

# THE CLOSED CONVEX HULL OF THE INTERPOLATING BLASCHKE PRODUCTS

KNUT ØYMA

## Abstract

---

The closed convex hull of the interpolating Blaschke products contains any bounded analytic function of sufficiently small norm.

---

Let  $H^\infty$  be the set of bounded analytic functions on the open unit disc  $D$ . Each  $f \in H^\infty$  has a non-tangential limit  $f(e^{i\theta})$  at almost every point  $e^{i\theta} \in T = \partial D$ . An inner function  $f$  is a bounded analytic function satisfying  $|f(e^{i\theta})| = 1$ , a.e.  $e^{i\theta} \in T$ .

If  $\{z_n\} \subset D$  satisfies  $\sum(1 - |z_n|) < \infty$ , then

$$\prod_{n=1}^{\infty} \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - \bar{z}_n z}$$

is called the Blaschke product with zeros  $\{z_n\}$ . If  $z_n = 0$ , replace  $\frac{-\bar{z}_n}{|z_n|}$  by 1. A Blaschke product is an inner function. These products have a lot of interesting properties: If  $f \in H^\infty$  then there is a Blaschke product  $B$  and a non-vanishing  $g \in H^\infty$  such that  $f = Bg$ . If  $I$  is an inner function then

$$\frac{I - \lambda}{I - \bar{\lambda}I}$$

is a Blaschke product for  $\lambda$  in a dense subset of  $D$ . Therefore the Blaschke products are dense in the inner functions in the uniform norm. The closed convex hull of the Blaschke products is the unit ball of  $H^\infty$ . See [G] for the proof of these and many other properties of the Blaschke products.

A square is a set of the form

$$Q = \{re^{i\theta} : 1 - h < r < 1, \theta_0 < \theta \leq \theta_0 + h\}.$$

The length of  $Q$  is  $|Q| = h$ . The pseudohyperbolic metric on  $D$  is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|.$$

A sequence  $\{z_n\} \subset D$  is called interpolating if the mapping  $H^\infty \rightarrow \ell^\infty : f(z) \rightarrow \{f(z_n)\}$  is onto. Carleson, see [G], proved that  $\{z_n\}$  is interpolating if and only if

$$\inf_{m \neq n} \rho(z_n, z_m) > \delta > 0,$$

and

$$\sum_{z_n \in Q} (1 - |z_n|^2) < C|Q|$$

for all squares.

Given an square  $Q$ , its top half is  $T(Q) = \{re^{i\theta} \in Q : r < 1 - \frac{1}{2}|Q|\}$ .

A Blaschke product whose zero set is an interpolating sequence is called an interpolating Blaschke product. These products have been studied in great detail, and they play a significant role in the theory of  $H^\infty$ , see [G]. An important open problem is whether or not they are dense in the set of inner functions. In this paper we study the closed convex hull  $K$  of the interpolating Blaschke products.

**Theorem.** *If  $\|f\| \leq 10^{-7}$ , then  $f \in K$ .*

The proof of this modest result is long and technical. The ideas may disappear in the formalism, so I delete details at some points.

Let  $B(z)$  be a Blaschke product with zeros  $\{z_n\}$ . Assume that  $\rho(z, z_n)$  is close to 1 for all  $n$ . Then

$$\log \frac{1}{|B(z)|^2} = (1 + o(1)) \sum \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - \bar{z}_n z|^2}.$$

Assume that the zeros are contained in disjoint squares  $Q_k$  and that  $|Q_k| \ll (1 - |z|)$ . Choose  $\xi_k \in \overline{Q_k} \cap T$ . By the density of  $Q$  w.r.t.  $B$  we mean

$$\frac{1}{|Q|} \sum_{z_n \in Q} (1 - |z_n|^2).$$

Assume that the density of all  $Q_k$  is smaller than a number  $d$ . Then

$$\begin{aligned} \log \frac{1}{|B(z)|^2} &= (1 + o(1)) \sum_k \sum_{z_n \in Q_k} \frac{(1 - |z|^2)(1 - |z_n|^2)}{|1 - \bar{\xi}_k z|^2} \\ &\leq (1 + o(1)) \sum_k \frac{(1 - |z|^2)|Q_k|d}{|1 - \bar{\xi}_k z|^2} \\ &\leq \left( (1 + o(1))d \int_0^{2\pi} \frac{1 - r^2}{|1 - re^{i(\theta-t)}|^2} dt \right) = (1 + o(1))d 2\pi. \end{aligned}$$

