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SAMPLING AND INTERPOLATION IN THE PALEY-WIENER SPACES L^p_π, $0 < p \leq 1$

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Abstract

Following Beurling's ideas concerning sampling and interpolation in the Paley-Wiener space L^∞_τ, we find necessary and sufficient density conditions for sets of sampling and interpolation in the Paley-Wiener spaces L^p_τ for $0 < p \leq 1$.

1. Introduction

This work is inspired by Beurling's lectures on balayage of Fourier-Stieltjes transforms and interpolation for an interval on $\mathbb{R}$. In our terms, his problem concerned the so called Paley-Wiener space L^p_τ with $p = \infty$. This space consists of entire functions of exponential type at most τ, bounded on the real axis. Beurling proved that a discrete set of real numbers is a set of sampling for this space if and only if its lower uniform density is bounded by τ/π, and the set is a set of interpolation if and only if its upper uniform density does not exceed τ/π. We prove that the same density results are valid for sampling and interpolation for functions which belong to L^p_τ, $0 < p \leq 1$. The Paley-Wiener spaces with $1 < p < \infty$ have different properties. The density restrictions turn out to be sufficient but not necessary conditions for sampling and interpolation. In [**4**], Lyubarskii and Seip describe complete interpolating sequences in the Paley-Wiener spaces for $1 < p < \infty$. Their work shows that the difference between $0 < p \leq 1$ and $p = \infty$ on the one hand and $1 < p < \infty$ on the other is related to the problem of boundedness of the Hilbert transform.

To see how to proceed in our case, we have been guided by Seip's analysis of corresponding problems for the Bargmann-Fock space [**5**].

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2. Main results

For $0 < p < \infty$ the space L^p_τ is defined to be the collection of entire functions f of exponential type at most τ for which

$$\|f\|_p = \left(\int |f(x)|^p \, dx\right)^{1/p} < \infty.$$

Unlike the regular L^p spaces, these spaces are nested, i.e., $L^p_\tau \subseteq L^q_\tau$ for $0 < p \le q$. According to classical results $|f(x+iy)| \le Ce^{\tau|y|}\|f\|_p$ so $\|f\|_\infty \le C\|f\|_p$ where C depends only on τ and p. Basic facts about entire functions can be found in e.g. [**8**]. L^p_τ is a Banach space for $1 \le p \le \infty$. For $0 < p < 1$ the norm $\|\cdot\|_p$ is a quasinorm and L^p_τ is complete with respect to this quasinorm [**3**]. We fix the type of the functions to be π for the rest of the paper. The other cases are handled by a change of variables.

For $f \in L^p_\pi$ and $\Lambda = \{\lambda_j\}$ a discrete set of real numbers, we write $\|f|\Lambda\|^p_p = \sum_{\lambda_j \in \Lambda} |f(\lambda_j)|^p$. The set Λ is said to be a set of sampling if there exist positive numbers A and B such that

$$A\|f\|^p_p \le \|f|\Lambda\|^p_p \le B\|f\|^p_p$$

for all $f \in L^p_\pi$. Λ is said to be a set of interpolation if to every sequence $w = \{w_j\} \in l^p$ we can find a function $f \in L^p_\pi$ such that $f(\lambda_j) = w_j$ for all j. We will consider sampling and interpolation for functions in L^p_π for $0 < p \le 1$.

For the description of the density of a set of real numbers, we use the following concept introduced by Beurling. Let $\Lambda = \{\lambda_j\}$ be a uniformly discrete set, i.e. there exists $\delta > 0$ such that $|\lambda_i - \lambda_j| \ge \delta$, $i \ne j$. Let $\overline{n}(r)$ and $\underline{n}(r)$ denote respectively the largest and smallest number of points in any interval $[x, x+r]$ for $r > 0$. We define the upper uniform density of Λ (u. u. d.(Λ)) and the lower uniform density of Λ (l. u. d.(Λ)) to be

$$\mathrm{u.\,u.\,d.}(\Lambda) = \lim_{r\to\infty} \frac{\overline{n}(r)}{r} \quad \text{and} \quad \mathrm{l.\,u.\,d.}(\Lambda) = \lim_{r\to\infty} \frac{\underline{n}(r)}{r},$$

where the limits exist because of the subadditivity of the function $r \mapsto \overline{n}(r)$ and the superadditivity of the function $r \mapsto \underline{n}(r)$.

The following two theorems are our main results.

Theorem 2.1. *A discrete set Λ is a set of sampling for L^p_π if and only if it can be expressed as a finite union of uniformly discrete sets and contains a uniformly discrete subset Λ' for which* l. u. d.(Λ') > 1.

Theorem 2.2. *A discrete set Λ is a set of interpolation for L^p_π if and only if it is uniformly discrete and* u. u. d.(Λ) < 1.

3. Auxiliary results

The following classical result of Plancherel-Pólya [**8**, p. 97] states that the upper sampling inequality holds for any uniformly discrete set.

Lemma 3.1. *Let $f \in L^p_\pi$ and $\{\lambda_k\}$ be an increasing sequence of real numbers such that $|\lambda_k - \lambda_j| \geq \epsilon > 0$, $k \neq j$, then*

$$\|f|\Lambda\|_p \leq B\|f\|_p,$$

where B is a constant depending only on p and ϵ.

