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A NOTE ON THE RELLICH FORMULA IN LIPSCHITZ DOMAINS

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Abstract

Let L be a symmetric second order uniformly elliptic operator in divergence form acting in a bounded Lipschitz domain Ω of $\mathbb{R}^N$ and having Lipschitz coefficients in Ω. It is shown that the Rellich formula with respect to Ω and L extends to all functions in the domain $\mathcal{D} = \{u \in H^1_0(\Omega);\ L(u) \in L^2(\Omega)\}$ of L. This answers a question of A. Chaïra and G. Lebeau.

1. Introduction

Let $L = \sum_{1\leq i,j\leq N} \partial_i(a_{ij}\partial_{j.})$ be a uniformly elliptic operator in divergence form in $\mathbb{R}^N$, the coefficients a_{ij} being (real) Lipschitz continuous functions in $\mathbb{R}^N$ such that $a_{ij} = a_{ji}$ for $1 \leq i,\, j \leq N$. Let $\mathcal{A}$ denote the matrix $\{a_{ij}\}$.

If $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain in $\mathbb{R}^N$, if V is a C^1 vector field in $\overline{\Omega}$ and if $u \in H^2(\Omega)$, then the following so-called Rellich formula holds (for references see Nečas [**N**, p. 224]).

$$(1)\quad \int_{\partial\Omega} \left\{ \partial_{\nu_L}(u)\partial_V(u) - \frac{1}{2}\|\nabla u\|_L^2 \langle V,\nu\rangle \right\} d\sigma$$

$$= \int_\Omega \left\{ du(V)L(u) + du(\partial_{\mathcal{A}\nabla u} V) - \frac{1}{2}\operatorname{div}(V)\|\nabla u\|_L^2 - \frac{1}{2} q'_{L,V}(\nabla u) \right\} dx$$

where ν is the unit exterior normal field along $\partial\Omega$ and $\nu_L = \mathcal{A}(\nu)$ is the conormal field; ∂_U denotes the differentiation operator in the direction U, and we have let $\|U\|_L^2 = \langle \mathcal{A}U, U\rangle$ and $q'_{L,V}(U) = \langle \partial_V(\mathcal{A})(U), U\rangle =$

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$\sum_{i,j} U_i U_j da_{ij}(V)$ when $V \in \mathbb{R}^N$. At least if $u \in C^2(\overline{\Omega})$, the formula follows from the Stokes formula $\int_{\partial\Omega} W.\nu\, d\sigma = \int_\Omega \operatorname{div}(W)\, dx$ on taking $W = du(V)\mathcal{A}(\nabla u) - \frac{1}{2}\|\nabla u\|_L^2 V$. The general case follows from an approximation argument. Of course the Lipschitz regularity of Ω is only needed in a neighborhood of $\operatorname{supp}(V) \cap \partial\Omega$.

In this note it is shown that the Rellich formula extends to all functions u in the domain of L, that is $u \in H_0^1(\Omega)$ with $L(u) \in L^2(\Omega)$; this amounts ([**N**]) to a continuity property of the gradient of L-solutions with respect to perturbations of Ω (see Theorem 1 below). This extension of Rellich formula answers a question raised to me by A. Chaïra and G. Lebeau [**CL**] (see also [**N**, Problème 2.2, p. 258)] and is useful in some problems in control theory for the wave equation ([**C**], [**CL**]). The proof relies on well-known results and methods of the Potential theory in Lipschitz domains (in particular [**D**], [**A1**], [**JK1**] and [**A2**]).

2. Notations and preliminaries

In this section we fix some notations and recall several basic properties of the Potential theory in Lipschitz domains with respect to elliptic second order operators.

2.1. Let N be a fixed integer ≥ 2 and let $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a function such that $\varphi(0) = 0$ and $|\varphi(x) - \varphi(y)| \leq k|x - y|$ for x, $y \in \mathbb{R}^{N-1}$ and a positive constant k. For $x \in \mathbb{R}^N$, we note $x = (x', x_N)$ the decomposition of x in $\mathbb{R}^{N-1} \times \mathbb{R}$ and let $\Sigma = \{(x', \varphi(x'));\, x' \in \mathbb{R}^{N-1}\}$. For $P = (P', P_N) \in \Sigma$, we set

$$\begin{aligned} &T(P,r) = \{(x', x_N) \in \mathbb{R}^N;\, |x' - P'| < r,\, |P_N - x_N| < 10kr\} \\ &\omega(P,r) = \{(x', x_N) \in T(P,r);\, x_N < \varphi(x')\}, \\ &A(P,r) = (P', P_N - 5kr) \quad \text{and} \\ &\Sigma(P,r) = \{(x', x) \in \Sigma;\, |x' - P'| < r\}. \end{aligned} \tag{2}$$

In the sequel, the dependence on N of the various constants is not made explicit. We note $\delta(x) = d(x, \Sigma)$ for $x = (x', x_N) \in \mathbb{R}^N$.

2.2. For $0 < \alpha \leq 1$ and $M > 0$, we denote $\Lambda(\alpha, M)$ the class of elliptic operators L in $\mathbb{R}^N$ in the form

$$L(u) = \sum_{1 \leq i,j \leq N} a_{ij}\partial_{ij}^2(u) + \sum_{1 \leq j \leq N} b_j \partial_j u + \gamma u \tag{3}$$

where a_{ij}, b_i and γ are bounded borel functions on $\mathbb{R}^N$ such that when x, $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$,

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq M^{-1}\sum_j \xi_j^2, \quad a_{ij}(x) = a_{ji}(x) \tag{4}$$

$$\begin{gathered}\Big(\sum_{i,j}\|a_{ij}\|_\infty\Big) + \Big(\sum_j \|b_j\|_\infty\Big) + \|\gamma\|_\infty \leq M,\\ \sum_{i,j}|a_{ij}(x) - a_{ij}(y)| \leq M|x-y|^\alpha.\end{gathered} \tag{4'}$$