Here  $z = re^{i\theta}$ . This proves

**Lemma 0.** *Under the above assumptions,  $|B(z)| \geq e^{-\pi d(1+o(1))}$ .*

**Lemma 1.** *Given a square  $Q$ , and constants  $\kappa < 1$ ,  $1/2 > \delta > 0$ ,  $1 > a \gg 1 - |Q|$ , there exist a finite Blaschke product  $B$  with zeros on  $(|z| = a) \cap Q$  and a region  $R$  of the unit disc, such that*

1. *The zeros of  $B$  are uniformly distributed on  $|z| = a$*
2. *The density of  $Q$  w.r.t.  $B \leq (1 + o(1))\frac{1}{\pi} \log \frac{1}{\delta}$*
3. *If  $z \in Q$ ,  $|z| < a$ , then  $\rho(z, R) \leq A < 1$ ,  $A = A(\kappa, \delta)$*
4. *If  $z \in R$ , then  $|B(z)| \leq \delta^\kappa$ .*

Below we consider only the dyadic squares:

$$Q = \left\{ re^{i\theta} : \frac{2\pi k}{2^n} < \theta < \frac{2\pi(k+1)}{2^n}, 1 - 2^{-n} \leq r < 1 \right\}.$$

Let

$$B_{a,N}(z) = \frac{z^N - a^N}{1 - a^N z^N},$$

where  $a$  is close to 1 and  $a^N = \delta$ . The zeros are  $z_k = ae^{\frac{2\pi i}{N}k}$ . Now  $N \log a = \log \delta$  and

$$N \sim \frac{\log \frac{1}{\delta}}{1 - a}$$

since  $a$  is close to 1. Also

$$\frac{1}{2\pi} \sum (1 - |z_k|^2) < \frac{1}{\pi} \sum (1 - |z_k|) = \frac{1}{\pi} N(1 - a) = (1 + o(1))\frac{1}{\pi} \log \frac{1}{\delta}.$$

If  $Q$  is a square such that  $|Q| \gg 1 - a$ , then the density of  $Q$  is  $(1 + o(1))\frac{1}{\pi} \log \frac{1}{\delta}$  since  $\{z_k\}$  are uniformly distributed.

Also,  $|B_{a,N}(z)| = \rho(z^N, a^N) \leq \rho(-|z|^N, a^N)$ . Assume that  $|z| = a^M$ ,  $M > 1$ . Then

$$|B_{a,N}(z)| \leq \rho(-a^{NM}, a^N) = \rho(-\delta^M, \delta) \leq \delta + \delta^M.$$

Let  $R \subset Q$  be the region bounded by  $\Gamma = \{z \in Q : |z| = a^M\}$  and a circle intersecting  $\Gamma$  in a small angle  $\gamma$ , as in Figure 1.

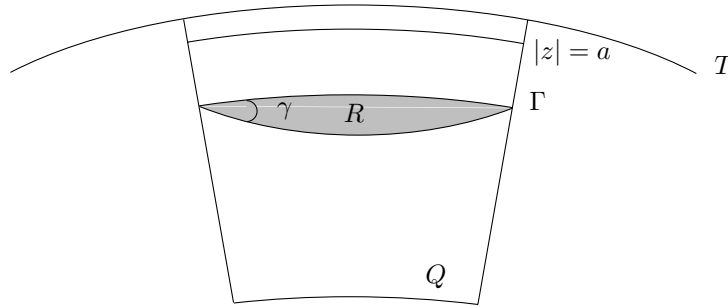


Figure 1.  $R$  is bounded by  $\Gamma$  and another circle. The angle of intersection,  $\gamma$ , is small. The corners of  $R$  are far, but not too far, from the endpoints of  $\Gamma$ .

The figure does not tell the truth: Most of  $Q \cap \{z : |z| < a\}$  is contained in  $R$ , that is: All points in  $Q \cap \{|z| < a\}$  are “close” to  $R$  in the pseudohyperbolic metric. The argument proving Lemma 0 shows that for  $z \in R$  the contribution to  $|B_{a,N}(z)|$  coming from the zeros outside  $Q$  is close to 1, provided the corners of  $R$  are moved away from the endpoints of  $\Gamma$ . By making  $\gamma$  small and increasing the distance from the corners of  $R$  to the endpoints of  $\Gamma$ , we obtain  $|B(z)| \leq (\delta + \delta^M)^{\kappa'}$  for  $z \in R$ ,  $\kappa'$  close to 1. It is also easy to see that if  $z \in Q$ ,  $|z| < a$ , then  $\rho(z, R) < A = A(\delta, \kappa') < 1$ .