Moreover we have the following lemma.

Lemma 3.2. *There exists a positive constant B such that*

$$\|f|\Lambda\|_p \leq B\|f\|_p$$

for all $f \in L^p_\pi$ if and only if Λ can be expressed as a finite union of uniformly discrete sets.

Proof: The "only if" follows from Lemma 3.1. For the converse suppose that such a B exists and that there is no bound on the number of points from Λ to be found in translates of a unit interval, $I_n = \{x : n \leq x < n+1\}$, $n \in \mathbb{Z}$. This means that we can find a sequence $\{n_j\}$ such that $\#(\Lambda \cap I_{n_j}) \to \infty$. Pick $x_j \in I_{n_j}$ and let $h_{x_j}(z) = \left[\frac{\sin(\pi(x_j - z)/m)}{\pi(x_j - z)/m}\right]^m$. Choose m a positive integer such that $h_{x_j}(z) \in L^p_\pi$. Then

$$\|h_{x_j}|\Lambda\|_p \to \infty,$$

which is a contradiction. We conclude that there has to be a bound, say N, on the number of points found in I_n. The set Λ can be divided into $2N$ uniformly discrete sets by letting Λ_k for $k = 1, 2, \ldots, N$ consist of point number k in I_n for n even, and Λ_k for $k = N+1, N+2, \ldots, 2N$ consist of point number $k - N$ in I_n for n odd. ■

We conclude that every set of sampling is a finite union of uniformly discrete sets and we do not have to consider the upper sampling inequality which always holds for such sets.

For a discrete set Λ, let $K(\Lambda) = K(\Lambda, p)$ denote the smallest number K such that

$$\|f\|_p \leq K\|f|\Lambda\|_p \tag{1}$$

for all $f \in L^p_\pi$. We shall refer to $K(\Lambda)$ as the sampling constant.

From now on, we assume every set of sampling to be uniformly discrete. The following lemma shows that this assumption can be made without loss of generality. The lemma was proved by Seip in [**6**, p. 141], for the space L^2_π.

Lemma 3.3. *If Λ is a set of sampling for L^p_π, then it contains a uniformly discrete subset $\Lambda' \subset \Lambda$ which is also a set of sampling for L^p_π.*

Proof: The Paley-Wiener spaces L^p_π, $p > 0$ are closed under differentiation, i.e. $\|f'\|_p \leq C\|f\|_p$, (see e.g. [**8**, p. 99]). Seip's proof for L^2_π is thus valid for any $p > 0$. Suppose that

$$\Lambda = \bigcup_{n=1}^{N} \Lambda^{(n)} \quad \text{where} \quad \inf_{\substack{\lambda\in\Lambda^{(n)}\\ i\neq j}} |\lambda_j - \lambda_i| \geq \delta > 0 \quad \text{for all} \quad n.$$

Let $0 < \epsilon < \delta/4$ and construct a uniformly discrete subset $\tilde{\Lambda} \subset \Lambda$ s.t. $|\lambda_i - \tilde{\lambda}| < \epsilon$ for every $\lambda_i \in \Lambda$. Then for arbitrary $\lambda_i^{(n)} \in \Lambda^{(n)}$, we can find a point $\tilde{\lambda}_i^{(n)} \in \tilde{\Lambda}$ s.t. $|\lambda_i^{(n)} - \tilde{\lambda}_i^{(n)}| < \epsilon$. The points $\tilde{\lambda}_i^{(n)}$ are distinct for fixed n. The mean value theorem gives

$$|f(\tilde{\lambda}_i^{(n)}) - f(\lambda_i^{(n)})| = |f(\mu_i^{(n)})|\, |\tilde{\lambda}_i^{(n)} - \lambda_i^{(n)}|,$$

where $\mu_i^{(n)}$ is a point between $\tilde{\lambda}_i^{(n)}$ and $\lambda_i^{(n)}$. We use the inequality $\|f+g\|_p^p \leq 2^p(\|f\|_p^p + \|g\|_p^p)$ instead of the triangle inequality for $p \in (0,1]$. This yields

$$\begin{aligned}\|f|\Lambda\|_p^p \leq 2^p N \|f|\tilde{\Lambda}\|_p^p + 2^p\,\epsilon \sum_{n=1}^{N} \|f'|\{\mu_i^{(n)}\}\|_p^p \\ \leq 2^p N \|f|\tilde{\Lambda}\|_p^p + C(\delta, N, p)\,\epsilon\,\|f\|_p^p,\end{aligned}$$

where the last step follows from an application of the Plancherel-Pólya inequality and the fact that L^p_π is closed under differentiation. The set $\tilde{\Lambda}$ is a set of sampling provided $\epsilon < 1/(C(\delta, N, p)K)$ where K is the sampling constant for the set Λ. ■

For a given closed set Q and for $t > 0$ let $Q(t)$ denote the set of points which are a distance less than or equal to t from Q. The Fréchet distance $[R, Q]$ between two closed sets R and Q is the smallest number t such that $Q \subset R(t)$ and $R \subset Q(t)$. Let Q_i be a sequence of closed sets. Q_i converges weakly to Q, denoted by $Q_i \rightharpoonup Q$, if for every finite interval $L = [-l, l]$ we have $[(Q_n \cap L) \cup \{-l, l\}, (Q \cap L) \cup \{-l, l\}] \to 0$. If Q is a uniformly discrete set, then every sequence of translates $Q + x_n$ contains a subsequence converging weakly to another uniformly discrete set. Let $W(Q)$ be the collection of weak limits of translates of Q. The next lemma implies that for a given set of sampling Λ, every set Λ' in $W(\Lambda)$ will be a set of sampling for L^p_π.

