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QUADRATIC VECTOR FIELDS WITH A WEAK FOCUS OF THIRD ORDER

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Abstract

We study phase portraits of quadratic vector fields with a weak focus of third order at the origin. We show numerically the existence of at least 20 different global phase portraits for such vector fields coming from exactly 16 different local phase portraits available for these vector fields. Among these 20 phase portraits, 17 have no limit cycles and three have at least one limit cycle.

1. Introduction

A vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $X = (P, Q)$ where $P = \sum a_{ij} x^i y^j$ and $Q = \sum b_{ij} x^i y^j$, $0 \leq i+j \leq n$, is called a *planar polynomial vector field of degree* n if $\sum_{i+j=n}(|a_{ij}|+|b_{ij}|) \neq 0$. The $M = (n+1)(n+2)$ real numbers a_{ij}, b_{ij} are called the *coefficients of* X. The space of these vector fields, endowed with the structure of affine $\mathbb{R}^M$-space in which X is identified with the M-tuple $(a_{00}, a_{10}, \ldots, a_{0n}, b_{00}, b_{10}, \ldots, b_{0n})$ of its coefficients, is denoted by $\mathcal{P}_n(\mathbb{R}^2)$. In particular, the polynomial vector fields of degree 2 are called *quadratic vector fields*.

A *weak focus* of a planar vector field is defined as a critical point such that the real part of its eigenvalues is zero. Several levels of weakness are defined for these points. In particular, for quadratic vector fields, we may have up to three levels of weakness. For more details see Section 3.

The main open problem in the study of the planar polynomial vector fields is the number of limit cycles and their distribution, the so called Hilbert's sixteen problem, see [**Hi**]. This problem remains open even for the quadratic vector fields, where it seems that the answer must be the following: A quadratic vector field has at most 4 limit cycles, and when

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they exist, three are surrounding one focus and the other is surrounding a different focus. All the analytic examples realizing these four limit cycles are obtained perturbing a quadratic system having a weak focus of third order. This is the main reason which motivates our classification of the quadratic vector fields with a weak focus of third order.

Prior to 1950, there was some interest in the qualitative theory of quadratic vector fields, but a real impetus was given to the development of this subject in the fifties. In fact, more than eight hundred papers have been published on this subject, see Reyn [**R1**].

The *Poincaré compactification* of $X \in \mathcal{P}_n(\mathbb{R}^2)$ is defined to be the unique analytic vector field $p(X) \in \mathcal{P}_n(\mathbb{S}^2)$ tangent to the sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$, whose restriction to the northern hemisphere $\mathbb{S}^2_+ = \{(x,y,z) \in \mathbb{S}^2 : z > 0\}$ is given by $z^{n-1}Df_+(X)$, where f_+ is the central projection from $\mathbb{R}^2 \equiv \{(x,y,1) \in \mathbb{R}^3 : (x,y) \in \mathbb{R}^2\}$ to $\mathbb{S}^2_+$, defined by $f_+(x,y) = (x,y,1)/(x^2+y^2+1)^{1/2}$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1,y_2,y_3) \mapsto (y_1,y_2)$ is called the *Poincaré disc*. See Section 2 for more details.

Let $p(X) \in \mathcal{P}_n(\mathbb{S}^2)$. We define the *local phase portrait around the critical points* as the union of the set of all critical points of $p(X)$ (finite and infinite) together with their local phase portrait. We will say that two vector fields $p(X_1)$ and $p(X_2) \in \mathcal{P}_n(\mathbb{S}^2)$ have *equivalent local phase portrait* if there is a homeomorphism of $\mathbb{S}^2$ leaving the equator $\mathbb{S}^1$ invariant and such that restricted to some neighborhood of each finite or infinite critical points carries orbits of the flow induced by $p(X_1)$ onto orbits of the flow induced by $p(X_2)$. Two vector fields $p(X_1)$ and $p(X_2) \in \mathcal{P}_n(\mathbb{S}^2)$ having equivalent local phase portraits may have non-equivalent global phase portraits. This is due to the fact that the global behavior of their separatrices (different from critical points and limit cycles) is different, or to the fact that they can have different number or distribution of limit cycles.

In this paper we get the local phase portrait around the critical points on the Poincaré disc of all the quadratic polynomial vector fields with one weak focus of third order. More precisely our main result can be stated as follows.

Theorem 1.1. *A quadratic vector field with a weak focus of third order may have up to 16 different local phase portraits around the critical points. Two of these local phase portraits are realizable at least by three different global phase portraits. The remainder* 14 *different local phase portraits around the critical points are realized at least by one global phase portrait.*

After this we characterize numerically the global phase portraits from these local phase portrait by finding realizable cases from each one of them.

Numerical result. *There are at least* 20 *phase portraits for the quadratic vector fields having a weak focus of third order. All these phase portraits are shown in Figure 1.1.*

The study is divided in four different parametric normal forms. In the study of these parametric families we will find several phase portraits having a center instead of a weak focus. The different phase portraits for such systems having a center are given in Figure 1.2 and are denoted by $V_{\#}$. We use the classification of Vulpe [**V**].

This paper is an improved version of the Master Ph. D. of the first author under the direction of the second author [**A1**].

2. Basic definitions and results

In this section we introduce some notation and definitions that we will need later.

Given the differential system

$$
\begin{aligned}
x' &= P(x_0, y_0),\\
y' &= Q(x_0, y_0),
\end{aligned}
\tag{2.1}
$$

where P and Q are two real analytic functions in the variables x and y, we say that (x_0, y_0) is a *critical* or *singular point* if $P(x_0, y_0) = Q(x_0, y_0) = 0$. In order to classify it we denote by $\Delta(x_0, y_0)$ and by $\rho(x_0, y_0)$, the determinant and the trace of the linear part of system (2.1) at the critical point (x_0, y_0) respectively. Set also $\delta(x_0, y_0) = \rho(x_0, y_0)^2 - 4\Delta(x_0, y_0)$. If $\Delta(x_0, y_0) \neq 0$ then we say that the critical point is *non-degenerated* and *elementary*; and we can apply the results of [**ALGM**] (see Chapter IV) in order to determine its local behavior. If $\Delta(x_0, y_0) = 0$ but $\rho(x_0, y_0) \neq 0$ then we say that the critical point is *degenerated* and *elementary*; and we apply Theorem 65 of [**ALGM**] in order to determine its local behavior. If $\Delta(x_0, y_0) = 0$ and $\rho(x_0, y_0) = 0$ but the matrix of the linear part of (2.1) at (x_0, y_0) is not identically zero, then we say that the critical point is *degenerated* and *nilpotent*; and we apply the result of [**An**] in order to determine its local behavior. If the matrix of the linear part of (2.1) at (x_0, y_0) is identically zero, then we say that the critical point is *degenerated with linear part equal to zero*, and we need to make a particular study around this point by using blow up's in order to determine its local behavior [**ALGM**].

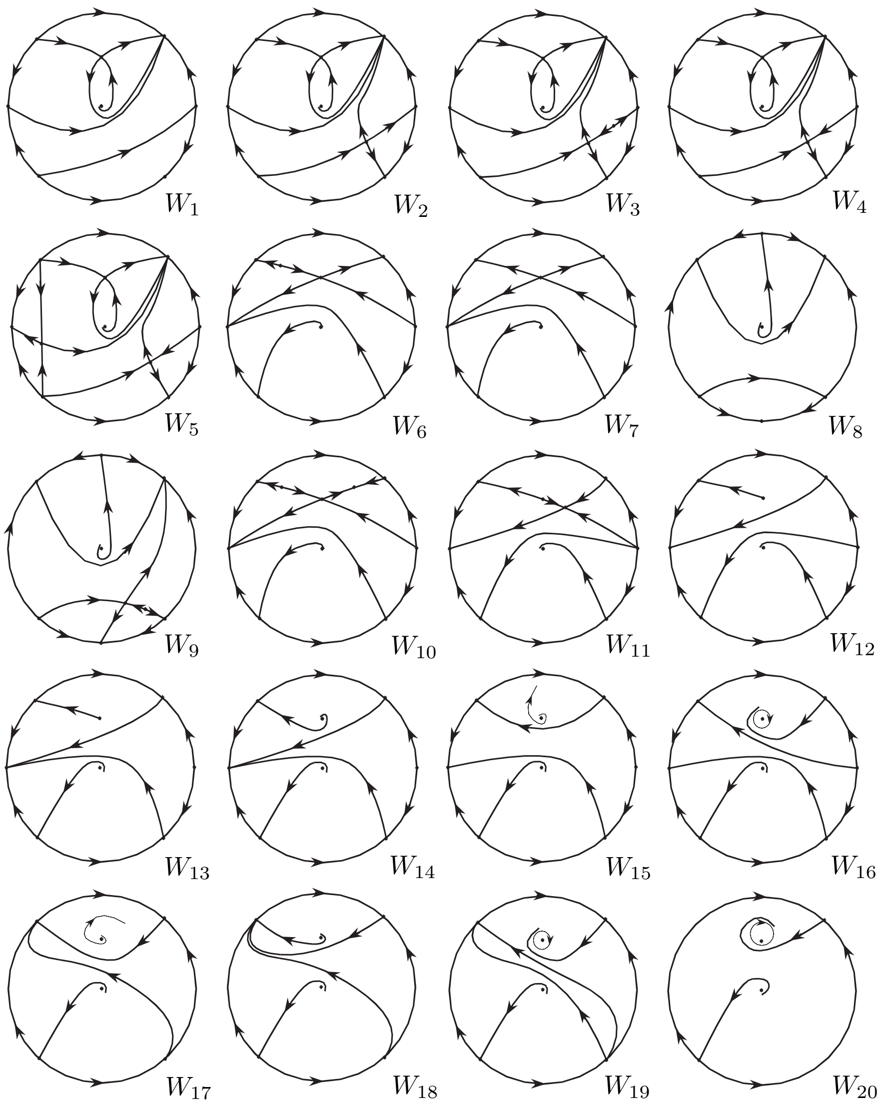

Figure 1.1. Phase portraits W_i for quadratic vector fields with a weak focus of third order.

