f) : A(X) \to A(Y)$.

Conversely, let A be an associative, commutative and idempotent $\mathbb{D}^U$-convex algebra that satisfies the Absorption Law. We define an order relation " $\leq$" on A by setting $a \leq b$ whenever $ab = a$. The resulting partially ordered set is denoted by $X(A)$. We claim that for all $\alpha_* \in \mathbb{D}^U$ and $a^* \in (U, A)$

$$(5.2.1) \qquad \langle \alpha_*, a^* \rangle = \bigvee \{a^v : \alpha_v = 1\}$$

holds. First we want to show that for any v with $\alpha_v = 1$ the equality $a^v \cdot \langle \alpha_*, a^* \rangle = a^v$ is satisfied. For this purpose let $u \in U$ be such that $u \neq v$. Define β_*^t, $t \in U$, by

$$\beta_*^t := \begin{cases} \alpha_*^{U \setminus \{v\}}, & \text{for } t = u \\ \delta_*^v, & \text{for } t = v \\ \text{anything}, & \text{otherwise.} \end{cases}$$

Then $\alpha_* = \langle \delta_\square^u + \delta_\square^v, \beta_*^\square \rangle$. Hence by the Absorption Law

$$\begin{aligned} a^v \cdot \langle \alpha_*, a^* \rangle &= \langle \alpha_*, a^v \cdot a^* \rangle \\ &= \big\langle \langle \delta_\square^u + \delta_\square^v, \beta_*^\square \rangle, a^v \cdot a^* \big\rangle \\ &= \big\langle \delta_\square^u + \delta_\square^v, \langle \beta_*^\square, a^v \cdot a^* \rangle \big\rangle \\ &= \langle \beta_*^u, a^v \cdot a^* \rangle + \langle \beta_*^v, a^v \cdot a^* \rangle \\ &= a^v \cdot \langle \beta_*^u, a^* \rangle + a^v \langle \delta_*^v, a^* \rangle \\ &= \langle \beta_*^u, a^* \rangle \cdot a^v + a^v = a^v. \end{aligned}$$

In other words, $a^v \leq \langle \alpha_*, a^* \rangle$ for all v with $\alpha_v = 1$. On the other hand, if $a^v \leq b$ for all v with $\alpha_v = 1$, i.e., $a^v \cdot b = a^v$, then by (3.5)

$$\langle \alpha_*, a^* \rangle = \langle \alpha_*, b \cdot a^* \rangle = b \langle \alpha_*, \alpha^* \rangle$$

and therefore $\langle \alpha_*, a^* \rangle \leq b$. This proves (5.2.1). In particular we obtain that $X(A)$ is a complete join-semilattice and thus a complete lattice. Moreover we have $a \wedge b = ab$. Finally

$$b \wedge \bigvee \{a^v : \alpha_v = 1\} = b \cdot \langle \alpha_*, a^* \rangle = \langle \alpha_*, b \cdot a^* \rangle = \bigvee \{b \wedge a^v ; \alpha_v = 1\},$$

which is the infinite distributive law for $X(A)$. Hence $X(A)$ is a frame. If $g : A \to B$ is a homomorphism of $\mathbb{D}^U$-convex algebras. Then by (3.4)

$$g(\langle \alpha_*, a^* \rangle) = \langle \alpha_*, g^U(a^*) \rangle, \quad \alpha_* \in \mathbb{D}^U \text{ and } a^* \in (U, A),$$

which shows that g preserves arbitrary joins. Since g preserves products, it preserves finite meets. Hence g is a frame morphism $X(g) : X(A) \to X(B)$. It is routine to verify that the resulting functors $A : \mathcal{F}rm \to \mathbb{D}^U \mathcal{A}lg_F$ and $X : \mathbb{D}^U \mathcal{A}lg_F \to \mathcal{F}rm$ are inverses to each other. ■