Lemma 3.4. *Let Λ' be a uniformly discrete set. $\Lambda_n \rightharpoonup \Lambda'$ implies $K(\Lambda') \le \underline{\lim}\, K(\Lambda_n)$.*

Proof: By Lemma 3.1, given $f \in L^p_\pi$ and $\epsilon > 0$, we can find $T > 0$ such that

$$\|f|\tilde{\Lambda} \cap \{|x| > T\}\|_p \le \epsilon$$

for any uniformly discrete set $\tilde{\Lambda}$. We can of course assume that $K(\Lambda_n)$ is finite for all n. The set Λ_n is a set of sampling, so we know that $\|f\|_p \le K(\Lambda_n)(\|f|\Lambda_n \cap [-T,T]\|_p + \epsilon)$. Since $[-T,T]$ is compact, the Fréchet distance $[(\Lambda_n \cap [-T,T]) \cup \{-T,T\}, (\Lambda' \cap [-T,T]) \cup \{-T,T\}] \to 0$ and we have $\underline{\lim}\, \|f|\Lambda_n \cap [-T,T]\|_p \le \|f|\Lambda' \cap [-T,T]\|_p$. This gives the inequality

$$\|f\|_p \le \underline{\lim}\, K(\Lambda_n)(\|f|\Lambda' \cap [-T,T]\|_p + \epsilon)$$

for all n. Letting $\epsilon \to 0$ we get $K(\Lambda') \le \underline{\lim}\, K(\Lambda_n)$. ■

The sampling inequality (1) gives a bound for the density of the sampling set.

Lemma 3.5. *If Λ is a uniformly discrete set of sampling for L^p_π, then* $\mathrm{l.u.d.}(\Lambda) > 0$.

Proof: Let Λ be uniformly discrete with $\mathrm{l.u.d.}(\Lambda) = 0$ and suppose that $\|f\|_p \le A\|f|\Lambda\|_p$, $\forall f \in L^p_\pi$. Let T_x be the translation operator, $T_x f(y) = f(y-x)$. Since $\|T_x f\|_p = \|f\|_p$ and $\|T_x f|\Lambda\|_p = \|f|\Lambda - x\|_p$, it follows that for all $x \in \mathbb{R}$ we have

$$\|f\|_p = \|T_x f\|_p \le A\|T_x f|\Lambda\|_p = A\|f|\Lambda - x\|_p, \quad \forall f \in L^p_\pi. \tag{2}$$

The fact that $\mathrm{l.u.d.}(\Lambda) = 0$ implies that we can find an arbitrarily large interval $I(x_R, R)$ for which $I(x_R, R) \cap \Lambda = \emptyset$. Choose n such that the function $g(z) = \left[\frac{\sin(\pi z)/n}{\pi z/n}\right]^n$ is in L^p_π. We can make the right-hand side of the above inequality arbitrarily small for this function by choosing R large, so (2) does not hold for g. We conclude that $\mathrm{l.u.d.}(\Lambda) > 0$. ■

A set Λ is said to be a set of uniqueness if every function $f \in L^p_\pi$ that vanishes on Λ vanishes identically. The sampling inequality (1) implies that a set of sampling is also a set of uniqueness. Beurling showed that for the sampling problem in L^∞_π even more is true (see [**1**, p. 345]):

Theorem 3.6. *The set Λ is a set of sampling for L^∞_π if and only if every set $\Lambda_0 \in W(\Lambda)$ is a set of uniqueness.*

Beurling's density result for L^∞_π (Theorem 5 [**1**, p. 346]) is crucial in our analysis.

Theorem 3.7. *The uniformly discrete set Λ is a set of sampling for L^∞_π if and only if $d = \text{l.u.d.}(\Lambda) > 1$.*

If $\Lambda = \{\lambda_j\}$ is a set of interpolation for L^p_π, standard arguments based on the open mapping theorem for Fréchet spaces [**7**, p. 75] shows that the interpolation is stable. This means that there exists a positive number K such that for every sequence $\{w_j\} \in l^p$ we can find $f \in L^p_\pi$ such that

$$\|f\|_p \leq K\|f|\Lambda\|_p. \tag{3}$$

The smallest such K is denoted by $K_0(\Lambda)$.