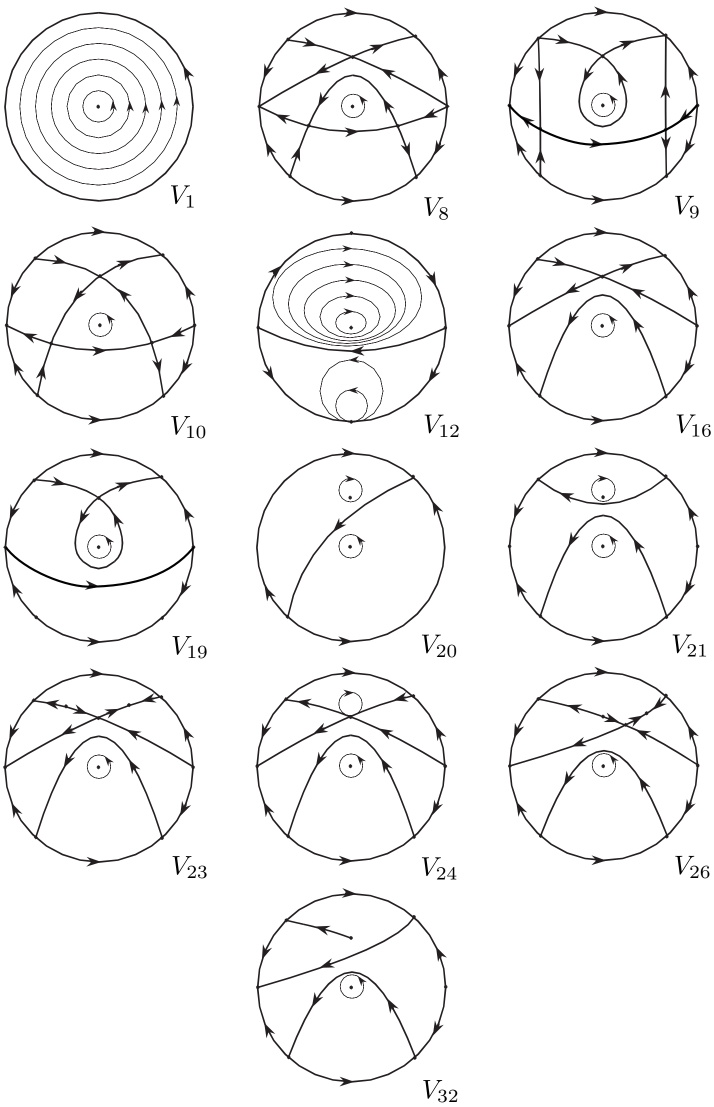

Figure 1.2. Phase portraits V_i for quadratic vector fields with a center which appear during the study of systems with a weak focus of third order.

For $X \in \mathcal{P}_n(\mathbb{R}^2)$ the *Poincaré compactified vector field* $p(X)$ *corresponding to* X is a vector field induced on $\mathbb{S}^2$ as follows (see, for instance [**G**], [**S**] and [**ALGM**]). Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ the tangent space to $\mathbb{S}^2$ at point y. Consider the central projections $f_+ : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2_+ = \{y \in \mathbb{S}^2 : y_3 > 0\}$ and $f_- : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2_- = \{y \in \mathbb{S}^2 : y_3 < 0\}$. These maps define two copies of X, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector field defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ by $Df_+ \circ X$ and $Df_- \circ X$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. In order to extend X' to an analytic vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that X satisfies suitable hypotheses. In the case that $X \in \mathcal{P}_n(\mathbb{R}^2)$, the *Poincaré compactification* $p(X)$ of X is the only analytic extension of $y_3^{n-1}X'$ to $\mathbb{S}^2$. The set of all compactified vector fields $p(X)$ with $X \in \mathcal{P}_n(\mathbb{R}^2)$ is denoted by $\mathcal{P}_n(\mathbb{S}^2)$.

For the flow of the compactified vector field $p(X)$, the equator $\mathbb{S}^1$ is invariant. On $\mathbb{S}^2 \setminus \mathbb{S}^1$ there are two symmetric copies of X, and knowing the behavior of $p(X)$ around $\mathbb{S}^1$, we know the behavior of X at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*, and it is denoted by $\mathbb{D}^2$.

As $\mathbb{S}^2$ is a differenciable manifold for computing the expression of $p(X)$, we can consider the following six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$, and the diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ which are the inverses of the central projections from the vertical planes tangents at points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)$ respectively. We denote by $z = (z_1, z_2)$ the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$. So z represents different things according to the local chart under consideration. Some easy computations give for $p(X)$ the following expressions

$$z_2^n \cdot \Delta(z) \left[Q\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) - z_1 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right), -z_2 P\left(\frac{1}{z_2}, \frac{z_1}{z_2}\right) \right], \tag{2.2}$$

$$z_2^n \cdot \Delta(z) \left[P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), -z_2 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right], \tag{2.3}$$

$$\Delta(z) \left[P(z_1, z_2), Q(z_1, z_2) \right],$$

in the local charts U_1, U_2 and U_3 respectively. Here $\Delta(z) = ({z_1}^2 + {z_2}^2 + 1)^{-\frac{n-1}{2}}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates and in the local charts U_i and V_i for $i = 1, 2$, $z_2 = 0$ denotes always the points of $\mathbb{S}^1$. In what follows we omit the factor $\Delta(z)$ by rescaling the vector field $p(X)$.

So, in each local chart we obtain a polynomial vector field. We denote by $\mathcal{P}_n(\mathbb{S}^2)$ all the polynomial vector fields $p(X)$ on $\mathbb{S}^2$ defined as above and endowed with the coefficient topology.

A critical point q of $X \in \mathcal{P}_n(\mathbb{R}^2)$ is called *infinite* (respectively *finite*) if it is a critical point of $p(X)$ in $\mathbb{S}^1$ (respectively in $\mathbb{S}^2 \setminus \mathbb{S}^1$). Then to compute the infinite critical points of X we take the nonzero terms of the expressions (2.2) and (2.3) with $z_2 = 0$ and we obtain

$$F(z_1) = Q_n(1, z_1) - z_1 P_n(1, z_1), \tag{2.4}$$

$$G(z_1) = P_n(z_1, 1) - z_1 Q_n(z_1, 1), \tag{2.5}$$

respectively, where P_n and Q_n are the homogeneous part of degree n of P and Q. Thus, the infinite critical points of X are the points $(z_1, 0)$ satisfying

$$\begin{aligned} F(z_1) &= 0 \quad \text{if } (z_1, 0) \in U_1, \\ G(z_1) &= 0 \quad \text{if } (z_1, 0) \in U_2. \end{aligned}$$

We remember that the equation $F(z_1) = 0$ gives us all the infinite critical points of X except, perhaps, the origin of coordinates of U_2.

In order to study the local behavior at the infinite critical points $(z_1, 0)$ we need its linear part given by the Jacobian matrix

$$\begin{pmatrix} \dfrac{dF(z_1)}{dz_1} & * \\ 0 & -P_n(1, z_1) \end{pmatrix} \quad \text{in} \quad U_1, \tag{2.6}$$

and

$$\begin{pmatrix} \dfrac{dG(z_1)}{dz_1} & * \\ 0 & -Q_n(z_1, 1) \end{pmatrix} \quad \text{in} \quad U_2. \tag{2.7}$$

3. Weak focus of third order

We will start from the expression given by Ye Yanqian (see [**Y1**] or [**Y2**]) of the quadratic vector fields which have a weak focus at the origin:

$$\begin{aligned} x' &= -y + lx^2 + mxy + ny^2, \\ y' &= x(1 + ax + by). \end{aligned}$$

From the papers of Bautin [**B**], we know how to classify a weak focus in four types, but here we use the notation of Li [**Li1**]. More precisely, given the constants:

$$
\begin{aligned}
L_1 &= m(l+n) - a(b+2l),\\
L_2 &= ma(5a-m)[(l+n)^2(n+b) - a^2(b+2l+n)],\\
L_3 &= ma^2[2a^2+n(l+2n)][(l+n)^2(n+b) - a^2(b+2l+n)],
\end{aligned}
$$

we say that the origin of the above system is

(1) a *weak focus of first order* if $L_1 \neq 0$,

(2) a *weak focus of second order* if $L_1 = 0$ and $L_2 \neq 0$,

(3) a *weak focus of third order* if $L_1 = L_2 = 0$ and $L_3 \neq 0$,

(4) a *center* if $L_1 = L_2 = L_3 = 0$.