2.3. Harnack boundary principle. If $L \in \Lambda(\alpha, M)$, if $P \in \Sigma$, if u and v are two positive L-solutions in $\omega(P,r)$ vanishing on $\Sigma(P,r)$ and if $A = A(P,r)$, $r \leq r_0$, then

$$c^{-1}\frac{u(x)}{u(A)} \leq \frac{v(x)}{v(A)} \leq c\frac{u(x)}{u(A)} \tag{5}$$

for all $x \in \omega(P, r/2)$, where $c = c(k, \alpha, M, r_0) > 0$ ([**A1**], see [**A3**] and references there for other related results). More generally, under the same assumptions on L and u, if v is positive L_1-harmonic on $\omega(P,r)$ for some $L_1 \in \Lambda(\alpha, M)$ having on $\Sigma(P,r)$ the same second order part than L, and if $v = 0$ on $\Sigma(P,r)$, inequalities (5) hold on $\omega(P, r/2)$ for some $c = c(k, \alpha, M, r_0)$ ([**A2**]).

2.4. Ratios of positive harmonic functions near the boundary. The Harnack boundary principle (5) when combined with the maximum principle implies a stronger continuity statement for the ratios of harmonic functions [**JK1**]. If u and v are positive L-solutions on $\omega(P,r)$, $P \in \Sigma$, $r \leq r_0$, vanishing on $\Sigma(P,r)$, and if $A_\theta = A(P, r\theta)$, $0 < \theta < 1$, then

$$\left|1 - \frac{u(x)}{u(A_\theta)} : \frac{v(x)}{v(A_\theta)}\right| \leq c\theta^\beta \tag{6}$$

for $x \in \omega(P, r\theta/2)$. Here c and β are > 0 constants (depending only on k, M, α and r_0).

2.5. Uniform decay property. The following consequence of 2.3 is also needed. There is a constant $\eta = \eta(\alpha, M, k, r_0)$, $0 < \eta \leq 1/4$, such that if u is positive L-harmonic in $\omega(P,r)$, $P \in \Sigma$, $r \leq r_0$, and $u = 0$ on $\Sigma(P,r)$, then $u(x) \leq \frac{1}{2}u(A(P,r))$ for $x \in \omega(P, \eta r)$. It follows that $u(x) \leq C[\delta(x)]^\gamma u(A(P,r))$, $\gamma = \log(2)/|\log(\eta)|$, for some constant $C = C(\alpha, M, k, r_0)$ and $x \in \omega(P, \frac{r}{2})$. The opposite estimate, $u(x) \geq C[\delta(x)]^{\gamma'} u(A(P,r))$ for $x \in \omega(P, cr)$ and with another constant $\gamma' > 0$ follows from the local Harnack inequalities.

2.6. Fatou's Theorem. Denote μ_A^Ω the harmonic measure of $A = A(P,r)$ in $\Omega = \omega(P,r)$ with respect to L. If s is positive and L-superharmonic in Ω then s admits a *fine* limit at μ_A^Ω almost every point $P \in \Sigma(P,r)$, this fine limit being zero μ_A^Ω-a.e. if s is a potential. If s is L-harmonic in Ω then s admits a *non-tangential* limit at μ_A^Ω almost every point $P \in \Sigma(P,r)$. The last property is related to the first by the following fact. If $U \subset \Omega$ is the union of a sequence of balls $B(x_j, \varepsilon\delta(x_j)) \subset \Omega$ where $\varepsilon > 0$ is fixed and $x_j \to Q \in \Sigma(P,r)$, then U is not minimally thin (in Ω) at P (ref. [**A1**]).

2.7. Density of harmonic measure. Let Ω be a domain such that $\Omega \cap T(P,r) = \omega(P,r)$ for some $P \in \Sigma$ and $r \le r_0$ and let $A = A(P,r)$. Let $L \in \Lambda(1,M)$ be formally self-adjoint. The L-harmonic measure μ_x^Ω of $x \in \Omega$ is equivalent on $\Sigma(P,r)$ to the natural area-measure σ. In fact, on $\Sigma' = \Sigma(P,r/2)$, $\mu_A^\Omega = f_A.\sigma$ with $\|f_A\|_{L^2(\Sigma')} \le C\{\sigma(\Sigma')\}^{-\frac{1}{2}}$ where $C = C(k,M,r_0) > 0$. This follows from the Rellich formula (see also [**D**], [**JK2**], [**A2**]). Also, $f_x > 0$ a.e. on Σ' (the argument of [**D**] for $L = \Delta$ is easily extended). The Harnack boundary principle shows that the density f_A satisfies also a reverse Hölder inequality. For each $a = (a', a_d) \in \Sigma'$ and each positive t with $t < \frac{1}{2}r$:

$$(7) \qquad \int_{\Sigma(a,t)} f_A(x)\, d\sigma(x) \ge C\sqrt{\sigma(\Sigma(a,t))}\left(\int_{\Sigma(a,t)} |f_A(x)|^2\, d\sigma(x)\right)^{\frac{1}{2}}$$

where $C = C(k,M,r_0) > 0$. By a theorem of Gehring ([**G**]), it follows that $f_A \in L^p(\Sigma')$ for some $p = p(k,M,r_0) > 2$ with a uniform bound $\|f_A\|_{L_p(\Sigma')} \le C\{\sigma(\Sigma')\}^{\frac{1}{p}-1}$, $C = C(k,M,r_0)$.

The above extends to wider classes of divergence type elliptic operators (see [**FKP**] and references there), and also to every L in $\Lambda(\alpha, M)$, $0 < \alpha \le 1$ ([**A4**]), but this will not be needed here.

3. Non-tangential differentiability property

From now on (Section 3, 4, 5) we consider an operator $L \in \Lambda(1,M)$,

$$L = \sum_{1 \le i,j \le N} a_{ij}(x)\partial_i\partial_j + \sum_{1 \le j \le N} b_j(x)\partial_j + \gamma$$

verifying (4) and (4′) with $\alpha = 1$. As a first step for the proof of Theorem 1 we prove the next lemma which is probably known but an explicit reference seems difficult to locate (see [**KP**] for L^p estimates of the non-tangential maximal function of the gradient and a variant of L^p convergence, compare also [**A2**]). We give a proof which relies on Fatou theorem (2.6 above).