Lemma 3.8. *Every set of interpolation for L^p_π is uniformly discrete.*

Proof: Choose $f(\lambda_k) = 1$ for some arbitrary k and let $f(\lambda_j) = 0$, $\forall j \neq k$. Then $\|f|\Lambda\|_p = 1$. We know that we can find f such that $\|f\|_p \leq K_0$. By Bernstein's inequality [**8**, p. 84] for L^∞_π and the fact that $\|f\|_\infty \leq C\|f\|_p$, we get

$$1 = \|f(\lambda_j) - f(\lambda_k)\|_\infty \leq \|f\|_\infty \pi|\lambda_j - \lambda_k| \leq CK_0\pi|\lambda_j - \lambda_k|. \quad \blacksquare$$

Lemma 3.9. *Let Λ be a uniformly discrete set. Then $\Lambda_n \rightharpoonup \Lambda$ implies $K_0(\Lambda) \leq \underline{\lim}\, K_0(\Lambda_n)$.*

Proof: Let $\Lambda = \{\lambda_k\}$ and $\Lambda_n = \{\lambda_k^{(n)}\}$. We may assume without loss of generality that $K_0(\Lambda_n) < \infty$ for all n, and thus there exists a solution $f^n_w \in L^p_\pi$ to every interpolation problem $f^n_w(\lambda_k^{(n)}) = w_k$ such that

$$\|f^n_w\|_p \leq K_0(\Lambda_n)\|w\|_p.$$

Choose a subsequence Λ_{n_i} for which $K_0(\Lambda_{n_i}) \to \underline{\lim}\, K_0(\Lambda_n)$. Then there exists a subsequence of n_i, say n_{i_j}, where $f_w^{n_{i_j}} \to f_w$ and $f_w(\lambda_k) = w_k$, $\|f_w\|_p \leq \underline{\lim}\, K_0(\Lambda_n)\|w\|_p$. $\blacksquare$

Assume that Λ is a set of interpolation and a set of uniqueness. Every function is then uniquely determined by its values on Λ. This implies that (3) holds for every $f \in L^p_\pi$ and thus Λ is a set of sampling. A set of interpolation can only be a set of uniqueness if it is also a set of sampling. The key lemma in the next section shows that this can not be the case, i.e., there are no discrete sets which are both sets of sampling and sets of interpolation for L^p_π.

4. The key lemma

The following lemma is our main auxiliary result.

Lemma 4.1. *For $0 < p \le 1$ there is no discrete subset of $\mathbb{R}$ that is both a set of sampling and a set of interpolation for L^p_π.*

Proof: We argue by contradiction. Suppose that such a set Λ exists. Choose a $\lambda_0 \in \Lambda$ and consider the unique function $g_0 \in L^p_\pi$ satisfying

$$g_0(z) = \begin{cases} 1 & \text{if } z = \lambda_0, \\ 0 & \text{if } z \in \Lambda \setminus \{\lambda_0\}. \end{cases}$$

Let $g(z) = (z-\lambda_0)g_0(z)$. It is clear that the function $f_\lambda(z) = g(z)/(z-\lambda)$ lies in L^p_π for arbitrary $\lambda \in \Lambda$. The fact that Λ is a set of sampling implies that

$$\|f_\lambda\|_p \le K\|f_\lambda|\Lambda\|_p = K\left(\sum_{\lambda_k\in\Lambda} |f_\lambda(\lambda_k)|^p\right)^{1/p} = K|g'(\lambda)|$$

because $g(\lambda_k) = 0$, for all $\lambda_k \in \Lambda$. The sampling constant K is independent of the choice of λ. Using the subharmonicity of $|f_\lambda(z)|^p$ we get (with $z = x + iy$)

$$|f_\lambda(\lambda)|^p = |g'(\lambda)|^p \le C(\epsilon) \iint_{R(\lambda,\epsilon,1)} |g(z)|^p \, dx\, dy$$

where $R(w,a,1) = \{z : w - a \le x \le w + a, -1 \le y \le 1\}$, $w, a \in \mathbf{R}$ and ϵ is chosen so small that $\inf |\lambda_i - \lambda_k| > 4\epsilon$ for $\lambda_i \ne \lambda_k$. The fact that a set of interpolation is uniformly discrete implies that we have a finite number of points from Λ in any $R(w,a,1)$. Collecting our results we obtain independently of w

$$\inf_{\xi\in R(w,T,1)} \sum_{\lambda\in\Lambda\cap R(w,T,1)} |\xi-\lambda|^{-p} \iint_{R(w,T,1)} |g(z)|^p \, dx\, dy \tag{4}$$

$$\le \sum_{\lambda\in\Lambda\cap R(w,T,1)} \int_{-B}^{B}\int_{-\infty}^{\infty} |f_\lambda(z)|^p \, dx\, dy$$

$$\le \sum_{\lambda\in\Lambda\cap R(w,T,1)} C|g'(\lambda)|^p \tag{5}$$

$$\le C \iint_{R(w,T,1)} |g(z)|^p \, dx\, dy, \tag{6}$$

where we have used a classical result by Plancherel-Pólya [**8**, p. 94]

$$\int_{-B}^{B}\int_{-\infty}^{\infty}|f_\lambda(x+iy)|^p\,dx\,dy \le C_1(B,p)\int_{-\infty}^{\infty}|f_\lambda(x)|^p\,dx.$$

Inequality (6) implies that the sum is bounded even when T tends to infinity, i.e.,

$$\inf_{|\Im z|\le 1}\sum_{\lambda\in\Lambda}|\lambda - z|^{-p} \le C. \tag{7}$$

But in Lemma 3.5 we found that $\text{l.u.d.}(\Lambda) > 0$, so the sum does not increase if we exchange the λ's by points on a grid where the separation is large enough. Choose e.g. the grid $\{cn\}_{n\in\mathbf{Z}}$. Since $\sum_{n\in\mathbf{Z}} 1/n^p$ diverges for $p \le 1$ we have a contradiction. This concludes the proof. ■

From the key lemma we see that the spaces L^p_π, $0 < p \le 1$ are fundamentally different from L^p_π, $1 < p < \infty$ for which there exist sets which are both sampling and interpolation sets. (See Lyubarskii and Seip [**4**].)