A center is a singular point such that all the orbits in a neighborhood of it are periodic. We also know that if the origin is a focus of order k (with $k = 1, 2, 3$), its stability is given by the sign of L_k; so we have that the origin is a *stable* weak focus of order k (respectively *unstable*) if and only if $L_k < 0$ (respectively > 0). If a quadratic system has a center, then it is integrable, see [**C**] and [**LS**].

As we want to study the quadratic vector fields with a weak focus of third order, we must force certain conditions to the coefficients l, m, n, a and b. In concrete, $L_1 = L_2 = 0$, and $L_3 \neq 0$. So we have that:

$$
\begin{aligned}
m &\neq 0,\\
a &\neq 0,\\
2a^2 + n(l+2n) &\neq 0,\\
(l+n)^2(n+b) - a^2(b+2l+n) &\neq 0.
\end{aligned}
$$

This, with the fact that $L_2 = 0$, implies that $m = 5a$. Substituting in $L_1 = 0$ we get that $5a(l+n) - a(b+2l) = 0$. As $a \neq 0$ we have $b = 3l+5n$.

In short, a quadratic vector field with a weak focus of third order at the origin may be written in the form

$$
\begin{aligned}
x' &= -y + lx^2 + 5axy + ny^2,\\
y' &= x + ax^2 + (3l+5n)xy,
\end{aligned}
$$

with $L_3 = 5a^3[2a^2 + n(l+2n)][(l+n)^2(6n+3l) - a^2(5l+6n)] \neq 0$. This vector field depends on three parameters; we can simplify it to two.

If $n \neq 0$, then with the change $x = n^{-1}X$, $y = n^{-1}Y$, and naming $l' = ln^{-1}$ and $a' = an^{-1}$, we get

$$
\begin{aligned}
X' &= -Y + l'X^2 + 5a'XY + Y^2,\\
Y' &= X + a'X^2 + (3l'+5)XY.
\end{aligned}
\tag{3.1}
$$

If $n = 0$ and $a \neq 0$, then with the change $x = a^{-1}X$, $y = a^{-1}Y$, and naming $l' = la^{-1}$, we get

$$\begin{aligned} X' &= -Y + l'X^2 + 5XY, \\ Y' &= X + X^2 + 3l'XY. \end{aligned} \tag{3.2}$$

If $n = a = 0$, then $L_3 = 0$, and the origin is a center. If $l \neq 0$, then with the change $x = l^{-1}X$, $y = l^{-1}Y$, we get

$$\begin{aligned} X' &= -Y + X^2, \\ Y' &= X + 3XY. \end{aligned} \tag{3.3}$$

Finally, if $l = 0$, then

$$\begin{aligned} X' &= -Y, \\ Y' &= X. \end{aligned} \tag{3.4}$$

These two last vector fields are integrable and so we know their phase portraits corresponding to phase portraits V_{12} and V_1 respectively. So we only have to study vector fields (3.1) and (3.2). In order to do that, we will split the plane of parameters (a', l') of the vector field (3.1), or the line of parameters (l') of the vector field (3.2) in convenient regions (from now on we omit the primes and change X, Y by x, y). These regions are limited by curves which mean when critical points, finite or infinite, appear or disappear, when any of the eigenvalues of the Jacobian matrix at a critical point is zero, or when the vector field is integrable. With this we can control the local behavior at critical points of all these vector fields. In order to obtain the global phase portraits we need to study the existence of limit cycles and the different α- and ω-limit sets for the separatrices of the systems.

4. Study of the normal form 3.1

The phase portraits of Figure 1.1 represent all the local phase portraits for normal form (3.1) changing a and l (except phase portraits W_8 and W_9 which are only realizable under normal form (3.2)). These local phase portraits have been included in a global phase portrait which is provided from it and that we have checked numerically its existence.

The plane $\mathbb{R}^2$ with parameters a and l has been divided in a collection of regions of dimensions 0, 1 and 2, so that for all the points (a, l) belonging to the same region, the local phase portrait is the same.

We will denote by (R), (A) or (C), and (P) or (Q), with the corresponding subindex, convenient regions of dimension 2, 1 and 0, respectively.

In Lemma 4.1 we will see that we only need to study the half-plane $a \geq 0$.

From Lemmas 4.2, 4.3, 4.4 and 4.5 we will see that the curves that separate regions will be:

(1) $l(3l+5)^2 - 3a^2(5l+8) = 0$, which is union of C_2, C_5, C_6, C_9, $P_2 = (0,0)$, $P_6 = (0,-8/5)$ and $Q_1 = (\sqrt{3}/3, -2)$;

(2) $3a^2 - l(l+2) = 0$, which is union of C_1, C_{11}, $P = (0,-2)$ and P_2;

(3) $25a^2 + 12(l+2) = 0$, which is union of C_{13} and P;

(4) $l = -2$, which is union of C_7, C_8, Q_1 and P;

(5) $a = 0$, which is union of A_1, A_3, A_5, A_7, A_{14}, A_{15}, P_2, $P_4 = (0,-1)$, P_6, P and $P_{15} = (0,-5/2)$;

(6) $125a^4 + a^2(25l^2 + 170l + 262) + (2l+5)^3 = 0$, which is union of C_{15}, C_{16}, P_{15} and $Q_2 \approx (0.054684523, -2.4085)$;

(7) $2a^2 + l + 2 = 0$, which is union of C_{12} and P;

(8) $3(l+1)^2(l+2) - a^2(5l+6) = 0$, which is union of C_3, C_4, C_{10}, P_4 and P.

We will see in Lemma 4.7 that a given local phase portrait has three different possible global phase portraits so that there must be a curve of unknown equation which we will name C_{14} having Q_2 and P as ends.

In Figure 4.1 we see the regions in which we have divided the semiplane $a \geq 0$. We have also written the code for the phase portrait corresponding to each region.

In Table 4.1 we describe the finite and infinite critical points in each region, and we also note the global phase portrait that we have found for a concrete case in that region. We use the next notation:

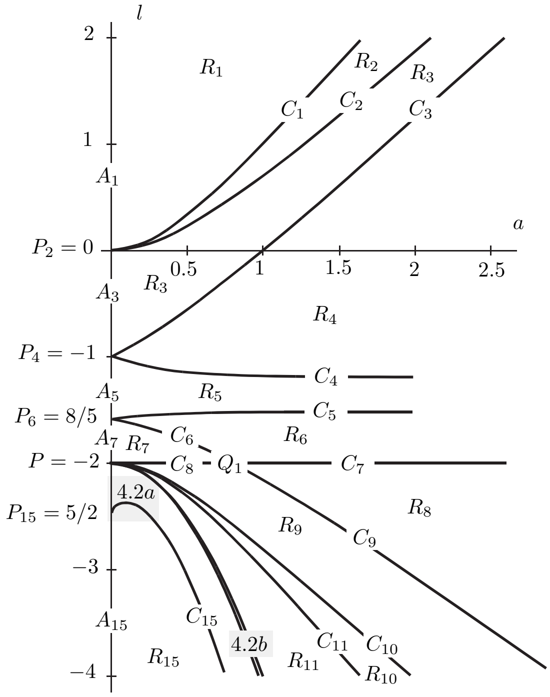

Figure 4.1. Bifurcation diagram for the normal form (3.1).

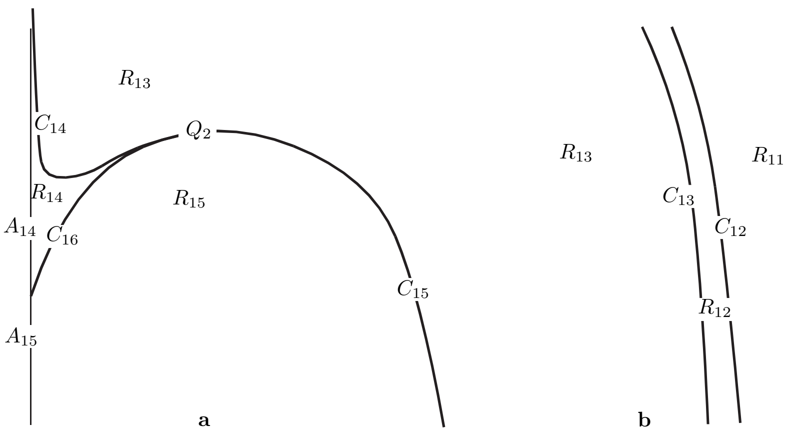

Figure 4.2. Magnifications of Figure 4.1.

C : center,
WF : weak focus,
SF : strong focus,
S : non-degenerate saddle,
N : elementary node,
A : stable, or attractor,
R : unstable, or repellor,
SN : elementary saddle-node (A or R will specify the behavior of the node),
SS : elementary degenerate saddle,
H : nilpotent critical point formed by one hyperbolic sector and one elliptic sector.