Lemma 1. *Let Ω be a Lipschitz domain in $\mathbb{R}^N$. If u is a solution of $Lu = 0$ in Ω vanishing on an open subset S of $\partial\Omega$, then ∇u admits a non-tangential limit at almost every point $P \in S$.*

Proof: It is enough to consider the case where $\Omega = \omega(0, r)$ and $S = \Sigma(0, r/2)$, $r > 0$ (with the notations in 2.1 and with respect to some Lipschitz continuous function $\varphi : \mathbb{R}^{N-1} \to \mathbb{R}$ such that $\varphi(0) = 0$). We let $\Omega' = \omega(0, r/2)$, $\Omega'' = \omega(0, 3r/4)$ and assume as we may that u is continuous and positive on $\overline{\Omega}$ with $u = 0$ on $\Sigma(0, r)$. Then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p < \infty$ ([**LU**, p. 203–205]) and $u_{|\Omega''} \in H^1(\Omega'')$ (see Remark 1.2 below).

Set $L_0 = \sum_{1\le i,j\le N} \partial_i(a^0_{ij}(x)\partial_j)$ where $a^0_{ij}(x) = a_{ij}(x', \varphi(x'))$ and let w be the solution to the problem $L_0 w = 0$ in Ω, $w = 1$ on $\partial\Omega \setminus \Sigma$, and $w = 0$ on Σ (compare [**A2**]). Note that $L_0 \in \Lambda(1, M')$ for some $M' = M'(k, M) > 0$ and that L_0 is self-adjoint. Observe also that $(-\partial_N w)$ is L_0-harmonic in Ω (because L_0 is independent of x_N) and positive (by the maximum principle). It follows from Harnack inequalities, the uniform decay property 2.5 and the interior gradient estimates that for $x \in \Omega''$ ([**A2**])

$$0 < -\partial_N w(x) \le |\nabla w(x)| \le c\frac{w(x)}{\delta(x)} \le -C\,\partial_N w(x). \tag{8}$$

By the boundary Harnack principle (5) we have if $x \in \Omega''$

$$|\nabla u(x)| \le c\frac{u(x)}{\delta(x)} \le c'\frac{w(x)}{\delta(x)} \le -C\,\partial_N w(x). \tag{9}$$

The argument is now broken into three steps. First we note that the distribution $L_0(\partial_k u)$ (which is defined as an element of $H^{-1}_{\mathrm{loc}}(\Omega)$ since $\partial_k u \in H^1_{\mathrm{loc}}(\Omega)$) belongs to $H^{-1}(\Omega')$, i.e. to the dual of $H^1_0(\Omega')$. Since $Lu = 0$,

$$\begin{aligned} L_0(\partial_k u) = \sum_{1\le i,j\le N} \partial_k[(a^0_{ij} - a_{ij})\partial_i\partial_j u] - \partial_k\left[\sum_{j=1}^N b_j\,\partial_j u + \gamma\,u\right] \\ - \sum_{1\le i,j\le N} (\partial_k a^0_{ij})(\partial_i\partial_j u) + \sum_{1\le i,j\le N} (\partial_i a^0_{ij})(\partial_k\partial_j u). \end{aligned}$$

Using the Hardy inequality (Remark 1.1) and $|a_{ij}(x) - a^0_{ij}(x)| \le c\,\delta(x)$, it is seen that $(a^0_{ij} - a_{ij})\partial_j\partial_j u \in L^2(\Omega')$. In fact, on a ball $B =$

$B(x, \delta(x)/(40\,k))$, $x \in \Omega'$, we have the standard inner estimate ([**LU**, p. 205])

$$\int_{B'} |D^2 u(z)|^2\,dz \le C \int_B \delta(z)^{-4} u(z)^2\,dz$$

where $B' = B(x, \delta(x)/(80\,k)))$. By a Whitney covering argument, it follows that

$$\int_{\Omega'} |(a_{ij} - a^0_{ij})(\partial_i \partial_j u)|^2\,dy \le C \int_{\Omega''} \delta^{-2}\,u^2\,dy < +\infty.$$

This entails that $\sum_{i,j} \partial_k[(a^0_{ij} - a_{ij})\partial_i\partial_j u] \in H^{-1}(\Omega')$. Similarly, using the boundedness of $\partial_k(a^0_{ij})$ it is seen that $(\partial_k a^0_{ij})(\partial_i\partial_j u) \in H^{-1}(\Omega')$. In fact for $v \in H^1_0(\Omega')$,

$$\int_{\Omega'} |v\,\partial_i\partial_j u|\,dx \le C \left(\int_{\Omega'} \delta^{-2}|v|^2|\,dx\right)^{1/2} \left(\int_{\Omega''} \delta^{-2}|u|^2|\,dx\right)^{1/2}$$
$$\le C\|\nabla u\|_{L^2(\Omega'')}\|\nabla v\|_{L^2(\Omega')},$$

where we have first used that

$$\int_{B'\cap\Omega'} |v\,\partial_i\partial_j u|\,dz \le C \left[\int_{B'\cap\Omega'} \delta^{-2}\,v^2\,dx\right]^{1/2} \left[\int_B \delta^{-2}\,u^2\,dx\right]^{1/2}$$

for $x \in \Omega$ as above, and then a Whitney partition, Schwarz and Hardy's inequalities. In the same time we have also shown that $(\partial_i a^0_{ij})(\partial_k\partial_j u) \in H^{-1}(\Omega')$.

Second step: Introduce the function $v \in H^1_0(\Omega')$ which is such that $L_0(v) = L_0(\partial_k u)$ in Ω'. Since $v \in H^1_0(\Omega')$, a well-known projection argument shows that there is a L_0-supersolution $p \in H^1_0(\Omega')$ such that $|v| \le p$. By Fatou's theorem (2.6) applied to p and L_0, v converges finely (w.r. to L_0) to zero at almost every point $P \in S$. Writing $\partial_k u = v + h$, h is a L_0-solution on Ω', and by (9) we have that $|h| \le p - C\,\partial_N w$ on Ω'. Since $-\partial_N w$ is a > 0 L_0-solution in Ω this means that $|h| \le -C\,\partial_N w$ and h is hence a difference of two positive L_0 solutions in Ω'. By 2.6, h converges finely (and non-tangentially) almost everywhere on S. Thus, $\partial_k u$ converges finely at almost all point $P \in S$.