By increasing  $M$ , decreasing  $\kappa'$  slightly and making  $\gamma$  small, we finish the proof. ■

**Lemma 2.** Any Blaschke product  $B$  can be factored  $B(z) = B_1(z) \cdot B_2(z)$  where

$$\limsup_{r \rightarrow 1} \min_{|z|=r} |B_i(z)| = 1$$

for  $i = 1, 2$ .

*Proof:* See [Ø]. ■

**Lemma 3.** *Let  $B(z)$  be a  $B$ -product with zeros  $\{z_n\}$ . Assume that there exists constants  $A < 1$ ,  $\eta > 0$  such that for all  $n$  there exists  $z_n^*$  satisfying  $\rho(z_n, z_n^*) < A$ ,  $|B(z_n^*)| > \eta$ . Then  $B(z)$  is a finite product of interpolating  $B$ -products.*

*Proof:* See [G-N]. ■

**Lemma 4.** *Any finite product of interpolating  $B$ -products can be uniformly approximated by an interpolating  $B$ -product.*

*Proof:* See [M-S]. A consequence of Lemma 4 is that the closed convex hull of the interpolating  $B$ -products is closed under multiplication. ■

By Lemmas 4 and 2, we may assume that the conclusion of Lemma 2 holds.

Given  $\delta > 0$  close to 1. Let  $0 < \delta < \beta < 1$  to be chosen later. Choose  $r_1 < 1$ ,  $r_1$  close to 1 such that  $|B(z)| > \beta$  if  $|z| = r_1$ . We may assume that  $|B(0)| > \beta$ .

Let  $Q_1^1, Q_1^2, \dots$  be the maximal dyadic squares such that

1.  $\inf_{z \in T(Q)} |B(z)| < \delta$ .
2.  $T(Q) \cap \{z : |z| < r_1\} \neq \emptyset$ .

These are the squares of the first generation. The proof of Lemma 0 shows that

$$\sum_{B(z_n)=0, z_n \in Q_1^k} (1 - |z_n|^2) < C(\delta) |Q_1^k|$$

where

$$C(\delta) \rightarrow 0 \text{ when } \delta \rightarrow 1.$$

Choose  $r_2 < 1$ ,  $(1 - r_1)/(1 - r_2)$  large, such that  $|B(z)| > \beta$  if  $|z| = r_2$ . The second generation are the maximal dyadic squares such that

1.  $\inf_{z \in T(Q)} |B(z)| < \delta$ .
2.  $T(Q) \cap \{z : |z| < r_2\} \neq \emptyset$ .
3.  $T(Q) \cap \{z : |z| < r_1\} = \emptyset$ .

Continue inductively. If  $\beta$  is large then

$$(1) \quad \sum_{k: Q_n^k \subset Q_{n-1}^j} |Q_n^k| < \varepsilon |Q_{n-1}^j|.$$

(See [G, p. 332].)

Denote  $(Q_n^k \setminus T(Q_n^k)) \cap \{z : |z| \leq r_n\} = S_{1,n}^k \cup S_{2,n}^k$  and  $\Gamma_{i,n}^k = S_{i,n}^k \cap \{z : |z| = r_n^M\}$ , where  $M$ ,  $\Gamma_{i,n}^k$  and  $R_{i,n}^k$  come from Lemma 1 and its proof. See figure below.

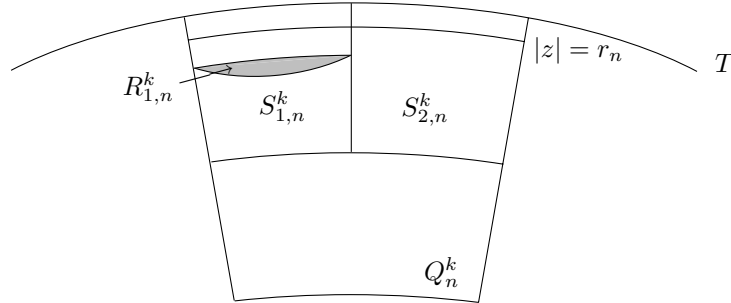


Figure 2

Decompose  $B(z) = B_1(z)B_2(z)$  where  $B_i(z)$  has zeros in  $\bigcup_{n,k} S_{i,n}^k$ . Construct  $B_{1,n}^k$  as in Lemma 1 with  $a = r_n$ ,  $Q = S_{1,n}^k$ . All  $B_{1,n}^k$  of generation  $n$  share the same  $a (= r_n)$ .