For a saddle-node at infinity we will distinguish:

SN1: the two hyperbolic sectors are in the same side of the infinity (we also add A if in the local chart $U_1 \cap \mathbb{D}^2$ we have a behavior of stable node, R if it is of unstable node, SA if it is of saddle with stable finite separatrix and SR if it is unstable).

SN2: the two hyperbolic sectors are in different sides of the infinity (we also add A or R depending on the behavior of the nodal component of the saddle-node in the local chart $U_1 \cap \mathbb{D}^2$).

We will write in capital letters the critical points at the infinite and in small letters the critical points at the finite plane.

Moreover we will write the points at infinity in order of decreasing slopes and only in the local chart U_1, indicating only the character of the nodes in this chart. In the finite case, we will write first the point $(0, 0)$, after the point $(0, 1)$, and after the other two if they do exist. In the regions where we have proved the existence of a limit cycle we will add the symbol ∃ (of existence), if it may exist but we have not proved it, we will write P∃ (for possible existence).

$A_1 \to V_{19}$ c, s, NA, S, NA.	$R_1 \to W_1$ fdr, s, NA, S, NA.
	$C_1 \to W_2$ fdr, s, snr, NA, S, NA.
$P_2 \to V_{19}$ c, s, NA, SS, NA.	$R_2 \to W_3$ fdr, s, s, nr, NA, S, NA.
	$C_2 \to W_4$ fdr, s, s, NA, SN1R, NA.
$A_3 \to V_9$ c, s, s, s, NA, NR, NA.	$R_3 \to W_5$ fdr, s, s, s, NA, NR, NA.
	$C_3 \to V_8$ c, s, s, s, NA, NR, NA.
$P_4 \to V_{10}$ c, s, s, s, NA, NR, NA.	$R_4 \to W_5$ fda, s, s, s, NA, NR, NA.
	$C_4 \to V_9$ c, s, s, s, NA, NR, NA.

$A_5 \to V_8$ c, s, s, s, NA, NR, NR.	$R_5 \to W_5$ fdr, s, s, s, NA, NR, NA.
	$C_5 \to W_4$ fdr, s, s, NA, NR, SN1SR.
$P_6 \to V_{16}$ c, s, SN1A, NR, SN1SR.	$R_6 \to W_3$ fdr, s, s, nr, NA, NR, S.
$C_6 \to W_6$ fdr, s, nr, SN1A, NR, S.	$C_7 \to W_2$ fdr, snr, s, NA, NR, S.
$A_7 \to V_{23}$ c, s, nr, na, S, NR, S.	$Q_1 \to W_7$ fdr, snr, SN1A, NR, S.
$R_7 \to W_{10}$ fdr, s, nr, na, S, NR, S.	$R_8 \to W_3$ fdr, nr, s, s, NA, NR, S.
$C_8 \to W_{11}$ fdr, snr, na, S, NR, S.	$C_9 \to W_6$ fdr, nr, s, SN1A, NR, S.
	$R_9 \to W_{10}$ fdr, nr, s, na, S, NR, S.
	$C_{10} \to V_{26}$ c, nr, s, na, S, NR, S.
	$R_{10} \to W_{10}$ fda, nr, s, na, S, NR, S.
	$C_{11} \to W_{11}$ fda, nr, sna, S, NR, S.
$P \to V_{24}$ c, h, S, NR, S.	$R_{11} \to W_{12}$ fda, nr, S, NR, S.
	$C_{12} \to V_{32}$ c, nr, S, NR, S.
	$R_{12} = C_{13} \to W_{13}$ fdr, nr, S, NR, S.
	$R_{13} \to W_{14}$ fdr, ffr, S, NR, S, P∃.
$C_{14} \to W_{15}$ fdr, ffr, S, NR, S, P∃.	
$A_{14} \to V_{21}$ c, c, S, NR, S.	$Q_2 \to W_{17}$ fdr, ffr, S, SN2R, P∃.
$R_{14} \to W_{16}$ fdr, ffr, S, NR, S, ∃.	$C_{15} \to W_{18}$ fdr, ffr, S, SN2R, P∃.
$P_{15} \to V_{20}$ c, c, SS.	$C_{16} \to W_{19}$ fdr, ffr, S, SN2R, ∃.
$A_{15} \to V_{20}$ c, c, S.	$R_{15} \to W_{20}$ fdr, ffr, S, ∃.

Table 4.1

For the graphic representation of the phase portraits, of this part and the next one, we have only drawn the separatrices of the critical points with an hyperbolic sector. In order to distinguish the hyperbolic sectors from the elliptic ones, in the rare cases we get an elliptic sector, we have drawn one or several curves which go out and return to the point. Moreover, in the cases where we have a center or a limit cycle, we have drawn a periodic orbit around it. All these orbits, and some other for which we will need to remark any interesting fact, will distinguish from the separatrices because the former are thinner than the latter.

All the phase portraits of Figure 1.1 have been got numerically with the program SDQ-SOFT (see [**A3**]).

Now we show the different lemmas needed for the classification of normal form (3.1).

Lemma 4.1. *Once we know the phase portrait of a vector field of normal form (3.1) for a given* (a, l) *with* $a \geq 0$*, the phase portrait given* *by $(-a,l)$ is the symmetric with respect to the y-axis and changing the sense of the orbits.*

The proof of this lemma is immediate.

Now we study the number of finite critical points of the normal form (3.1) in function of a and l.

Lemma 4.2. *The following statements hold.*

(i) *The points $(0,0)$ and $(0,1)$ are always critical points of the normal form (3.1).*

(ii) *If $(a,l) \in \{(0,-5/3),(0,0),(0,-2),(\sqrt{3}/3,-2)\}$ or if $3a^2 - l(l+2) < 0$, then the unique critical points of the normal form (3.1) are the $(0,0)$ and $(0,1)$.*

(iii) *If $3a^2 - l(l+2) = 0$ and $(a,l) \notin \{(0,0),(0,-2)\}$, then the point*

$$\left(\frac{a(9+6l)}{l(3l+5)^2 - 3a^2(5l+8)}, \frac{-l(3l+5)+3a^2}{l(3l+5)^2 - 3a^2(5l+8)}\right),$$

is the only critical point of the normal form (3.1) different from $(0,0)$ and $(0,1)$.

(iv) *If $(a,l) \notin \{(0,-5/3),(0,0),(\sqrt{3}/3,-2)\}$ and $l(3l+5)^2 - 3a^2(5l+8) = 0$, then the point*

$$\left(\frac{2+l}{2a(3+2l)}, \frac{-8-5l}{2(3+2l)(3l+5)}\right),$$

is the only critical point of the normal form (3.1) different from $(0,0)$ and $(0,1)$.

(v) *If $l = -2$ and $a(3a^2-1) \neq 0$, then the point*

$$\left(\frac{-3a}{3a^2-1}, \frac{-1}{3a^2-1}\right),$$

is the only critical point of the normal form (3.1) different from $(0,0)$ and $(0,1)$.

(vi) *If we are not in the cases (ii)-(v), then the points*

$$S = \left(\frac{9a+6al+(3l+5)D}{l(3l+5)^2 - 3a^2(5l+8)}, \frac{-l(3l+5)+3a^2-aD}{l(3l+5)^2 - 3a^2(5l+8)}\right),$$

$$R = \left(\frac{9a+6al-(3l+5)D}{l(3l+5)^2 - 3a^2(5l+8)}, \frac{-l(3l+5)+3a^2+aD}{l(3l+5)^2 - 3a^2(5l+8)}\right),$$

where $D = \sqrt{9a^2 - 3l(l+2)}$, are the only critical points of the normal form (3.1) different from $(0,0)$ and $(0,1)$.

Proof: (i) Trivial.

(iv) It is easy to see that the coordinate x of a critical point of the normal form (3.1) different from $(0,0)$ and $(0,1)$ is solution of the equation

$$x^2(l(3l+5)^2 - 3a^2(5l+8)) - 6a(2l+3)x + 3l + 6 = 0, \tag{4.1}$$

when l is not equal to $-5/3$. If the coefficient of x^2 is zero we obtain the result of (iv).

(ii) The discriminant of the equation (4.1) is of the form $(324a^2 - 108l(l+2))(l+5/3)^2$. If we take out what does not affect to the sign, we have $3a^2 - l(l+2)$. Then it follows easily (ii) and (iii). In order to complete the proof of (ii) we only have to check some concrete cases.

(vi) If we are not in the cases (ii)-(v), we have to isolate x.

(v) This is a particular case of (vi) in which one of the two critical points which come from the solution of (4.1) is the point $(0,1)$.

If $l = -5/3$ we cannot use the same arguments in the proof, but at the end, we get the same expression of (vi) with $l = -5/3$. ■

In the next lemma we will study the nature of the finite critical points.