The following three results are direct consequences of the key lemma.

Lemma 4.2. *We can remove a point from a set of sampling and still have a set of sampling.*

Proof: Removing a point from a set of sampling does not change the fact that the sampling operator has closed range. The operator is injective so the open mapping theorem yields the lower frame bound. ■

We shall need the following notion of distance from a point x on the real axis to the set Λ. For $x \in \mathbb{R}$, let $\rho(x;\Lambda) = \sup_f |f(x)|$, where f ranges over all functions $f(x) \in L^p_\pi$ vanishing on the set Λ and for which $\|f\|_p \le 1$ (see [**1**, p. 352]).

Lemma 4.3. *If Λ is a set of interpolation then $\rho(x;\Lambda) > 0$, $x \notin \Lambda$.*

Proof: If Λ is a set of interpolation, it is not a set of uniqueness as remarked in the last paragraph of section 3.

Given $x_0 \notin \Lambda$, pick $f \in L^p_\pi$, where $f|_\Lambda = 0$ and $f \not\equiv 0$. We can find an integer n, $n \ge 0$ such that the function

$$g(x) = \frac{f(x)}{(x-x_0)^n}$$

is analytic at x_0 and $g(x_0) \ne 0$. Hence $\rho(x_0;\Lambda) \ne 0$. ■

Lemma 4.4. *Adding a point to a set of interpolation yields another set of interpolation.*

Proof: Let w_0 be the value at x_0 and w_n at λ_n where $|w_0| \leq 1$ and $|w_n| \leq 1$. Using the result in the above lemma we can find a function $f_0 \in L^p_\pi$, such that $\|f_0\|_p \leq 1$, f_0 is vanishing on the set Λ and $f_0(x_0) = \rho(x_0) \neq 0$. If f solves the interpolation on Λ, the function

$$g(x) = f(x) + \frac{w_0 - f(x_0)}{\rho(x_0)} f_0(x)$$

solves the interpolation on $\Lambda \cup \{x_0\}$. ■

5. Sampling

5.1. The necessity part of Theorem 2.1. We assume that Λ is a set of sampling for L^p_π. Let $\Lambda_0 \in W(\Lambda)$. According to Lemma 3.4. every set in $W(\Lambda)$ is a set of sampling, so Λ_0 is a set of sampling and thus a set of uniqueness. We want to show that Λ_0 is a set of uniqueness not only for L^p_π but also for L^∞_π. If we can show this, we can use Beurling's results for L^∞_π as cited in Theorem 3.6 and Theorem 3.7.

Suppose that Λ_0 is not a set of uniqueness for L^∞_π, i.e., that there exits $g \in L^\infty_\pi$ such that $g \not\equiv 0$ but $g|_{\Lambda_0} = 0$. Define the function

$$f(z) = \frac{g(z)}{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)} \text{ where } \lambda_1, \ldots, \lambda_n \in \Lambda_0,\ np \geq 1+\epsilon.$$

This function is an entire function and $f(\lambda) = 0$ for every $\lambda \in \Lambda_0 \setminus \{\lambda_1, \ldots, \lambda_n\}$. Since $g \in L^\infty_\pi$, we have $\sup_{x\in\mathbb{R}} |g(x)|^p < \infty$ so $|f(x)|^p \sim O(|x|^{-1-\epsilon})$ which means that $f \in L^p_\pi$. According to Lemma 4.2, $\Lambda_0 \setminus \{\lambda_1, \ldots, \lambda_n\}$ is a set of sampling and thus a set of uniqueness. But $f \not\equiv 0$, so our original assumption about the set Λ_0 is false. The set Λ_0 is a set of uniqueness for L^∞_π. We conclude that every $\Lambda_0 \in W(\Lambda)$ is a set of uniqueness for L^∞_π. By Beurling's Theorem 3.6 this implies that Λ is a set of sampling for L^∞_π, and applying Theorem 3.7 we find that $\operatorname{l.u.d.}(\Lambda) > 1$. This completes the proof.