Third step: By [**LU**, p. 205], $\partial_k u$ has also the following uniform continuity property: for $x \in \Omega'$, and $y \in B(x, \delta(x)/2))$ one has $|\partial_k u(y) -$

$\partial_k u(x)| \le C\|u\|_{\infty,B}\ \delta^{-1-\alpha}(x)|x-y|^\alpha$, $B = B(x, \frac{3}{4}\delta(x))$, for some constants $\alpha = \alpha(M, r) \in]0, 1]$ and $C > 0$. Therefore, by (9) and Harnack inequalities,

$$|\partial_k u(y) - \partial_k u(x)| \le -C\, \partial_N w(x) \left(\frac{|x-y|}{\delta(x)}\right)^\alpha .$$

It follows that if $P \in S$ is such that $\partial_N w$ is non-tangentially bounded at P and $\partial_k u$ admits a fine limit ℓ at P, then $\partial_k u$ converges nontangentially to ℓ at P. If not, a positive number ε and points $x_j \in \Omega'$ converging non-tangentially to P could be constructed such that for each $j \ge 1$, $\inf\{|\partial_k u(x) - \ell|;\ |x - x_j| \le \varepsilon\,\delta(x_j)\} \ge \varepsilon$. This means that $\cup_{j\ge1} B(x_j, \varepsilon\delta(x_j))$ is thin at P, a contradiction (see 2.6 above). ∎

Remarks.

1.1. Hardy's inequality says that $\int_{\omega'} \delta(x)^{-2}|u(x)|^2 dx \le c \int_{\omega'} |\nabla u(x)|^2 dx$, where $c = c(k)$, $\omega' = \omega(P, \frac{r}{2})$, for $u \in H^1(\omega(P, r))$ with $u = 0$ on $\Sigma(P, r)$. (See [**KK**], [**STE**].)

1.2. If u is a (continuous) L-solution on $\omega(0, r)$ with $u = 0$ on $\Sigma(r)$, then $u_{|\Omega'} \in H^1(\Omega')$ and $\|\nabla u\|_{L^2(\Omega')} \le c\, r^{-1}\|u\|_{L^2(\Omega)}$ (e.g. extend u by 0 outside $\omega(0, r)$ and apply Lemme 5.2 in [**S**] to u_+ and u_-).

We shall also need the following observation.

Lemma 2. *If the function u in Lemma 1 is positive, then* $\nabla u(P) \neq 0$ *a.e. on S.*

Proof: We may assume as before that $\Omega = \omega(0, r_0/2)$, $S = \Sigma(0, r_0/4)$ and that u vanishes on $\Sigma \cap \partial\Omega$. By the Harnack boundary principle 2.3, we may also assume that $L = L_0$ (defined as above) and that $u = w$.

Let $\Omega_j = \{x \in \Omega;\ w(x) > \frac{1}{j}\}$ and $S_j = \{(x', x_N) \in \partial\Omega_j \bigcap \Omega;\ |x'| < r_0/2\}$. By (8) above and for j sufficiently large, Ω_j is of the form $\Omega_j = \Omega \cap \{(x', x_N);\ x_N < \varphi_j(x')\}$ where $\varphi_j : \mathbb{R}^{N-1} \to \mathbb{R}$ is C-Lipschitz for some constant $C = C(M, r_0, k)$ and of class $C^{1,\alpha}$ for all $\alpha < 1$.

On S_j, the harmonic measure of $A = A(0, r_0/2)$ w.r. to L_0 and Ω_j is $\mu_j = -\partial_{\nu_L}\big(G_j(., A)\big).d\sigma_{S_j}$; here $\nu_L = \mathcal{A}(\nu)$ on S_j, where ν is the exterior unit normal field along S_j, and G_j is the Green's function w.r. to L_0 in Ω_j. This follows from the Stokes formula $\int_{\partial\Omega_j} \langle W, \nu\rangle\, d\sigma = \int_{\Omega_j} \operatorname{div}(W)\, dx$ which is valid for each vector field W of class $W^{1,p}(\Omega_j)$, $p > N$; with $W = \varphi\,\mathcal{A}(\nabla G_j(., A)) - G_j(., A)\,\mathcal{A}(\nabla\varphi)$, where φ is smooth and of support in $T(0, r_0/4)$ one gets that $\psi(A) = -\int_{S_j} \varphi\langle\nabla G_j(., A), \nu_L\rangle\, d\sigma$ for $\psi = \varphi + G_j(L_0(\varphi))$, i.e. ψ is the solution to $L_0(\psi) = 0$ and $\psi = \varphi$ on $\partial\Omega_j$.

Thus, by the Harnack boundary principle $C^{-1}|\partial_N(w)|.\sigma \le \mu_j \le C|\partial_N(w)|.\sigma$ on S_j; in particular $\|\partial_N w\|_{L^2(S_j)} \le C'$ for j large. Since the L_0-harmonic measure μ of A in Ω is the weak limit of μ_j and since the Lipschitz constants of the graphs S_j are uniformly bounded $C''^{-1}|\partial_N(w)|.\sigma \le \mu \le C''|\partial_N(w)|.\sigma$ on S for some constant $C'' = C(M,k,r)$. Since the density f_A of μ is > 0 a.e. on S it follows that $\partial_N w \neq 0$ a.e. in S. ■

Remark 2. From 2.3 and the uniform bound $\|f_A\|_{L^p(S)} \le c(k,M,r)$ (with $p = p(M,k) > 2$, $S = \Sigma(0,r/2)$), it follows that every positive L-harmonic function u in $\omega(0,r)$ vanishing on $\Sigma(0,r)$ verifies $\|\nabla u\|_{L^p(S)} \le c'(k,M,r)\, u(A(0,r))$. Note also that under the assumptions of Lemma 1, and if $u \in H^1(\Omega)$, a simple limit argument shows that ∇u coincides on S with the weak gradient $\tilde\nabla u \in H^{1/2}_{\rm loc}(S)$ (defined by $\int_S \langle \tilde\nabla u, \mathcal{A}\nu\rangle\, d\sigma = \int_\Omega \{\langle \mathcal{A}\nabla u, \nabla f\rangle + L_0(u)\, f\}\, dx$ for all $f \in H^1(\Omega)$ with $[\operatorname{supp} f] \cap \partial\Omega \subset S$).