Lemma 4.3. *The following statements hold.*

(i) *The point $(0,0)$ is an unstable weak focus of third order if $3(l+1)^2(l+2)-a^2(5l+6) > 0$, $2a^2+l+2 > 0$ and $a \neq 0$, or $2a^2+l+2 < 0$ and $a \neq 0$; stable if $3(l+1)^2(l+2)-a^2(5l+6) < 0$, $2a^2+l+2 > 0$ and $a \neq 0$; and a center if $3(l+1)^2(l+2) - a^2(5l+6) = 0$, or $2a^2+l+2 = 0$, or $a = 0$.*

(ii) *The point $(0,1)$ is a saddle if $l > -2$; a center if $a = 0$ and $l < -2$; an elementary saddle-node if $l = -2$ and $a \neq 0$; a nilpotent point formed by an elliptic and an hyperbolic sector if $l = -2$ and $a = 0$, an unstable strong focus if $l < -25a^2/12-2$; and an unstable node in the rest of the cases.*

(iii) *If we are in the conditions of Lemma 4.2 (iii), then the critical point is an elementary saddle-node.*

(iv) *If we are in the conditions of Lemma 4.2 (iv), then the critical point is an unstable node if $-5/3 > l > -2$, or a saddle in the rest of the cases.*

(v) *If we are in the conditions of Lemma 4.2 (v), then the critical point is a saddle if $a > \sqrt{3}/3$, and an unstable node if $a < \sqrt{3}/3$.*

(vi) *If we are in the conditions of Lemma 4.2 (vi), then the critical point S is an unstable node if $l(3l+5)^2-3a^2(5l+8)>0$ and $l>0$; a stable node if $l(3l+5)^2-3a^2(5l+8)<0$ and $l<-5/3$, and a saddle in the rest of the cases; and the point R is an unstable node if $l(3l+5)^2-3a^2(5l+8)>0$ and $-8/5>l>-2$; or $l(3l+5)^2-3a^2(5l+8)<0$ and $-5/3>l>-2$, and a saddle in the rest of the cases.*

Proof: (i) It comes immediately from Section 2.

(ii) When the point is elementary the computations are easy. For the degenerate cases it is easy to apply the corresponding results mentioned in Section 2. It is easy to see that next curves are important: $\Delta_{(0,1)} = -(3l+6)=0$ and $\delta_{(0,1)} = 25a^2+12(l+2)=0$.

In order to prove (iii) and (iv) we must carry out a long computations to reduce our equations to a single parameter, but they do not add any theoretical complexity. Moreover, as we are in a curve, we may get the same results studying the adjacent regions and applying continuity criteria.

The case (v) is easier because the conditions are linear.

To demonstrate (vi) we must carry out again a long computations to get next results for S:

$$\Delta(S) = \frac{2(9a^2-3l(l+2))(3l+5)+6a(2l+3)\sqrt{9a^2-3l(l+2)}}{l(3l+5)^2-3a^2(5l+8)},$$

$$\rho(S) = 5\frac{10al+3al^2+3a^3+9a+(3l^2+8l+5-a^2)\sqrt{9a^2-3l(l+2)}}{l(3l+5)^2-3a^2(5l+8)}.$$

Then it is easy to decide when we have a saddle. In the cases in which we have a focus or a node we must decide about its stability. In order to decide if we have a node or a focus we must find $\delta(S)$.

$$\begin{aligned}\delta(S) = {} & (2a(-75a^4+510a^2l^2+1466a^2l+1014a^2+9l^4+306l^3+1370l^2\\ & +2150l+1125)\sqrt{9a^2-3l(l-2)}+450a^6+2265a^4l^2+8334a^4l\\ & +7740a^4-324a^2l^4-1008a^2l^3+2014a^2l^2+9240a^2l+7650a^2\\ & -27l^6-414l^5-2370l^4-6300l^3-7875l^2-3750l)/\\ & (l(3l+5)^2-3a^2(5l+8))^2.\end{aligned}$$

If we make this equal to zero and eliminate the square root we get:

$$\begin{aligned}
&-4860000a^{10}l^2 - 15552000a^{10}l - 12441600a^{10} + 5394375a^8l^4 \\
&+19808100a^8l^3 + 7978140a^8l^2 - 38256192a^8l - 35852544a^8 \\
&-1282500a^6l^6 - 1077480a^6l^5 + 27567516a^6l^4 + 88464936a^6l^3 \\
&+81585216a^6l^2 - 12083904a^6l - 36288000a^6 - 89910a^4l^8 \\
&-2411856a^4l^7 - 13276044a^4l^6 - 21397488a^4l^5 + 25879082a^4l^4 \\
&+1234421440a^4l^3 + 134411400a^4l^2 + 36072000a^4l - 12960000a^4 \\
&-18468a^2l^{10} - 390744a^2l^9 - 3813372a^2l^8 - 20058552a^2l^7 \\
&-59800380a^2l^6 - 98862600a^2l^5 - 73522500a^2l^4 + 14055000a^2l^3 \\
&+58500000a^2l^2 + 27000000a^2l - 729l^{12} - 22356l^{11} - 299376l^{10} \\
&-2302560l^9 - 11258550l^8 - 36585000l^7 - 80122500l^6 \\
&-117000000l^5 - 109265625l^4 - 59062500l^3 - 14062500l^2 = 0.
\end{aligned}$$

From this expression we have not managed to get any analytical conclusion but we know some numerical behaviors. So, for values (a, l) for which $\delta(S)$ is defined, we have seen that $\delta(S) > 0$ always, so we conjecture that if S is not a saddle, then it must be a node.

Similar functions and arguments work for the point R. ■

Lemma 4.4. *The normal form (3.1) has at infinity six critical points if we have that* $(a, l) \in \{(a, l) \in \mathbb{R}^2 : 125a^4 + a^2(25l^2 + 170l + 262) + (2l + 5)^3 > 0\}$, *four if we have* $(a, l) \in \{(a, l) \in \mathbb{R}^2 : 125a^4 + a^2(25l^2 + 170l + 262) + (2l + 5)^3 = 0, a \neq 0\}$ *and two otherwise.*

Proof: From Section 2 we have that the point $(0, 1, 0)$ of S^2 is never critical and so all the critical points will come from the equation:

$$F(z) = z^3 + 5az^2 - (2l + 5)z - a = 0.$$

The discriminant of this polynomial is

$$-(125a^4 + a^2(25l^2 + 170l + 262) + (2l + 5)^3),$$

and the rest is immediate. ■

We denote by K_1, K_2 and K_3 with $K_1 > K_2 > K_3$ the three real roots of $F(z) = 0$. In studying the equation $F(z) = 0$ we see that if two roots disappear, the remainder one corresponds to the highest one, i.e. K_1. We suppose that if we have two roots they will be K_1 and K_2.

Lemma 4.5. *The following statements hold.*

(i) *If* $l(3l+5)^2 - 3a^2(5l+8) > 0$ *and* $l > 0$*, then* K_1 *and* K_3 *are stable nodes and* K_2 *is a saddle.*

(ii) *If* $l(3l+5)^2 - 3a^2(5l+8) = 0$ *and* $l > 0$*, then* K_1 *and* K_3 *are stable nodes and* K_2 *is an elementary saddle-node, SN1, with the nodal behavior in the semiplane* $z_2 > 0$ *of the chart* U_1*.*

(iii) *If* $a = l = 0$*, then* K_1 *and* K_3 *are stable nodes and* K_2 *is an elementary degenerated saddle.*

(iv) *If* $l(3l+5)^2 - 3a^2(5l+8) < 0$ *and* $l > -5/3$*, then* K_1 *and* K_3 *are stable nodes and* K_2 *is an unstable node.*

(v) *If* $l(3l+5)^2 - 3a^2(5l+8) = 0$ *and* $-8/5 > l > -5/3$*, then* K_1 *is a stable node,* K_2 *is an unstable node and* K_3 *is an elementary saddle-node, SN1, with the hyperbolic component in the semiplane* $z_2 > 0$ *of the chart* U_1*.*

(vi) *If* $l(3l+5)^2 - 3a^2(5l+8) > 0$ *and* $l < -8/5$*, then* K_1 *is a stable node,* K_2 *is an unstable node and* K_3 *is a saddle.*

(vii) *If* $a = 0$ *and* $l = -5/3$*,* K_2 *is an unstable node, then* K_1 *is an elementary saddle-node, SN1, with the nodal component in the semiplane* $z_2 > 0$ *of the chart* U_1*, and* K_3 *is an elementary saddle-node, SN1, with the hyperbolic component in the semiplane* $z_2 > 0$ *of the chart* U_1*.*

(viii) *If* $l(3l+5)^2 - 3a^2(5l+8) = 0$ *and* $l < -5/3$*, then* K_1 *is an elementary saddle-node, SN1, with the nodal component in the semiplane* $z_2 > 0$ *of the chart* U_1*,* K_2 *is an unstable node, and* K_3 *is a saddle.*

(ix) *If* $l(3l+5)^2 - 3a^2(5l+8) < 0$*,* $125a^4 + a^2(25l^2 + 170l + 262) + (2l+5)^3 > 0$ *and* $l < -5/3$*, then* K_1 *and* K_3 *are saddles and* K_2 *is an unstable node.*

(x) *If* $125a^4 + a^2(25l^2 + 170l + 262) + (2l+5)^3 = 0$ *and* $a \neq 0$*, then* K_1 *is a saddle and* K_2 *is an elementary saddle-node, SN2, with the nodal component in the semiplane* $z_1 > 0$ *of the chart* U_1*.*

(xi) *If* $125a^4 + a^2(25l^2 + 170l + 262) + (2l+5)^3 < 0$*, then* K_1 *is a saddle and its infinite separatrix is stable.*

(xii) *If* $a = 0$ *and* $l = -5/2$*, then* K_1 *is an elementary degenerated saddle and its infinite separatrix is stable.*

Note that all the results of the Lemma 4.5 are refereed to the local chart U_1. With the Poincaré s compactification we may get the results for the chart V_1.