5.2. The sufficiency part of Theorem 2.1.

We suppose now that $a = \operatorname{l.u.d.}(\Lambda) > 1$. By Theorem 3.7, Λ is a set of sampling for every space $L^\infty_{\pi+\epsilon}$, $\epsilon < a-1$. If, say, $\epsilon = \frac{1}{2}(a-1)$, we get

$$\|f\|_\infty \leq C\|f|\Lambda\|_\infty \quad \text{for all} \quad f \in L^\infty_{\pi+\epsilon}.$$

The set of sequences $\{f(\Lambda) : f \in L^\infty_{\pi+\epsilon}\}$ is a closed subspace of c_0 ($c_0 \subset l^\infty$ consists of all bounded sequences converging to zero). The mapping $f(\Lambda) \mapsto f(x)$ with x fixed is a bounded linear functional on this space. By the Riesz representation theorem, we have therefore

$$f(x) = \sum_k c_k(x) f(\lambda_k) \quad \text{where} \quad \sum |c_k| < M\|f\|_\infty. \tag{8}$$

Given $f \in L^p_\pi$, we put $g_x(t) = f(t)\left[\frac{\sin(\epsilon(t-x))/n)}{\epsilon(t-x)/n}\right]^n \in L^\infty_{\pi+\epsilon}$, and apply (8) to this function. Since $g_x(x) = f(x)$, we obtain by integrating and using the inequality $(\sum |a_k|)^p \leq \sum |a_k|^p$

$$\begin{aligned}\int |f(x)|^p\,dx &= \int \left|\sum c_k(x) f(\lambda_k)\left[\frac{\sin(\epsilon(\lambda_k - x)/n)}{\epsilon(\lambda_k - x)/n}\right]^n\right|^p dx \\ &\leq \int \sum \left|c_k(x) f(\lambda_k)\left[\frac{\sin(\epsilon(\lambda_k - x)/n)}{\epsilon(\lambda_k - x)/n}\right]^n\right|^p dx \\ &= \sum |f(\lambda_k)|^p \int |c_k(x)|^p \left|\frac{\sin(\epsilon(\lambda_k - x)/n)}{\epsilon(\lambda_k - x)/n}\right|^{np} dx \\ &\leq C\|f|\Lambda\|_p.\end{aligned}$$

The last inequality follows from the fact that $\{c_k(x)\} \in l^1$. We have thus proved that $\text{l.u.d.}(\Lambda) > 1$ implies $K(\Lambda) < \infty$.

6. Interpolation

6.1. The necessity part of Theorem 2.2. The next lemma corresponds to Lemma 5 in [**1**, p. 354]. Its proof is a bit simpler than for L^∞_π, since all functions in L^p_π, $0 < p \leq 1$ tend to zero when $|x| \to \infty$.

Lemma 6.1. *Given δ_0, k_0 there exists a constant $C = C(\delta_0, k_0)$ such that if Λ is a set of interpolation with $K_0(\Lambda) \leq k_0$ and if* $\text{dist}(x, \Lambda) \geq \delta_0$, *then*

$$\rho(x; \Lambda) \geq C.$$

Proof: If the lemma is false, there exists a sequence of sets Λ_n, all with $K_0(\Lambda_n) \leq k_0$, and points x_n with $\text{dist}(x_n, \Lambda_n) \geq \delta_0$ such that $\rho(x_n, \Lambda_n) \to 0$. By translating Λ_n for all n we may assume that $x_n = 0$ for all n and that $\Lambda_n \rightharpoonup \Lambda'$. By Lemma 3.9

$$K_0(\Lambda') \leq \underline{\lim}\, K_0(\Lambda_n) \leq k_0.$$

We have that the point $0 \notin \Lambda'$. We arrive at a contradiction by proving that $\rho(0,\Lambda_n)$ has a positive lower bound. Fix y, $y \notin \Lambda'$, $y \neq 0$. By Lemma 4.4 we have $K_0(\Lambda' \cup \{y\}) < \infty$. The distance from zero to $\Lambda' \cup \{y\}$ is $\rho(0,\Lambda' \cup \{y\}) = \gamma > 0$. Hence we can find $f \in L^p_\pi$ such that $\|f\|_p \leq 1$, $f(\lambda') = 0$, for all $\lambda' \in \Lambda'$. Since $\Lambda_n \rightharpoonup \Lambda'$, f vanishes on Λ' and moreover $f(x) \to 0$ when $|x| \to \infty$, it follows that $\|f|\Lambda_n\|_p = \epsilon_n \to 0$ when $n \to \infty$. Choose $f_n \in L^p_\pi$ such that $f_n(\lambda) = f(\lambda)$ for all $\lambda \in \Lambda_n$. This implies that $\|f_n\|_p \leq k_0\epsilon_n$. Define the function

$$g_n(x) = \frac{f(x) - f_n(x)}{\|f\|_p + k_0\epsilon_n}.$$

Then $g_n \in L^p_\pi$ with $\|g_n\|_p \leq 1$ and g_n vanishes on Λ_n. Hence

$$\rho(0,\Lambda_n) \geq |g_n(0)| = \frac{|f(0) - f_n(0)|}{\|f\|_p + \epsilon_n k_0}.$$

Since the supremum norm is bounded by the L^p_π-norm, we have $|f_n(x)| \leq Ck_0\epsilon_n$, and thus

$$\varliminf \rho(0,\Lambda_n) > \frac{f(0)}{\|f\|_p} = \frac{\gamma}{\|f\|_p} > 0.$$