5.3. Addendum.

Let $\mathcal{F}(A)$ be the free $\mathbb{D}^U$-convex module on the set A. Then the corresponding partial order relation (see proof of (5.2)) satisfies the infinite distributive law. In particular, $\mathcal{F}(A)$ always carries an associative, commutative, idempotent $\mathbb{D}^U$-algebra structure satisfying the Absorption Law. Moreover, if $f : \mathcal{F}(A) \to X$ is a surjective homomorphism of $\mathbb{D}^U$-convex modules then for any $x', x'' \in X$

$$f(\bigvee\{f^{-1}(x')\} \wedge \bigvee\{f^{-1}(x'')\}) = x' \wedge x''. \tag{5.3.1}$$

Proof:

The elements of $\mathcal{F}(A)$ (see proof of (4.1)) are certain maps $\varphi : A \to R$. In addition, $\varphi_1 + \varphi_2$, as defined above, is the map given by $(\varphi_1 + \varphi_2)(a) := \varphi_1(a) + \varphi_2(a), a \in A$, that is $\varphi_1 + \varphi_2$ is the pointwise sum. Since $\mathbb{D}^U$ is the set of all maps $U \to \mathbb{D}$ whose support is a set, $\varphi_1 \leq \varphi_2$ is equivalent to $\varphi_1(a) \leq \varphi_2(a)$. Hence arbitrary joins and meets are formed pointwise. This shows that the validity of the infinite distributive law needs to be checked only pointwise, which means in $\mathbb{D}$ itself. $\mathbb{D}$, however, satisfies the infinite distributive law. Now let X be any $\mathbb{D}^U$-convex module and let $f : \mathcal{F}(A) \to X$ be a surjective homomorphism of $\mathbb{D}^U$-convex modules. Due to (5.2.1), f preserves arbitrary joins. In particular, $f(\bigvee\{f^{-1}(x)\}) = x$ for all $x \in X$. Hence, if $\varphi \in \mathcal{F}(A)$ satisfies $\varphi \leq \bigvee\{f^{-1}(x')\} \wedge \bigvee\{f^{-1}(x'')\}$ then

$$\begin{aligned} f(\varphi) &\leq f(\bigvee\{f^{-1}(x')\} \wedge \bigvee\{f^{-1}(x'')\}) \\ &\leq f(\bigvee\{f^{-1}(x')\}) \wedge f(\bigvee\{f^{-1}(x'')\}) = x' \wedge x''. \end{aligned}$$

On the other hand if $y \leq x'$ then there is a $\bar{y} \in X$ with $y + \bar{y} = x'$. Suppose $\psi, \bar{\psi} \in \mathcal{F}(A)$ satisfy $f(\psi) = y$ and $f(\bar{\psi}) = \bar{y}$. Then $f(\psi + \bar{\psi}) = f(\psi) + f(\bar{\psi}) = y + \bar{y} = x'$ and thus $\psi \leq \bigvee\{f^{-1}(x')\}$. This means that f maps $\{\varphi : \varphi \leq \bigvee\{f^{-1}(x')\} \wedge \bigvee\{f^{-1}(x'')\}\}$ onto $\{y : y \leq x' \wedge x''\}$. As a consequence we obtain formula (5.3.1). ■

6. Classification of convexity theories over $\mathbb{D}$

By a *level* we mean either the cardinal number of a set or the cardinal number of U. As usual we define for two levels λ_1 and λ_2, $\lambda_1 \preceq \lambda_2$ provided there are sets A_1 and A_2 with $\operatorname{card}(A_i) = \lambda_i$, $i = 1, 2$, for which there is an injective map $A_1 \to A_2$; we write $\lambda_1 \prec \lambda_2$ in case $\lambda_1 \preceq \lambda_2$ and $\lambda_1 \neq \lambda_2$.

6.1. Proposition.

Let Γ be a big convexity theory over $\mathbb{D}$. Then there exists a unique level λ, which is either 1 or infinite, such that Γ is one of the following:

$$\begin{aligned}
(\lambda_1) &= \{\alpha_* \in \mathbb{D}^U : \quad \operatorname{card}(\operatorname{supp}\alpha_*) \preceq \lambda\} && \textit{and} \quad \lambda \neq \operatorname{card} U; \\
(\lambda_2) &= \{\alpha_* \in \mathbb{D}^U : 1 \preceq \operatorname{card}(\operatorname{supp}\alpha_*) \preceq \lambda\} && \textit{and} \quad \lambda \neq \operatorname{card} U; \\
(\lambda_3) &= \{\alpha_* \in \mathbb{D}^U : \quad \operatorname{card}(\operatorname{supp}\alpha_*) \prec \lambda\} && \textit{and} \quad \lambda \quad \textit{is infinite}; \\
(\lambda_4) &= \{\alpha_* \in \mathbb{D}^U : 1 \preceq \operatorname{card}(\operatorname{supp}\alpha_*) \prec \lambda\} && \textit{and} \quad \lambda \quad \textit{is infinite}.
\end{aligned}$$