Proof: The proof of the lemma implies to compute the roots of the third degree equation in an algebraic way and substituting them in the jacobian of the vector field written in the corresponding form of the chart U_1. This means a long and tedious calculus.

In other way, we may find out the same if we compute the curves of the degenerated points, that is, if we solve the systems:

$$\begin{cases} F(z) = 0 \\ F'(z) = 0 \end{cases}, \quad \text{and} \quad \begin{cases} F(z) = 0 \\ P_2(1, z) = 0 \end{cases}.$$

Isolating and substituting z we get respectively $125a^4 + a^2(25l^2 + 170l + 262) + (2l + 5)^3 = 0$, and $l(3l + 5)^2 - 3a^2(5l + 8) = 0$. These two curves and $a = 0$ limit some convenient regions. We now do a particular study of each region and we may generalize the results to the whole region. ■

Now we have found the curves and points of bifurcation for the normal form (3.1) refereed to the number and nature of all critical points, finite or infinite. These curves limit a certain well-defined regions of dimension 2. We also see that these curves intersect among them in certain concrete points. These points differentiate the pieces of the curves such that different points in a same piece will imply different phase portraits. We must separate these curves in homeomorphic sets to the interval $(0, 1)$ such that they do not intersect each other. It remains the unexplained curve (C_{14}) which we will comment later on.

Now we see that these curves cannot intersect in a point different from the origin.

The curve C_1 does not cross C_2 because C_1 increases with slope $\sqrt{3}/3$ and C_2 increases with slope $\sqrt{5/3}$.

The curve C_2 does not cross C_3 because even though they increase with the same slope, C_3 begins in a slower position.

The curve C_4 does not cross C_5 because C_4 is asymptotic to $l = -5/6$ and C_5 is asymptotic to $l = -8/5$.

The curve C_9 does not cross C_{10} for the same reason that C_2 does not cross C_3.

The curve C_{10} does not cross C_{11} for the same reason that C_1 does not cross C_2.

The curve C_{11} does not cross C_{12} because C_{11} increases as a straight line and C_{12} increases as a parabola of coefficient -2.

The curve C_{12} does not cross C_{13} because C_{13} increases as a parabola of coefficient $-25/12 < -2$.

We note that with respect to the possible intersection between C_{13} and C_{15} the arguments of growing are not sufficient. In [**QSC**] it is proved for our vector fields that, if we have an only infinite critical point, then the point $(0,1)$ is always a focus. So, from Lemmas 4.3 and 4.4 C_{13} can not cross C_{15}.

We separate the points $(a,l) \in \mathbb{R}^2$ in several regions so that we can assure that if two vector fields are in the same region, then they have the same number of critical points, either finite or infinite, with the same local phase portrait. In order to determine if two vector fields in one of these regions have topologically equivalent phase portraits, we need to know their limit cycles and the global behavior of their separatrices.

The determination of the limit cycles for a polynomial vector field of degree n is an open problem. In fact, to know the maximum number of limit cycles in function of n is still open. In particular, for the case $n = 2$ it seems that this maximum number is 4. An example with at least 4 limit cycles can be obtained perturbing the phase portraits corresponding to the region R_{15} of Figure 4.1. Such phase portraits have a limit cycle around the strong focus and three small limit cycles can bifurcate from the third order weak focus at the origin under small perturbations.

As a result of the classification we are given, we obtain the regions where there exist this limit cycle around the strong focus. We only know that there must exist at least one limit cycle, but we cannot prove that it is unique. In order to prove this we have made a numerical study of the region R_{15}.

If we cut the region R_{15} with an horizontal line $l =$ constant with $l < -5/2$, we find three interesting points: the point $(0,l)$, the point where the line cuts the curve C_{15}, and the point where the line cuts the curve C_{13}.

The point $(0,l)$ corresponds to an integrable system where the only pair of saddles at infinity connect their separatrices, and the two finite critical points are centers. If we move to the right along our line, we have that the point $(0,1)$ becomes a strong focus and a limit cycle appears around it. It appears when the unstable separatrix of the infinity disconnects from the stable one and tries to go to the point $(0,1)$, forcing the appearance of a limit cycle because the point $(0,1)$ is an unstable focus. We know that when we arrive to the curve C_{13} this limit cycle has disappeared because we cannot have a limit cycle around a node. As the point $(0,1)$ does not change from unstable to stable anywhere close to where we are now, the limit cycle cannot die at this point and, so, it has to die in an infinite graphic in the same way that it has appeared. As usual a *graphic* of $p(X) \in \mathcal{P}_n(\mathbb{S}^2)$ denotes a closed simple curve which is the union of a finite number of critical points and separatrices, and at least in one of the two sides of the curve is defined a return or Poincaré map.

When we cross the curve C_{15} there appears a new infinite critical point which is an elementary saddle-node with nodal sectors in the two sides of the infinity; that is, it is of type SN2. The separatrix of the infinite saddle which rotated around the point $(0,1)$ now goes to the nodal sector of the saddle-node which we have at U_2, and the separatrix of this saddle-node is the one which goes to the point $(0,1)$, but since this separatrix is stable, the limit cycle can disappear. More concretely, the previous limit cycle can dissapear at infinity when the saddle-node appears; or perhaps it can remain and a new limit cycle appears simultaneously with the saddle-node at infinity. The numerical results detect only the first possibility.

For example, now we do the global study for a concrete region. For the systems of region R_{15} we know that we have two separatrices which come from the infinity, a stable one and an unstable one. The Poincaré-Bendixson Theorem says that the separatrices of any system may only end at a critical point, a periodic orbit or a graphic. So, we could have that the stable separatrix of the infinity comes from the critical points $(0,1)$, or $(0,0)$, or from any periodic orbit that there may exist, or that it connects with the unstable separatrix at the infinity. In order to analyze these different possibilities we will use the following results.

[**Yu**]: If a quadratic vector field has a limit cycle, then it surrounds one and only one critical point, and this point is a focus.

Therefore the limit cycles cannot be anywhere in the plane. They may only be around the points $(0,0)$ and $(0,1)$.

[**Li2**]: There are no limit cycles around a weak focus of third order.

Then only around the point $(0,1)$ we may have limit cycles. Moreover, the unstable separatrix of the infinity may not turn around the critical point $(0,0)$ because this is an unstable weak focus, and it should have to turn around a limit cycle which we know it can not exist.

[**SP**]: If in a quadratic vector field the separatrix of an infinite saddle connects with the symmetric one, then they form an invariant straight line.

Consequently, we can see that the two separatrices do not connect. Studying the normal form (3.1) we see that we cannot have an invariant straight line of the form $x = c$ because the north pole is not a critical point. It should be of the form $y = cx+d$, which would imply $y' = cx'$ on the points of this straight line. So, it should have the following equalities:

$$
\begin{aligned}
a + (3l+5)c &= c(l + 5ac + c^2),\\
1 + (3l+5)d &= c(-c + 5ad + 2cd),\\
0 &= c(-d + d^2).
\end{aligned}
$$

This implies either $c = 0$, or $d = 0$, or $d = 1$. But $d = 0$ or $d = 1$ imply that the line crosses the point $(0,0)$ or $(0,1)$ respectively, which is impossible because these points are focus. The possibility $c = 0$ tells us that $a = 0$ and the region R_{15} has no contact with this line. So, the only possible phase portrait for the systems of region R_{15} is W_{20} as it is shown in Figure 1.1 with an unknown number of limit cycles around the point $(0,1)$. We may say that we will have at least one limit cycle because the point is unstable and the separatrix which turns around it is also unstable.

A similar study should be made for each region so we limit the number of possible phase portraits in each one, but in this paper we will limit ourselves to give the local behavior and a phase portrait numerically found for each region, and will leave this study for latter.

Only for the regions where we have a center, we determine analytically the global phase portrait (see the appendix).

We also need the next two lemmas.

Lemma 4.6. *In all the regions where the point* $(0,1)$ *is not a focus (i.e., except in the regions* R_{13}*,* C_{14}*,* R_{15}*,* C_{15}*,* C_{16} *and* Q_2*) we cannot have any limit cycle.*

The proof is immediate from the results [**Li2**], [**Yu**] and from the Lemma 4.3.

Lemma 4.7. *For all* $(a,l) \in R_{15}$*,* R_{14} *or* C_{16}*, there exists at least one limit cycle around the focus* $(0,1)$*.*

Proof: The lemma was proved for the systems defined by the region R_{15}.