4. The local $C^{0,1}$ approximation

Recall that we have fixed $L \in \Lambda(1,M)$ with (3), (4), (4′) and $\alpha = 1$. We set $L_0' = \sum a^0_{ij}(x)\,\partial_i\partial_j$ where $a^0_{ij}(x) = a_{ij}(x',\varphi(x'))$. The operator L_0' is slightly more convenient now than L_0 (as defined in Section 3) because its solutions are at least of class $C^{2,1}$. Fix $r_0 > 0$, let $\Omega = \omega(0,r_0)$, $\Omega' = \omega(0,r_0/2)$ (see notations in 1.1) and let w denote now the solution of $L_0' w = 0$ in Ω such that $w = 1$ on $\partial\Omega \setminus \Sigma$ and $w = 0$ on $\Sigma(0,r_0)$. We observe that a local $C^{0,1}$ approximation of Ω at 0 is provided by the level sets $U(w,\varepsilon) = \{w > \varepsilon\} \cap \Omega'$, $\varepsilon > 0$. Let $D(r) = \{x' \in \mathbb{R}^{N-1};\, |x'| \le r\}$.

Lemma 3. *For $\varepsilon > 0$ small enough, we may write*

$$U(w,\varepsilon) = \{(x',x_N);\, |x'| < r_0/2,\, -5\,k \times r_0 < x_N < \varphi_\varepsilon(x')\}$$

where $\varphi_\varepsilon : D(r_0/2) \to \mathbb{R}$ is of class $C^{2,1}$ and C-Lipschitz for some $C > 0$ independent of ε; also $-k \times r_0 < \varphi_\varepsilon(x') < \varphi(x)$. Moreover, when ε decreases to zero, φ_ε increases to φ uniformly on $D(r_0/2)$, and $\lim_{\varepsilon\to 0} D\varphi_\varepsilon(x') = D\varphi(x')$ for almost all $x' \in D(r_0/2)$.

Proof: The first claim follows by the arguments used in the proof of Lemma 2. As before

$$0 < -\partial_N w(x) \le |\nabla w(x)| \le c\frac{w(x)}{\delta(x)} \le -C\,\partial_N w(x) \tag{10}$$

when $x \in \Omega'$. Hence if $\varepsilon > 0$ is so small that $w(x) > \varepsilon$ for $|x'| \le r_0/2$ and $x_N \le -kr_0$, the implicit function theorem shows that the region $U(w,\varepsilon) \cap \Omega'$ is as required by the first claim above. Recall that by Schauder's theory w is locally of class $C^{2,1}$ in Ω. That $\varphi_\varepsilon \to \varphi$ uniformly on $D(r_0/2)$ is then obvious, since w is continuous on $\Omega \cup \Sigma(0, r_0)$.

The last part of the proposition follows now from Lemma 1, Lemma 2 (applied to w) and Lemma 4 below. If φ is differentiable at $a' \in D(r_0/2)$ and if $\nabla w(x)$ admits a non tangential limit α at $(a', \varphi(a')) = a$ with $\alpha = (\alpha', \alpha_N) \ne 0$ (that is $\alpha_N \ne 0$ by (10)), then

$$\partial_j \varphi(a') = \lim_{\varepsilon\to 0} \partial_j \varphi_\varepsilon(a') = -\alpha_j/\alpha_N, \text{ for } j = 1, \dots, N-1.$$

To see this, fix $\eta > 0$ and j, $1 \le j \le N-1$, and apply Lemma 4 below to the function $f(t) = \varphi_\varepsilon(a' + t\, e_j) - \varphi(a')$ with $\beta = \partial_j\varphi(a')$, $u < 0 < v$ being the closest to zero with $\varphi_\varepsilon(a' + ue_j) = \varphi(a) + (\beta+\eta)u$, $\varphi_\varepsilon(a' + ve_j) = \varphi(a) + (\beta-\eta)v$. It follows that for each small enough $\varepsilon > 0$, there is a point $x'(\varepsilon) \in D(r_0/2)$ with the following properties:

(a) the i-th coordinate $x'_i(\varepsilon)$ satisfies $x'_i(\varepsilon) = a'_i$ if $i \ne j$, $1 \le i \le N-1$,
(b) $\varphi_\varepsilon(x'(\varepsilon)) \le \varphi(a) + (\partial_j\varphi(a') \pm \eta)\,(x'_j(\varepsilon) - a'_j)$,
(c) $|\partial_j\varphi(a') - \partial_j\varphi_\varepsilon(x'(\varepsilon))| \le \eta$.

Now from (b) it follows that when $\varepsilon \to 0$ the point $x_\varepsilon = \big(x'(\varepsilon), \varphi_\varepsilon(x'(\varepsilon))\big)$ converges non tangentially to a in Ω as well as $a_\varepsilon = (a', \varphi_\varepsilon(a'))$. Hence, since ∇w has a non tangential limit $\alpha = (\alpha', \alpha_N)$ at a such that $\alpha_N \ne 0$,

$$\begin{aligned}\partial_j\varphi_\varepsilon(x'(\varepsilon)) &= -\partial_j w(x_\varepsilon)/\partial_N w(x_\varepsilon)\\ &= -\partial_j w(a_\varepsilon)/\partial_N w(a_\varepsilon) + o(1)\\ &= \partial_j\varphi_\varepsilon(a') + o(1),\end{aligned}$$

and $\limsup_{\varepsilon\to 0} |\partial_j\varphi_\varepsilon(a') - \partial_j\varphi(a)| \le \eta$ by (c).