Let

$$B_{1,n} = \prod_k B_{1,n}^k, \quad B_1^* = \prod_n B_{1,n}.$$

The estimate (1) shows  $B_1^*$  is an interpolating  $B$ -product and if  $\beta$  is close to 1, then  $|B_1^*(z)| \approx |B_{1,n}(z)|$  when  $z$  is close to  $\Gamma_{1,n}^k$ .

By choosing  $\delta$  close to 1 we obtain

$$\sum_{\substack{B_1 B_1^*(z_i)=0 \\ z_i \in Q_n^k}} (1 - |z_i|^2) \leq [1 + o(1)] \frac{1}{2\pi} \left( \log \frac{1}{\delta} \right) |Q_n^k|.$$

That is: We have control over the density of  $Q_n^k$  w.r.t.  $B_1 B_1^*$ . For  $z \in R_{1,n}^k$ ,  $|B_1^*(z)| < \delta^\kappa$ . Choose  $r < \kappa$ ;  $r$  slightly smaller. Then  $r$  is close to 1. By Frostman's theorem we can choose  $r$  such that

$$\frac{B_1 B_1^* - \delta^r}{1 - \delta^r B_1 B_1^*} = C_1$$

is a  $B$ -product.

**Claim.**  $C_1$  is a finite product of interpolating  $B$ -products.

The proof uses Lemma 3. Assume that  $C_1(w) = 0$ . Then  $B_1(w)B_1^*(w) = \delta^r$ . Let  $w^* \in D$ ,  $\text{Arg } w^* = \text{Arg } w$  and  $10(1 - |w^*|) = (1 - |w|)$ .

Split  $B_1(z)B_1^*(z) = B_Q(z)B_Q^*(z)$  where  $B_Q$  takes care of the zeros inside  $Q_w$  (see Figure 3), and consider the following three different situations.

*Case 1:*  $\rho(w, z_n)$  is large for all  $n$  and every  $S_{1,n}^k$  meeting  $Q_w$  is small, say,

$$\frac{|S_{1,n}^k|}{1 - |w|} < \eta.$$

Since  $|B_Q^*(w)| > \delta^r$  one has

$$r \log \frac{1}{\delta} \geq \log \frac{1}{|B_Q^*(w)|} = (1 + o(1)) \frac{1}{2} \sum_{z_n \notin Q_w} \frac{(1 - |w|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n w|^2} = A$$

and

$$\begin{aligned} \log \frac{1}{|B_Q^*(w^*)|} &= (1 + o(1)) \frac{1}{2} \sum_{z_n \notin Q_w} \frac{(1 - |w^*|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n w^*|^2} \\ &< \frac{1}{4} A < \frac{r}{4} \log \frac{1}{\delta}. \end{aligned}$$

Therefore  $|B_Q^*(w^*)| > \delta^{\frac{r}{4}}$ . The world looks like this:

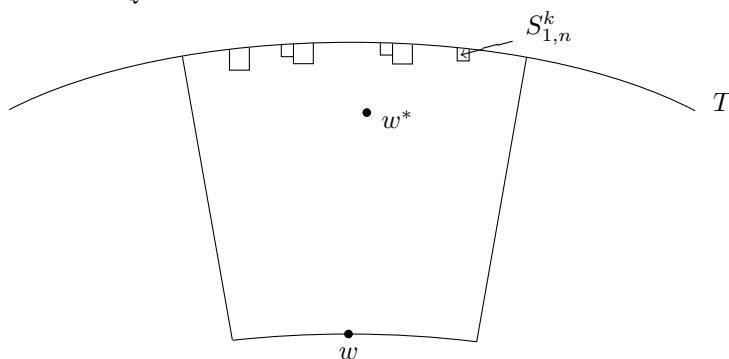


Figure 3. Situation in Case 1.