We study now the systems of the region R_{14}. Since the curve C_{14} is not defined yet, we may suppose that the region R_{13} includes for the moment, the regions R_{14} and C_{14}. The same study we have done for the region R_{15} may be done now for R_{13}. There are three possible ω-limits for the separatrix which comes from the infinite saddle of the local chart U_1 which are: The two nodes of the local chart U_2 or a limit cycle around the point $(0,1)$ (it cannot be the point because it is unstable). For points close to P_{15}, in the semiplane $a > 0$ and in R_{13}, it has been proved [**QSD**], that the separatrix of the infinite saddle turns around a limit cycle which has the critical point $(0,1)$ in its interior. For many other points far from here, it is numerically clear that the ω-limit of the separatrix is the node in local chart U_2 which lays upwards of the saddle there. For continuity, there exists a saddle-to-saddle connection in a curve that we will denote by C_{14}.

Numerically we have seen that the region R_{14} inside R_{13} and separated of this by the curve C_{14} (that its corresponding systems have at least one limit cycle) is quite bigger than the estimated analytically in [**QSD**], and it is of the form showed in Figure 4.2. We have not found the analytic expression of this curve, but numerically we may conjecture that it begins in the point $(0, -2)$ and ends in the curve defined by the equation $125a^4 + a^2(25l^2 + 170l + 262) + (2l + 5)^3 = 0$, in a point of approximated coordinates $(0.054684523, -2.4085)$ which we will denote by Q_2, which produces that the latter curve splits in the arcs C_{15} and C_{16}. Even though we do not know exactly the expressions of C_{14} and Q_2 we may guess the phase portraits of their corresponding systems by using continuity criteria. ■

So, *we conjecture that quadratic vector fields having a weak focus of third order can not have limit cycles anywhere except the corresponding ones to the regions* R_{14}, C_{16} *and* R_{15}, *and that those only have one limit cycle.*

5. Study of the normal form (3.2)

The phase portraits that we study now represent all the local behaviors we may get from the normal form (3.2) in changing l. These local behaviors have been drawn as a part of a global phase portrait which is provided from it but that we have checked numerically its existence.

The line $\mathbb{R}$ with parameter l has been divided in a series of regions of dimensions 0 and 1, so that for all the points l belonging to a same region the local behaviors are the same.

We denote by B and S, with the corresponding subindex to certain regions of dimension 1 and 0, respectively.

In Lemma 5.1 we show that it is sufficient to study the half-line $l \geq 0$.

From Lemmas 5.2, 5.3, 5.4 and 5.5, the points that separate regions will be: $\sqrt{3}$, $\sqrt{5/3}$ and 0.

Table 5.1 shows all the regions in which we have divided the half-line $l \geq 0$, taking into account the number and the different local phase portraits of the finite and infinite critical points, following the same rules as in Section 4. It also gives the corresponding phase portrait for each region.

No one of these cases may have limit cycles as we will prove in Lemma 5.4.

The examples of the global phase portraits for each one of these regions are included in Figure 1.1 and 1.2. We have obtained them with the same method as in Section 4.

$(\sqrt{3},\infty)$	$B_1 \to W_8$ fdr, S, NA, SN1A.
$\sqrt{3}$	$S_1 \to W_7$ fdr, snr, S, NA, SN1A.
$(\sqrt{5/3},\sqrt{3})$	$B_2 \to W_9$ fdr, s, S, NA, SN1A.
$\sqrt{5/3}$	$S_2 \to V_{16}$ c, s, SN1R, NA, SN1A.
$(0,\sqrt{5/3})$	$B_3 \to W_4$ fda, s, s, NR, NA, SN1A.
0	$S_3 \to V_{19}$ c, s, NR, NA, SS.

Table 5.1

We must study the system

$$
\begin{aligned}
X' &= -y + lx^2 + 5xy,\\
Y' &= x + x^2 + 3lxy.
\end{aligned}
\tag{5.1}
$$

Lemma 5.1. *Given the phase portrait of a vector field in the normal form (5.1) with $l \geq 0$, in order to get the phase portrait of (5.1) with $-l$ we must make a symmetry respect to the y-axis and change the sense of all the orbits.*

The proof of this lemma is immediate.

Now we study the number of finite critical points of system (5.1) in function of l.

Lemma 5.2. *For system (5.1) the following statements hold.*

(i) *The point* $(0,0)$ *is always critical.*

(ii) *If $l = 0$ we have only another critical point, namely the point* $(-1,0)$.

(iii) *If $l = \sqrt{5/3}$ we have only another finite critical point namely the point with coordinates* $(1/4, -\sqrt{5/3}/4)$.

(iv) *If $l = \sqrt{3}$ we have only another finite critical point namely the point with coordinates* $(1/2, -1/2\sqrt{3})$.

(v) *If $l > \sqrt{3}$ the only critical point is the point* $(0,0)$.

(vi) *If $0 < l < \sqrt{3}$ and $l \neq \sqrt{5/3}$ we have only two other critical points, which are*

$$R = \left(\frac{2-\sqrt{9-3l^2}}{3l^2-5}, \frac{3(1-l^2)+\sqrt{9-3l^2}}{3(3l^2-5)l} \right),$$

$$S = \left(\frac{2+\sqrt{9-3l^2}}{3l^2-5}, \frac{3(1-l^2)-\sqrt{9-3l^2}}{3(3l^2-5)l} \right).$$

Proof: (i) Trivial.

From (5.1) any critical point different from $(0,0)$ must be such that its second coordinate must satisfy next relation: $y^2(9l^3-15l)+(6l^2-6)y+l=0$. Then if $l=0$ and $l=\sqrt{5/3}$ we may have only one critical point different from $(0,0)$ and we obtain (ii) and (iii).

If we are not in the latter cases, we have a second degree equation with discriminant $\Delta = 36-12l^2$ from which we easily have (iv), (v) and (vi). ■

Now we study the local phase portrait of the finite critical points of system (5.1).

Lemma 5.3. *For system (5.1) the following statements hold.*

(i) *The critical point* $(0,0)$ *is an unstable weak focus of third order if* $l>\sqrt{5/3}$, *a linear center if* $l=\sqrt{5/3}$ *or if* $l=0$, *and a stable weak focus of third order if* $0<l<\sqrt{5/3}$.

(ii) *If* $l=0$, *then the critical point* $(-1,0)$ *is a saddle.*

(iii) *If* $l=\sqrt{5/3}$, *then the critical point* $(1/4,-\sqrt{5/3}/4)$ *is a saddle.*

(iv) *If* $l=\sqrt{3}$, *then the critical point* $(1/2,-1/2\sqrt{3})$ *is an elementary saddle-node.*

(v) *If* $0<l<\sqrt{3}$ *and* $l\neq\sqrt{5/3}$, *then the critical point* R *is a saddle.*

(vi) *If* $\sqrt{5/3}<l<\sqrt{3}$, *then the critical point* S *is an unstable node.*

(vii) *If* $0<l<\sqrt{5/3}$, *then the critical point* S *is a saddle.*

The proof is easy just substituting the critical points in their corresponding jacobians.

Now we analyze the infinite critical points of system (5.1).

Lemma 5.4. *System (5.1) has always three critical points at infinity which are:* $(K_1,0)$, $(K_2,0)$ *and* $(K_3,0)$ *in the local chart* U_2 *where* $K_1=-l+\sqrt{l^2+5}$, $K_2=0$, *and* $K_3=-l-\sqrt{l^2+5}$.

Proof: From Section 2 we have that the point $(0,0)$ of the local chart U_1 is never critical. So all the critical points come from the equation: $-z^3-2lz^2+5z=0$, and they will be in the local chart U_2. It is clear that this equation has the three mentioned roots. ■

Next we study the local phase portraits at the infinite critical points of system (5.1).

Lemma 5.5. *For system (5.1) the following statements hold.*

(i) *The point $(K_2, 0)$ is always elementary and its local phase portrait is either a saddle-node of type SN1, with the saddle part at the north pole if $l \neq 0$, or a saddle if $l = 0$.*

(ii) *The point $(K_1, 0)$ is always a stable node.*

(iii) *The point $(K_3, 0)$ is a saddle if $l > \sqrt{5/3}$, a stable node if $0 \leq l < \sqrt{5/3}$, and an elementary saddle-node of type SN1, if $l = \sqrt{5/3}$.*

The proof is similar to the one of Lemma 4.5.

Appendix 1

We are going to determine exactly the phase portraits of the quadratic systems with normal form (3.1)-(3.4) with a center. We will work mainly with the paper of Vulpe [**V**] where he studies all the quadratic systems having a center. Vulpe defined several invariants from the coefficients of the system. Depending on the values of these invariants, we get the different phase portraits.