Thus, $\lim_{\varepsilon\to 0} \partial_j\varphi_\varepsilon(a') = \partial_j\varphi(a) = \text{n.t.}\lim_{(a,\varphi(a))}\{-\partial_j w/\partial_N w\}$ (where n.t. means nontangential). ■

Lemma 4. *Let $f : I \to \mathbb{R}$ be a function of class C^1 on some interval $I = [u, v]$, $u < 0 < v$. Let $\beta \in \mathbb{R}$, $\eta > 0$, $\psi(t) = \inf\{(\beta+\eta)\,t, (\beta-\eta)\,t\}$ and assume that $f(t) \le \psi(t)$ on I, and $f(u) = \psi(u)$, $f(v) = \psi(v)$. Then, there exists $t \in I$ such that $|f'(t) - \beta| \le \eta$.*

Proof: Since $\frac{f(v)-f(u)}{v-u} = \beta - \eta\frac{v+u}{v-u}$ and $|\frac{v+u}{v-u}| \le 1$, the lemma follows at once from the mean value theorem. ■

5. The Rellich formula for u in the domain of L

The previous constructions are now used to obtain the following strong L^2 approximation property. It is well-known that the later implies the desired extension of the Rellich formula. Notations and assumptions are as in the previous section. It is also assumed for sake of simplicity that $\gamma \leq 0$. Recall that $\Omega' = \omega_\varphi(0, \frac{r_0}{2}) = \omega(0, \frac{r_0}{2})$ and that $D(r) = \{x' \in \mathbb{R}^{N-1}; |x'| \leq r\}$.

Theorem 1. *Let $\Omega_j = \{x \in \Omega'; w(x) > \varepsilon_j\}$, where $\varepsilon_j \to 0$, $\varepsilon_j > 0$ and let $\varphi_j = \varphi_{\varepsilon_j}$. Let u_j, $j \geq 1$, be L-harmonic on Ω_j vanishing on $\Sigma_j = \{w = \varepsilon_j\}$, $j \geq 1$. If u_j converges uniformly on Ω' to u (set $u_j = 0$ on $\Omega' \setminus \Omega_j$), the functions $f_j(x') = \partial_{\nu_L} u_j(x', \varphi_j(x'))$ converge strongly in $L^2(D(r_0/4))$ to $f(x') = \partial_{\nu_L} u(x', \varphi(x'))$ (and in fact in $L^p(D(r_0/4))$ for some $p = p(k, M) > 2$.)*

Here, $\partial_{\nu_L} u = \langle \mathcal{A}\nabla u, \nu\rangle$ denotes the conormal derivative of u along Σ (and ν is the unit exterior normal), $\partial_{\nu_L} u_j$ denotes the conormal derivative of u_j along $\Sigma_j = \{(x', \varphi_j(x'); x' \in D(0, r_0/2)\}$. Note that if the u_j are ≥ 0, then simple convergence in Ω already implies uniform convergence on $\omega(0, r')$, $r' < r_0/2$, by boundary Harnack property. Also an obvious decomposition of u_j shows that to prove Theorem 1 we may restrict to the case where $u_j \geq 0$.

Proof of Theorem 1: Consider first the special case of the sequence $v_j = w - \varepsilon_j$ with $L = L_0'$ and denote f_j^0, f^0, the corresponding functions f_j and f. Then by Lemma 1 $f_j^0(x') \to f^0(x') = \langle \nabla w(x), \nu_L(x)\rangle$ almost everywhere in $D' = D(r_0/4)$ (where $x = (x', \varphi(x'))$). Since there is a uniform bound on $\|f_j^0\|_{L^p(D')}$ for some $p > 2$, it follows that f_j^0 converge (strongly) to f^0 in $L^2(D')$. And the proposition follows for the case at hand. It is then clear that $g_j(x') = \partial_{\nu_L} v_j(x', \varphi_j(x'))$ tends to $f^0(x')$ a.e. in D' and in $L^2(D')$ (note that $f_j^0(x') = \partial_{\nu_{L_0'}} v_j(x', \varphi_j(x'))$).

In the general case (with $u_j \geq 0$, $u > 0$ in Ω'), consider $h_j = f_j/f_j^0$. By Lemma 5 below, this is a sequence of Hölder continuous functions on $\overline{D'}$ which is bounded in $C^\alpha(\overline{D'})$ for some α, $0 < \alpha < 1$. Moreover the function $h = f/f^0$ —which may be seen as a Hölder continuous function on $\overline{D'}$— is the unique cluster value of this sequence in $L^\infty(D')$. In fact, by Lemma 5, if H is such a cluster value and if $\eta > 0$ is small, both quantities

$$|1 - [H(x') : (u(A)/w(A))]| \text{ and } |1 - [\big(f(x')/f^0(x')\big) : (u(A)/w(A))\big]|,$$

where $x' \in D'$ and $A = (x', \varphi(x') - \eta)$, are bounded by $\leq c\,\eta^\delta$ for some positive real δ.

Thus, $h_j \to H$ in $L^\infty(D')$. Since $f_j^0 \to f^0$ almost everywhere on D' and in $L^2(D')$, it follows that $f_j \to f$ in $L^2(D')$ and almost everywhere on D' which proves the theorem. ■

Recall that for $a = (a', a_N) \in \Sigma$, $T(a,\eta) = \{(x', x_N);\ |x' - a'| < \eta,\ |x_N - a_N| < 10\,k\,\eta\}$. Let $S = \Sigma \cap \{(x', \varphi(x'));\ |x'| < r_0/4\}$ and $U_\varepsilon = \omega(0, r_0/2) \bigcap \{w > \varepsilon\}$. Set $w_\varepsilon = w - \varepsilon$ for each $\varepsilon > 0$.