Conversely, if λ is equal to 1 or λ is an infinite level, then each of the big sets $(\lambda_1) - (\lambda_4)$ is a big convexity theory over $\mathbb{D}$.

Proof:

One checks easily that $\{0_*\} \cup \{\delta_*^u : u \in U\}$ and $\{\delta_*^u : u \in U\}$ are convexity theories over $\mathbb{D}$. They correspond to (λ_1) and (λ_2) for $\lambda = 1$. Suppose that there is an $\alpha_* \in \Gamma$ with $1 \prec \operatorname{card}(\operatorname{supp} \alpha_*)$. Let A be any non-empty subset of U such that $\operatorname{card}(A) \preceq \operatorname{card}(\operatorname{supp} \alpha_*)$ holds. Then there is an injective map $j : A \to \operatorname{supp} \alpha_*$. Choose $a \in A$ and define $\beta_*^u, u \in U$, by

$$\beta_*^u := \begin{cases} \delta_*^a & \text{if } u = j(a) \quad \text{or } u \in \operatorname{supp} \alpha_* \setminus j(A), \\ \delta_*^b & \text{if } u = j(b) \quad \text{and } b \neq a. \end{cases}.$$

Then a simple computation leads to $\operatorname{supp}\langle \alpha_\square, \beta_*^\square \rangle = A$. Consequently, Γ contains all elements $\gamma_* \in \mathbb{D}^U$ with $1 \preceq \operatorname{card}(\operatorname{supp} \gamma_*) \preceq \operatorname{card}(\operatorname{supp} \alpha_*)$. In particular, Γ contains a $\gamma_* \in \mathbb{D}^U$ with $\operatorname{card}(\operatorname{supp} \gamma_*) = 2$, and consequently all $\gamma_* \in \mathbb{D}^U$ with non-empty finite support belong to Γ. The map

$$\Gamma \ni \alpha_* \longmapsto \operatorname{card}(\operatorname{supp} \alpha_*)$$

is either bounded by the cardinality of some set or for each set B there is an $\alpha_* \in \Gamma$ with $\operatorname{card}(B) \prec \operatorname{card}(\operatorname{supp} \alpha_*)$. In the latter case Γ is either $\mathbb{D}^U$ or $\mathbb{D}^U \setminus \{0_*\}$. Both of these big sets are convexity theories. They correspond to (λ_3) and (λ_4) in case $\lambda = \operatorname{card}(U)$. In the first case, however, $\lambda := \sup\{\operatorname{card}(\operatorname{supp} \alpha_*) : \alpha_* \in \Gamma\}$ is the cardinality of some set, and hence $\lambda \neq \operatorname{card}(U)$. If there is an $\alpha_* \in \Gamma$ with $\lambda = \operatorname{card}(\operatorname{supp} \alpha_*)$, then Γ is either of the type (λ_1) or of the type (λ_2). Otherwise we have $\operatorname{card}(\operatorname{supp} \alpha_*) \prec \gamma$ for all $\alpha_* \in \Gamma$, and Γ is either of the type (λ_3) or of the type (λ_4). This discussion also shows that λ is either equal to 1 or else infinite. A simple computation shows that for the stated values of λ each of the sets $(\lambda_1) - (\lambda_4)$ is a big convexity theory over $\mathbb{D}$. ■

6.2. Addendum.

Suppose that N is a set. Let Γ be a N-convexity theory over $\mathbb{D}$. Then there exists a unique cardinal number λ, which is either 1 or infinite and $\preceq \operatorname{card}(N)$, such that Γ is one of the following:

(λ_1^N)	$\{\alpha_* \in \mathbb{D}^U :$	$\operatorname{card}(\operatorname{supp} \alpha_*) \preceq \lambda\}$;	
(λ_2^N)	$\{\alpha_* \in \mathbb{D}^U :$	$1 \preceq \operatorname{card}(\operatorname{supp} \alpha_*) \preceq \lambda\}$;	
(λ_3^N)	$\{\alpha_* \in \mathbb{D}^U :$	$\operatorname{card}(\operatorname{supp} \alpha_*) \prec \lambda\}$	and λ is infinite;
(λ_4^N)	$\{\alpha_* \in \mathbb{D}^U :$	$1 \preceq \operatorname{card}(\operatorname{supp} \alpha_*) \prec \lambda\}$	and λ is infinite.

Conversely, if λ is equal to 1 or λ is an infinite cardinal number with $\lambda \preceq \operatorname{card}(N)$, then each of the sets $(\lambda_1^N) - (\lambda_4^N)$ is a N-convexity theory over $\mathbb{D}$.

Proof:

Nearly identical with the proof of (6.1). ■

6.3. Proposition.

Suppose that N is a set and that λ is a cardinal number, which is either 1 or infinite and $\preceq \operatorname{card}(N)$. Then for each $i = 1, \dots, 4$ the categories $(\lambda_i)\mathcal{C}$ and $(\lambda_i^N)\mathcal{C}$ are isomorphic.

Proof:

Under the stated assumptions we have $\lambda_i^N \cong \lambda_i \mid N$. Hence (3.17) leads to our statement. ■

The convexity theories listed in (6.2) are briefly mentioned and the associated convex module categories $(\lambda_i^N)\mathcal{C}$, $i = 1, \dots, 4$, are touched upon in [**4**]. Finally the proof of (5.2) shows that $(\operatorname{card}(U)_3)\mathcal{C} = (\mathbb{D}^U)\mathcal{C}$ is canonically isomorphic with the category $\mathcal{CSL}at$ of complete (join-)semilattices and their homomorphisms. (Observe that although the objects are in fact complete lattices, the homomorphisms are only required to preserve joins.) Denoting $\operatorname{card}(U)_4$ by $\mathbb{D}^{U_*} = \{\alpha_* \in \mathbb{D}^U : \alpha_* \neq 0_*\}$, it remains to describe $(\mathbb{D}^{U_*})\mathcal{C}$.

6.4. Proposition.

The category $(\mathbb{D}^{U_*})\mathcal{C}$ is isomorphic to the category $\mathcal{CSL}at^*$, whose objects are those ordered sets that have joins for all non-empty subsets and whose morphisms are those order-preserving maps that preserve these joins.

Proof:

Let X be a $\mathbb{D}^{U_*}$-convex module. As in the proof of (5.2) one defines an addition $X \times X \ni (x', x'') \mapsto x' + x'' \in X$ and a relation "$x' \leq x''$" on X, and shows as there that X equipped with this addition is a unital hemimodule over the hemiring $\mathbb{D}^* := \{1\}$ and that the relation $x' \leq x''$ is an order relation with respect to which X has joins for arbitrary non-empty subsets of X. The morphisms of $\mathbb{D}^{U_*}$-convex modules satisfy (3.4) and hence preserve joins since $\langle \alpha_*, x^* \rangle = \bigvee \{x^u : \alpha_u = 1\}$ holds. Conversely, if X is an ordered set with the property stated in (6.4), then $\langle \alpha_*, x^* \rangle := \bigvee \{x^u : \alpha_u = 1\}$ makes X a $\mathbb{D}^{U_*}$-convex module. If a map between two such ordered sets preserves joins then it satisfies, by definition of $\langle \alpha_*, x^* \rangle$, the defining property (3.4) of morphisms of $\mathbb{D}^{U_*}$-convex modules. ■

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Heinrich Kleisli:
Math. Institute
University of Fribourg
CH-1700 Fribourg
SWITZERLAND

Helmut Röhrl:
9322 La Jolla Farms Road
La Jolla, CA 92037
U.S.A.

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