For a general system of our type the invariants take next values:

$$
\begin{aligned}
I_1 &= \quad 0,\\
I_2 &= -2,\\
I_3 &= -bn - am - ln - lb - l^2,\\
I_4 &= -b^2 - m^2 - l^2 - 2bl,\\
I_5 &= -2bn - b^2 - m^2 - 2am - l^2,\\
I_6 &= mn - ab + lm - la,\\
I_7 &= -b^3n - 4m^2b^2 + 2abmn - am^3 - lb^2n - 3lm^2b + 2almn\\
&\quad + l^2bn - l^2m^2 + l^3n,\\
I_8 &= -3b^3n - 4m^2b^2 - 4amnb - 3am^3 + 2a^2n^2 + 3lb^2n - lm^2b\\
&\quad + 4almn - l^2bn - l^2m^2 + l^3n,\\
I_9 &= b^3n - 4m^2b^2 + am^3 + 3lb^2n - 5lm^2b + 3l^2bn - l^2m^2 + l^3n,\\
I_{10} &= -n^2 + bn + am - a^2 - ln + lb,\\
I_{13} &= mb^2n + 2mb^3 - 2m^3b - am^2b + 2lmnb + 4lmb^2 - lm^3 - lam^2\\
&\quad + l^2mn + 2l^2mb,\\
I_{16} &= -mbn^2 - an^3 - 2abn^2 + ab^2n - am^2n + 2a^2mn + a^2mb\\
&\quad + a^3n - lmn^2 - 2lmnb - 2lan^2 - 2albn + lab^2 - lam^2\\
&\quad + 3la^2m - 2l^2mn - l^2mb - l^3m + l^3a.
\end{aligned}
$$

First we study the centers which appear in the normal form (3.1). Thus, we know that if $a = 0$ we have a center. With these conditions the invariants are:

$$\begin{aligned}
I_1 &= 0,\\
I_2 &= -2,\\
I_3 &= -(5/2)(l+1)^2,\\
I_4 &= -(25/4)(l+1)^2,\\
I_5 &= -(1/4)(45+42l+13l^2) < 0,\\
I_6 &= 0,\\
I_7 &= -(25/8)(l+1)^2(l+5),\\
I_8 &= -(1/8)(31l^2+90l+75)(l+5),\\
I_9 &= (125/8)(l+1)^3,\\
I_{10} &= (3/2)(l+1)^2,\\
I_{13} &= 0,\\
I_{16} &= 0,\\
\beta &= (375/4)(l+1)^3(l+2),\\
\gamma &= (12/5)(l/(l+1)).
\end{aligned}$$

If $l \geq 0$ we are in the conditions of Case 19 of [**V**].
If $-1 < l < 0$ we are in the conditions of Case 9 of [**V**].
If $l = -1$ we are in the conditions of Case 10 of [**V**].
If $-5/3 < l < -1$ we are in the conditions of Case 8 of [**V**].
If $l = -5/3$ we are in the conditions of Case 16 of [**V**].
If $-2 < l < -5/3$ we are in the conditions of Case 23 of [**V**].
If $l = -2$ we are in the conditions of Case 24 of [**V**].
If $-5/2 < l < -2$ we are in the conditions of Case 21 of [**V**].
If $l \leq -5/2$ we are in the conditions of Case 20 of [**V**].

Another case where we have a center in the normal form (3.1) is when $l = -2a^2 - 2$. Then the invariants are:

$$\begin{aligned}
I_1 &= 0,\\
I_2 &= -2,\\
I_3 &= -(1/2)(5+25a^2+20a^4) < 0,\\
I_4 &= -(1/4)(25+125a^2+100a^4) < 0,\\
I_5 &= -(1/4)(13+65a^2+52a^4) < 0,
\end{aligned}$$

$$
\begin{aligned}
I_6 &= 0,\\
I_7 &= -(1/8)(75 + 750a^2 + 2475a^4 + 2700a^6) < 0,\\
I_8 &= -(1/8)(57 + 570a^2 + 1881a^4 + 2052a^6) < 0,\\
I_9 &= -(1/8)(125 + 1250a^2 + 4125a^4 + 4500a^6) < 0,\\
I_{10} &= (1/2)(3 + 15a^2 + 12a^4) > 0,\\
I_{13} &= -(a^3/4)(125 + 875a^2 + 1500a^4) \le 0,\\
I_{16} &= a^3(2 + 14a^2 + 24a^4) \ge 0.
\end{aligned}
$$

If $a = 0$ it corresponds to a case that we have seen before.

If $a > 0$ we have a center because it is true that $5I_3 - 2I_4 = 0$ and $13I_3 - 10I_5 = 0$, and as $I_{13} \neq 0$ we are in Case 32 of [**V**].

The third possibility of having a center for the normal form (3.1) corresponds to the equality $a^2 = 3(l+1)^2(l+2)/(5l+6)$. As the invariants are quite complicated we will write here only the needed ones.

$$
\begin{aligned}
I_1 &= 0,\\
I_2 &= -2,\\
I_3 &= -10(l + 1)^2(2l + 3)/(5l + 6),\\
I_4 &= -25(l + 1)^2(2l + 3)/(5l + 6),\\
I_6 &= 0,\\
I_9 &= -125(l + 1)^3(3l + 4)(2l + 3)^2/(5l + 6)^2,\\
I_{13} &= 0,\\
\beta &= -750(l + 1)^3(l + 2)(2l + 3)^2/(5l + 6)^2,\\
\gamma &= (6/5)(5l + 6)/(l + 1),\\
\gamma - 6 &= 6/(5(l + 1)),\\
\gamma - 4 &= (2/5)(5l + 8)/(l + 1).
\end{aligned}
$$

If $l > -1$ we are in the conditions of Case 8 of [**V**].

If $l = -1$ we are in a case we have already studied.

If $-6/5 < l < -1$ we are in the conditions of Case 9 of [**V**].

If $l = -2$ we are in a case we have already studied.

If $l < -2$ we are in the conditions of Case 26 of [**V**].

This proves that all the systems with a center that we had found numerically are the only ones which exist for the normal form (3.1).

In a similar way and more easily we can describe the centers for the normal forms (3.2), (3.3) and (3.4).

Appendix 2

For drawing of the global phase portrait of quadratic differential systems we use a computer program (see for more details [**A2**] and [**A3**]). Only in very particular cases the global phase portrait can be studied without the help of a computer, like for instance, those for which their first integral is known, or those for which certain algebraic methods allow their study.

The program is written in FORTRAN language, and its two versions run under the VMS system and DOS system.

The main goal of the program is to draw the solution curves of the quadratic vector field. In order to do so we simply need to find several "good" points where we will begin to integrate and a sense (positive or negative), and after we will integrate the quadratic vector field with a numerical method. Even its simplicity, it is the localization of these "good" points what carries the main part of the program.

We will also be interested in the critical points of the quadratic vector field, either finite or infinite. The study of the critical points determines the exact order of the weak focus. And in the case we have got a center, it will tell us the first integral of the system. What the program does not make, is the study of degenerated points with linear part identically zero but all other kind of points are perfectly studied.

In short, what the program does is to receive the coefficients of a quadratic vector field as:

$$\begin{aligned} x' &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \\ y' &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2, \end{aligned}$$

and return its phase portrait (either via screen or plotter) and a file where its critical points are described.

The program is divided in several files (subroutines and functions) in order to make easier the different paths.

Its organization can be seen in next diagram:

- SDQ
  - CONIQUES – BATSUB
  - TIPUS
    - THH
      - POSINI
      - SISTEM
      - INTEGRAR
        - EQU – RK78 – DERIV
        - INTEGRA – EQU
    - THS
      - POSINI
      - SISTEM
      - INTEGRAR
    - CENTRE
      - CANVI
      - INVARIANTS
      - INTPRIM
    - SELLA
      - POSINI
      - SISTEM
      - INTEGRAR
  - ARRINF
    - BATSUB
    - TIPUS
  - VARIETATS – EQU

We have only indicated the trees of subroutines which get out from each file first time they appear. It is clear that the same tree is repeated once a same file appears again.

Main program SDQ. This program studies any quadratic vector field satisfying the following conditions:

1) The system is not a linear differential system, it is to say, almost one of the quadratic coefficients is not zero.
2) The finite and infinite critical points of the quadratic vector fields are isolated.
3) The system has no finite or infinite degenerated critical point with linear part identically zero.

The reason for the restrictions 1) and 2) is obvious because, in these cases, the study is reduce to analyze a linear system and all of them are already known.

The reason for the restriction 3) is that the phase portrait of a quadratic system with a degenerated point with linear part identically zero is quite easy to study by hand and difficult to study with a program.

Even though, we do not close our minds, and maybe, we will prepare in a future, a couple of routines to deal with these cases.

Under these conditions, the program will find the finite critical points (subroutine CONIQUES), and will study each of them (subroutine TIPUS). We integrate each separatrix (different from a critical point) with a multi-step Runge-Kutta method of orders 7 and 8 (subroutine RK78). After this, it will do the same with the infinite critical points (subroutine ARRINF). Finally it will ask to us if we wish to integrate some other orbit. We can draw any orbit we want choosing the direction in time that we prefer.

We must pay attention that any system we draw, corresponds to a punctual case in a space of dimension 12. As every numerical program, it is sure to make rounding errors. These errors, in general, will make no important changes in the phase portrait. More concretely, the errors will be quite more important if the coefficients are unexact, if there exist a degenerated point, or if we have a connection of separatrices. In other words, if the system is structurable unstable.

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