Let us consider all the disjoint dyadic subsquares of  $Q_w$ , of length  $100\eta(1 - |w|)$ . Then the density of all these small, but not too small, subsquares of  $Q_w$  is less than

$$(1 + o(1)) \frac{1}{2\pi} \log \frac{1}{\delta}.$$

By Lemma 0,  $|B_Q(w^*)| \geq \delta^{\frac{1}{2}(1+o(1))} > \delta^{0.51}$ .

Hence  $|B_1(w^*)B_1^*(w^*)| > \delta^{\frac{r}{4}} \cdot \delta^{0.51} > \delta^{\frac{4}{5}r}$  since  $r$  is close to 1. Therefore  $|C_1(w^*)| = \rho(B_1(w^*)B_1^*(w^*), \delta^r) \geq \rho(\delta^r, \delta^{\frac{4}{5}r})$ . Lemma 3 does the trick.

*Case 2:*  $\rho(w, z_N)$  is small for some  $N$ .

Here small means not close to 1. Then  $z_N$  plays the same role as  $w^*$  in Case 1.

*Case 3:* One  $S_{1,N}^k$  meeting  $Q_w$  is “not small”, that is,  $\frac{|S_{1,n}^k|}{1-|w|} > \eta$ , for some  $n$ . Then there exists  $w^* \in R_{1,N}^k$  such that  $\rho(w, w^*)$  is “not large”. Lemma 1 gives

$$|B_1(w^*)B_1^*(w^*)| < |B_1^*(w^*)| < \delta^\kappa < \delta^r.$$

Use Lemma 3 again.

**Remark.** This argument will be repeated later, but we will not repeat the details. The parameter of the Frostman transform,  $\delta^r$ , satisfied two conditions:

1.  $\delta^r > \delta^\kappa \geq \max_{z \in R_{1,n}^k} |B_1^*(z)|$  for all  $n, k$ .
2. The density  $d$  of  $Q_n^k$  w.r.t.  $B_1B_1^*$  satisfied  $e^{-\pi d} > \delta^\kappa$ .

Therefore the argument with a Frostman parameter  $\delta^*$  works if

$$\max_{z \in R_{1,n}^k} |B_1^*(z)| < |\delta^*| < e^{-\pi d}.$$

It is easy to see that the unimodular constants belong to the closed convex hull  $K$  of the interpolating  $B$ -products. We have

$$B_1B_1^* = \frac{C_1 + \delta^r}{1 + \delta^r C_1} = \sum_0^\infty a_n C_1^n.$$

Since  $\delta > 0$ , a computation proves that  $\sum |a_n| = 1 + 2\delta^r$ , hence  $B_1B_1^* \in (1 + 2\delta^r)K$ .



**Alternatively.** Repeat the argument with  $\delta^r$  replaced by  $\delta^r e^{i\theta}$ . For almost all  $\theta$  the function

$$C_\theta = \frac{B_1 B_1^* - \delta^r e^{i\theta}}{1 - \delta^r e^{-i\theta} B_1 B_1^*}$$

is a finite product of interpolating  $B$ -products.

Approximating with Riemann sums we see that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{B_1 B_1^* - \delta^r e^{i\theta}}{1 - \delta^r e^{-i\theta} B_1 B_1^*} d\theta \in K.$$

This leads to  $B_1 B_1^* \in \frac{1}{1-\delta^{2r}} K$ . This estimate is better than the previous one if  $\delta^r$  is small.

We now want to repeat the argument with  $B_1^*$  replaced by

$$B_1^{**} = \frac{B_1^* - \sigma}{1 - \bar{\sigma} B_1^*}.$$

The zeros of  $B_1^{**}$  are close to the zeros of  $B_1^*$  if  $\sigma$  is not too large. The zeros of  $B_1^*$  are concentrated on arcs  $|z| = r_n$ , one arc for each generation

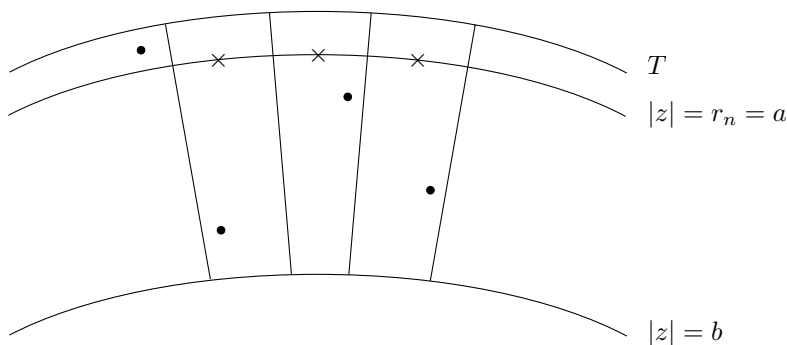


Figure 4. The zeros of  $B_1^*$  and  $B_1^{**}$  are marked by crosses and dots respectively.