Lemma 5. *There are constants $C > 1$ and $D \in (0,1)$ with the following property. If $P \in S$, if h is positive L-harmonic in $T(P,\eta) \cap U_\varepsilon$ vanishing on $T_P(\eta) \cap \partial U_\varepsilon$, $\varepsilon > 0$, and if $\eta > 0$ is small, then*

$$(11)\qquad (1 - C\,\eta^D)\frac{w_\varepsilon(x)}{w_\varepsilon(A'_\eta)} \le \frac{h(x)}{h(A'_\eta)} \le (1 + C\,\eta^D)\,\frac{w_\varepsilon(x)}{w_\varepsilon(A'_\eta)}$$

for $x \in T(P, \frac{\eta^2}{4C}) \cap U_\varepsilon$ and $A'_\eta = (P', P_N - 5\,k\,\eta^2)$.

Note that if $A_\eta \notin U_\varepsilon$, then $T(P, \frac{\eta^2}{4C}) \bigcap U_\varepsilon = \emptyset$ at least if η is small.

Proof: We use a construction from [**A2**]. Let $s = f(w)$ where $f(t) = \int_0^t e^{-\theta^\alpha}\,d\theta$, $u = g(w)$ where $g(t) = \int_0^t e^{\theta^\alpha}\,d\theta$ and $0 < \alpha < 1$. It is immediately checked, using (9), that if α is small, then s (resp. u) is L-superharmonic (resp. L-subharmonic) in $\Omega' \bigcap \{w < \varepsilon\}$ for $\varepsilon > 0$ small. In fact,

$$L(s) = f''(w)\left\{\sum a_{ij}\,\partial_i w\,\partial_j w\right\} + f'(w)\,L(w) + \gamma(f(w) - w f'(w))$$

so that using (9) and Schauder interior estimates, we have on Ω' near Σ,

$$\begin{aligned}\left\{\sum a_{ij}\,\partial_i w\,\partial_j w\right\}^{-1} L(s) &\le f''(w) + C|\nabla w|^{-2} f'(w)\{L(w) - L'_0(w)\}\\ &\quad + C|\nabla w|^{-2}\,w\\ &\le f''(w) + C'\frac{\delta^2}{w^2}\,f'(w)\,\left(\delta\frac{w}{\delta^2} + \frac{w}{\delta} + w\right) + C'\frac{\delta^2}{w}\\ &\le f''(w) + C'' w^{\beta-1}\,(f'(w) + 1)\end{aligned}$$

for some positive constants C'' and β, and where in the last line we have used 2.5. It follows that if we fix α in $(0,\beta)$, then s is L-superharmonic near Σ in Ω'. The subharmonicity of $u = g(w)$ is obtained similarly.

Let $s_\varepsilon = s - f(\varepsilon) = f(w) - f(\varepsilon)$ and $u_\varepsilon = u - g(\varepsilon) = g(w) - g(\varepsilon)$ for $\varepsilon > 0$. If $m = \sup\{w(x);\ x \in T(P,\eta) \cap U_\varepsilon\}$, $P \in S$,

$$e^{-m^\alpha} \le s_\varepsilon(x)/w_\varepsilon(x) \le e^{-\varepsilon^\alpha}$$

when $x \in \omega = U_\varepsilon \cap T(P,\eta)$. Similarly, we have $e^{\varepsilon^\alpha} \le u_\varepsilon(x)/w_\varepsilon(x) \le e^{m^\alpha}$ for $x \in \omega$. Observe also that

$$e^{-(\varepsilon^\alpha - m^\alpha)} \le e^{(m-\varepsilon)^\alpha} \le \exp(c'\,\eta^{b\alpha}) = e^{c'\,\eta^{\beta'}}$$

for some constants $b > 0$ and $c' > 0$, where we have applied again 2.5.

Now, let $\tilde{s}_\varepsilon = \frac{m-\varepsilon}{(f(m)-f(\varepsilon))} s_\varepsilon$ and $\tilde{u}_\varepsilon = \frac{m-\varepsilon}{(g(m)-g(\varepsilon))} u_\varepsilon$. For $\eta > 0$ and small, the function $\tilde{s}_\varepsilon$ (resp. $\tilde{u}_\varepsilon$) is positive L-superharmonic (resp. L-subharmonic) on $\omega = T(P,\eta) \cap U(\varepsilon)$, vanishes on $\Sigma_\varepsilon = \partial U_\varepsilon \cap T(P,\eta)$ and $\tilde{u}_\varepsilon \le \tilde{s}_\varepsilon$ in ω. Taking the smallest L-harmonic majorant of $\tilde{u}_\varepsilon$ in ω, we obtain a positive L-harmonic function h_1 on ω such that $\tilde{u}_\varepsilon \le h_1 \le \tilde{s}_\varepsilon$ on ω. Of course, h_1 vanishes on Σ_ε and by the previous estimates, we have in ω

$$(12) \qquad (1 - c\eta^{\beta'}) w_\varepsilon(x) \le h_1(x) \le (1 + c\eta^{\beta'})\, w_\varepsilon(x).$$

Finally, if h is any positive L-harmonic function on ω vanishing on Σ_ε, we know (see Section 2.4) that for some real $\beta'' \in (0,1]$

$$(13) \qquad (1 - c\,\eta^{\beta''}) \frac{h_1(x)}{h_1(A'_\eta)} \le \frac{h(x)}{h(A'_\eta)} \le (1 + c\,\eta^{\beta''}) \frac{h_1(x)}{h_1(A'_\eta)}$$

when $x \in U_\varepsilon \cap T(P, \frac{\eta^2}{4C})$. Combining (12) and (13) we obtain (11). ■

Let now $L \in \Lambda(1,M)$ be in the form $L = \sum \partial_i(a_{ij}\partial_j.)$ the a_{ij} satisfying (4) and (4′) with $\alpha = 1$. From Theorem 1, the desired generalization of Rellich formula (1) together with an extension of Theorem 1 itself are easily derived. In the next corollary notations are the same as in Theorem 1.