This contradiction concludes our proof. ■

Lemma 6.2. *Given* Λ *and suppose that* $K_0(\Lambda) \leq k_0 < \infty$. *Then there exists a positive constant* $C_1(k_0)$ *such that*

$$\int_0^1 \log \rho(x,\Lambda)\,dx \geq -\,C_1(k_0).$$

Proof: The lemma and its proof are identical to Lemma 6 in [**1**, p. 354]. Since Λ is uniformly discrete, we can find $x_0 \in (0,1)$ such that $\operatorname{dist}(x_0,\Lambda) \geq 1/\pi k_0$. According to the previous lemma, there exists a lower bound $C(k_0)$ for $\rho(x_0,\Lambda)$. This implies that we can choose $f \in L^p_\pi$, f vanishing on Λ but with $f(x_0) \geq C$. Denote by D the domain bounded by the circle $|z| = 3$ and the slit $(0,1)$. By the maximum principle $f(z_0) \geq C$ for some z_0 on the circle $|z_0| = 2$ and

$$\log C \leq \log|f(z_0)| \leq \frac{1}{2\pi}\int_{\partial D} \frac{\partial G}{\partial n}(z;z_0)\log|f(z)|\,ds$$

where G is Green's function of D and $\partial/\partial n$ is differentiation along the interior normal. Since $\partial G/\partial n \leq \text{const}$ and $|f(z)| \leq e^{\pi|y|}$, we have that

$$\log C \leq \text{const}\left[\int_0^1 \log|f(x)|\,dx + 3\pi\right] \leq \text{const}\left[\int_0^1 \log\rho(x,\Lambda)\,dx + 3\pi\right]$$

and the result follows. ■

Because of translation invariance we have

$$\int_t^{t+1} \log\rho(x,\Lambda)\,dx \geq -C_1(k_0). \tag{9}$$

Assume now that $K_0(\Lambda) < \infty$. Inequality (9) and the fact that $L^p_\pi \subset L^\infty_\pi$ enable us to copy Beurling's proof for L^∞_π. Its basic idea is to adjoin single points over large intervals, and summing estimates based on Jensen's formula over even larger intervals, to obtain sufficiently strong estimates.

We need the following construction. For every interval $(t,t+1)$, $t \in \mathbb{Z}$, choose a point x_t such that $\text{dist}(x_t,\Lambda) \geq \delta_0$ where δ_0 is independent of t. Fix x and choose t such that $x \in (t,t+1)$. Set $\Lambda_t = \Lambda \cup \{x_t\}$. By Lemma 4.4 and Lemma 6.1 $K_0(\Lambda_t) \leq k_1$, where k_1 is independent of t, and thus

$$\int_t^{t+1} \log\rho(x,\Lambda_t)\,dx \geq -C_1(k_1). \tag{10}$$

Choose $f \in L^p_\pi$ with $\|f\|_p \leq 1$, f vanishing on Λ_t and $|f(x)| = \rho(x,\Lambda_t)$. Denote by D the disk $|z-x| < r$. Let $z_1,\dots,z_n$ be the zeros of $f(z)$ in D. Jensen's formula applied to D gives

$$\log\rho(x,\Lambda_t) = \log|f(x)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(x+re^{i\theta})|\,d\theta - \sum_{k=1}^{n}\log\frac{r}{|z_k-x|} \tag{11}$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(x+re^{i\theta})|\,d\theta + \sum_{\lambda_k\in\Lambda\cap D}\log\frac{|\lambda_k-x|}{r} + \log\frac{|x_t-x|}{r} \tag{12}$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(x+re^{i\theta})|\,d\theta + \sum_{\lambda_k\in\Lambda\cap D}\log\frac{|\lambda_k-x|}{r} + \log\frac{1}{r}. \tag{13}$$

Using the fact that $|f(x+re^{i\theta})| \le e^{\pi r|\sin\theta|}$ we get that the integral in (13) is less than or equal to $2r$. The sum in (13) can be written as an integral with respect to the discrete measure $dn(s)$ where $n(t_2)-n(t_1) = \#$(points of Λ in (t_1,t_2))

$$\sum_{\lambda_k\in\Lambda\cap D} \log\frac{|\lambda_k - x|}{r} = -\int_{-\infty}^{\infty} \log^+\left|\frac{r}{s-x}\right| dn(s).$$

Collecting our results we get the following inequality

$$\int_{-\infty}^{\infty} \log^+\left|\frac{r}{s-x}\right| dn(s) \le 2r - \log r + \log\frac{1}{\rho(x,\Lambda_t)}. \tag{14}$$

This inequality holds for all $x \in (t, t+1)$, thus we can integrate over this interval and use (10)

$$\int_t^{t+1}\left[\int_{-\infty}^{\infty} \log^+\left|\frac{r}{s-x}\right| dn(s)\right] dx \le 2r - \log r + C_1(k_1).$$

Fix x_0 and R and sum the preceding inequality from $t = x_0 - r$ to $t = x_0 + r + R - 1$. We get

$$\begin{aligned}\int_{x_0-r}^{x_0+r+R}\left[\int_{-\infty}^{\infty} \log^+\left|\frac{r}{s-x}\right| dn(s)\right] dx& \\ \le (2r - \log r + C_1(k_1))(R+2r).&\end{aligned} \tag{15}$$

Noting that

$$\int_{s-r}^{s+r} \log^+\left|\frac{r}{s-x}\right| dx = 2r,$$

we obtain by changing the order of integration in (15)

$$\begin{aligned}2r(n(x_0+R)-n(x_0)) &= \int_{x_0}^{x_0+R}\left[\int_{s-r}^{s+r} \log^+\left|\frac{r}{s-x}\right| dx\right] dn(s)\\ &\le (2r - \log r + C_1(k_1))(R+2r).\end{aligned}$$