The zeros of  $B_1^*$  are marked by crosses. Close to these zeros  $|B_1^*(z)| \geq (1 + 0(1))|B_{1,n}(z)|$ . Figure 4 shows that if  $|\sigma| < \rho(a^N, b^N)$ ,  $a^N = \delta$ ,  $b = a^s$ ,  $s < 1$  then the zeros of  $B_1^{**}$  are concentrated as indicated by the dots. If  $a$  is close to 1 we obtain

$$|\sigma| < \rho(a^N, b^N) = \rho(\delta, \delta^s) = \frac{\delta - \delta^s}{1 - \delta^{s+1}}.$$

Hence

$$\sum_{z_i \in Q_n^k, B^{**}(z_i)=0} (1 - |z_i|^2) \leq (1 + o(1)) \frac{s}{2\pi} \log \frac{1}{\delta} |Q_n^k|.$$

If  $z \in R_{1,n}^k$  then  $|B_1^*(z)| < \delta^\kappa$ . Therefore

$$|B_1^{**}(z)| = \rho(\sigma, B_1^*(z)) \leq \rho(-|\sigma|, \delta^\kappa) = \frac{|\sigma| + \delta^\kappa}{1 + |\sigma|\delta^\kappa}.$$

Therefore the argument works for  $B_1^{**}$  replacing  $B_1^*$  if we choose the parameter  $\delta^*$  in the Frostman transform such that

$$(2) \quad \frac{|\sigma| + \delta^\kappa}{1 + |\sigma|\delta^\kappa} < |\delta^*| < e^{-\pi \frac{s}{2\pi} \log \frac{1}{\delta}} = \delta^{\frac{s}{2}}.$$

Then we obtain  $B_1 B_1^{**} \in (1 + 2\delta^*)K$ .

Choose  $\delta = 0.8$ ,  $s = 1.4$ ,  $\sigma = 0.1647$ ,  $\delta^* = 0.8524$ .

Then (2) is satisfied (since  $r$  and  $\kappa$  are close to 1 we may think of them as being equal to 1) and we obtain:

$$\begin{aligned} B_1 B_1^* &\in (1 + 2\delta)K = 2.6K \\ B_1 B_1^{**} &\in (1 + 2\delta^*)K \subset 2.8K. \end{aligned}$$

Hence

$$B_1(B_1^* - B_1^{**}) = B_1 \frac{\sigma - \sigma(B_1^*)^2}{1 - \sigma B_1^*} \in 5.4K.$$

This leads to  $B_1 \in 40.81K$ .

Therefore each of the subproducts of Lemma 2 belongs to  $(40.81)^2 K$ . Therefore every  $B$ -product belongs to  $(40.81)^4 K \subset 3 \cdot 10^6 K$ .

This proves the theorem since every function in the unit ball of  $H^\infty$  is contained in the closed convex hull of the  $B$ -products.

**Editor's note.** Professor Knut Øyma had formed links with our Department, and we note with sadness that he died suddenly on July 18, 1996. This article is based on a draft version found among his papers; we felt that his approach would be of interest to other people working in the area, and that it deserved to be published. We thank A. Nicolau and A. Stray for preparing the paper for publication, and the referee for many useful remarks.

## References

- [C] L. CARLESON, An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.
- [G] J. B. GARNETT, “*Bounded Analytic Functions*,” Academic Press, New York, 1981.
- [Ø] K. ØYMA, Approximation by interpolating Blaschke products, *Math. Scand.* **79** (1996), 255–262.
- [M-S] D. MARSHALL AND A. STRAY, Interpolating Blaschke products, *Pacific J. Math.* **173** (1996), 491–499.
- [G-N] J. B. GARNETT AND A. NICOLAU, Interpolating Blaschke products generate  $H^\infty$ , *Pacific J. Math.* **173** (1996), 501–510.

Høgskolen i Agder  
Tordenskjolds gate 65  
4604 Kristiansand  
NORWAY

Rebut el 20 de Gener de 1997