Corollary 1. *Let $v \in H^1_0(\Omega')$ be such $L(v) = f \in L^2(\Omega')$, and let $v_j \in H^1_0(\Omega'_j)$ be such that $L(v_j) = f$ in Ω'_j. Then, $\partial_{\nu_L} v_j(x', \varphi_j(x')) \to \partial_{\nu_L} v(x', \varphi(x'))$ in $L^2(D')$.*

Recall (ref. [**N**]) that if W be a bounded Lipschitz region in $\mathbb{R}^N$, then for $u \in H^1_0(W)$ such that $L(u) = f \in L^2(W)$, the weak conormal derivative $\partial_{\nu_L}(u)$ (defined as a member of $H^{-\frac{1}{2}}(\partial W)$) belongs to $L^2(\partial\Omega)$ and $\|\partial_{\nu_L}(u)\|_{L^2(\partial W)} \le C(W,M)\|f\|_{L^2(\partial W)}$. This follows from a natural approximation argument combined with Rellich formula for functions in H^2. By Theorem 1, if $f = 0$ in an open neighborhood V of $P \in \partial W$, then the weak and the strong conormal derivatives of v coincide in $V \cap \partial W$.

Proof of Corollary 1: By decomposing f_j into its positive and negative parts we may assume that $f_j \leq 0$ in Ω'_j, $j \geq 1$. By Rellich formula there is a uniform estimate $\|\partial_{\nu_L}(w)\|_{L^2(\Omega'_j)} \leq C\|f\|_{L^2(\partial\Omega'_j)}$ for $w \in H^1_0(\Omega'_j)$ with $L(w) = f \in L^2(\Omega'_j)$ and a constant C independent of j. Thus by a standard approximation argument we may also assume that $f_j = 0$ on a neighborhood V of Σ. Then, $v_j \to v$ simply on $V \cap \Omega'$ and the result follows from Theorem 1. ∎

Remarks.

5.1. It follows from Lemma 1 and Theorem 1 (and the obvious approximation argument) that in a given bounded region Ω the L-harmonic measure of $x_0 \in \Omega$ induces on a Lipschitz open piece S of $\partial\Omega$ the measure of density $-\partial_{\nu_L}(G(x_0,.)$, if G denotes the Green's function of L in Ω.

5.2. Corollary 1 is easily extended to the following case: $v \in H^1(\Omega')$ with $L(v) = f \in L^2(\Omega')$ in Ω' and $v = F$ on $\partial\Omega$ for some $F \in H^2(\Omega')$; the $v_j \in H^1(\Omega'_j)$ are such that $L(v_j) = f$ in Ω'_j and $v_j = F$ in $\partial\Omega'_j$. One has just to look to $v'_j = v_j - F$ and to notice that $\nabla F(x', \varphi_j(x')) \to \nabla F(x', \varphi(x'))$ a.s. in $D(r_0/4)$ and also in $L^2(D')$ (in fact in $H^{1/2}(D')$) since $\nabla F \in H^1(\Omega)$.

From Corollary 1 and Remark 5.2 above the extension of Rellich formula follows.

Corollary 2. *Let L be as before, let V be a C^1_0 vector field in $\mathbb{R}^N$ and let Ω be a domain in $\mathbb{R}^N$ which is Lipschitz in a neighborhood of each point P of $F = \operatorname{supp}(V) \bigcap \partial\Omega$. If $u \in H^1(\Omega)$ is such that $L(u) \in L^2(\Omega)$ and if $u = g$ in a neighborhood of F in $\partial\Omega$ for some $g \in H^2(\Omega)$ then the Rellich formula (1) holds.*

To state the next corollary, we assume that we are given a sequence of functions ψ_j in $D(r_0)$ such that $\psi_j \leq \varphi$, $|\psi_j(x) - \psi_j(y)| \leq k|x-y|$ for x, y in $D(r_0)$, $\lim_{j\to\infty} \|\psi_j - \varphi\|_\infty = 0$ and $\lim_{j\to\infty} D\psi_j(x') = D\varphi(x')$ for almost all $x' \in D(r_0)$. We let $\Omega' = \omega_\varphi(0, r_0/2)$ (as before) and $\Omega_j = \Omega' \cap \{(x', x_N);\, x_N < \psi_j(x')\}$. Set $\Sigma_j = \{(x', \psi_j(x'));\, x' \in D(r_0)\}$, $\Sigma = \{(x', \varphi(x'));\, x' \in D'(r_0)\}$, and ν_j (resp. ν) to denote the exterior unit normal field on Σ_j (resp. on Σ).

Corollary 3. *Let L be as in Corollary 1 and let $\{u_j\}$ be a sequence of functions such that $u_j \in H^1(\Omega_j)$, $u_j = 0$ on Σ_j and $L(u_j) = f_j \in L^2(\Omega_j)$. Assume that $f_j \to f$ in $L^2(\Omega')$ (set $f_j = 0$ in Ω_j^c) and that*

$u_j \to u$ in $H^1_{\mathrm{loc}}(\Omega')$. Then, $u \in H^1(\omega_\varphi(0, \frac{3r}{8}))$, $u = 0$ in $\Sigma(r_0/4)$, and $\partial_{\nu_j} u_j(x', \psi_j(x')) \to \partial_\nu u(x', \varphi(x'))$ in $L^2(D(r_0/4))$.

This follows from Rellich formula (Corollary 2) and the fact that in a Hilbert space $(H, \|.\|)$ every weakly convergent sequence θ_j such that $\lim_{j\to\infty} \|\theta_j\| = \| \lim_{j\to\infty} \theta_j\|$ is strongly convergent. Corollary 3 means that for a bounded Lipschitz domain Ω with a given $C^{0,1}$ approximation (ref. [**N**]) by a sequence of Lipschitz domains Ω_j the following holds: if $v_j \in H^1_0(\Omega_j)$, $v \in H^1_0(\Omega)$ are such $L(v_j) = f_j \in L^2(\Omega_j)$ converges strongly (in the appropriate sense) to $L(v) = f \in L^2(\Omega)$, then $\nabla v_{j\,|\partial\Omega_j} \to \nabla v_{|\partial\Omega}$ in the appropriate strong L^2 sense.

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