This gives

$$\begin{aligned}\frac{(n(x_0+R)-n(x_0))}{R} &\le \left(1 - \frac{\log r}{2r} + \frac{C_1(k_1)}{2r}\right)\left(1+\frac{2r}{R}\right)\\ &= 1 + C(r, R, C_1(k_1)).\end{aligned}$$

Let $R = r^2$, and choose r such that $C(r, r^2, C_1(k_1)) < 0$. This implies that $\frac{n(x_0+r^2)-n(x_0)}{r^2} < (1-\delta) < 1$. For this particular r and arbitrary x_0 every interval (x_0, x_0+r^2) will contain less than $(1-\delta)r^2$ points from Λ. δ does not depend on x_0. Every interval of length mr^2 can be expressed as a disjoint union of m intervals $[x_0, x_0+r^2)$ with less than $(1-\delta)r^2$ points from Λ. It follows that $\frac{\overline{n}(mr^2)}{mr^2} \le 1-\delta$. Letting $m\to\infty$ we get the desired density bound u. u. d.$(\Lambda) < 1$.

6.2. The sufficiency part of Theorem 2.2. We shall give a constructive proof, consisting of an explicit construction of a linear operator of interpolation, analogous to that of P. Beurling [**2**]. Assume that $\mathrm{u.\,u.\,d}(\Lambda) = d < 1$. We will show that $K_0(\Lambda) < \infty$, and begin by choosing a rational number d_1 between d and 1. By the definition of d, for large L, the largest number of points from Λ in an interval of length L is bounded by Ld_1. Without loss of generality we can choose a large L such that $m = Ld_1$ is an integer. Divide $\mathbb{R}$ into intervals $\{\omega_k\}_{-\infty}^{\infty}$ of length L and add points so that each interval ω_k contains exactly m points. The points we add are chosen in such a way that their distance from Λ is uniformly bounded away from zero. The resulting set is still called Λ.

The following lemma is verbatim Lemma 7 from Beurling [**1**, p. 357].

Lemma 6.3. *Let* Λ *be as above and assume* $0 \in \Lambda$. *The limit*

$$f(z) = \lim_{R\to\infty} \left\{ \prod_{\substack{0<|\lambda_k|<R \\ \lambda_k\in\Lambda}} \left(1 - \frac{z}{\lambda_k}\right) \right\}$$

exists for all $z \in \mathbb{C}$ *and* f *is an entire function, vanishing on* $\Lambda \setminus \{0\}$, $f(0) = 1$ *and*

$$|f(x+iy)| \le C(|z|+1)^{5m} e^{\pi d_2 |y|}, \quad d_2 < 1.$$

We use the lemma to carry out the interpolation on Λ.

Fix ϵ between 0 and $1 - d_2$ and choose

$$h(z) = \left(\frac{\sin \epsilon z/n}{\epsilon z/n}\right)^n, \quad \text{where} \quad n \ge \frac{2}{p}.$$

This function satisfies the inequality

$$|h(z)| \le \frac{C}{(|z|+1)^{5m+n}} e^{\epsilon |y|}.$$

For every $\lambda \in \Lambda$ we apply Lemma 6.3 to construct a function f_λ using λ as the origin such that $f_\lambda(\mu) = 0$ for $\mu \in \Lambda \setminus \{\lambda\}$, $f_\lambda(\lambda) = 1$ and

$$|f_\lambda(z)| \le C(1 + |z-\lambda|)^{5m} e^{\pi d_2 |y|}.$$

Define the function $g_\lambda(z) = f_\lambda(z) h(z-\lambda)$. This function has the following properties: $g_\lambda(\mu) = 0$ for $\mu \in \Lambda \setminus \{\lambda\}$, $g_\lambda(\lambda) = 1$ and

$$|g_\lambda(z)| \le \frac{C}{(|z-\lambda|+1)^n} e^{(\pi d_2 + \epsilon)|y|}.$$

Since $\pi d_2 + \epsilon < \pi$, we see that $g_\lambda \in L^p_\pi$, $\lambda \in \Lambda$. Their L^p-norms are all bounded by the same constant C'. Furthermore $\sum_{\lambda\in\Lambda} |g_\lambda(z)|^p < \infty$, for all $z \in \mathbb{C}$. If $\{w(\lambda)\} \in l^p$ is any sequence defined for $\lambda \in \Lambda$, then

$$g(z) = \sum_{\lambda\in\Lambda} w(\lambda) g_\lambda(z)$$

solves the interpolation problem. We see that

$$\begin{aligned}\int |g(x)|^p\,dx = \int \left|\sum_{\lambda\in\Lambda} w(\lambda) g_\lambda(x)\right|^p dx &\leq \sum_{\lambda\in\Lambda} |w(\lambda)|^p \int |g_\lambda(x)|^p\,dx \\ &\leq C' \sum_{\lambda\in\Lambda} |w(\lambda)|^p < \infty,\end{aligned}$$

so $g(z)$ is a function in L^p_π. This implies that $K_0(\Lambda, a) < \infty$. Our proof is finished.

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