MEAN GROWTH OF $H^p$ FUNCTIONS

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Abstract

A classical result of Hardy and Littlewood asserts that if $0 < p < q < 1$ and $f$ is a function which is analytic in the unit disc and belongs to the Hardy space $H^p$, then, if $\lambda \geq p$ and $\alpha = \frac{1}{p} - \frac{1}{q}$, we have

$$\int_0^1 (1 - r)^{\lambda \alpha - 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q \, d\theta \right)^{\lambda / q} \, dr < \infty.$$ 

We prove that this result is sharp in a very strong sense. Indeed, we prove that if $p$, $q$, $\lambda$ and $\alpha$ are as above and $\varphi$ is a positive, continuous and increasing function defined in $[0, 1)$ with $\frac{\varphi(x)}{x^p} \to \infty$, as $x \to \infty$, then there exists a function $f \in H^p$ such that

$$\int_0^1 (1 - r)^{\lambda \alpha - 1} \left( \int_I \varphi \left( |f(re^{i\theta})| \right) \, d\theta \right)^{\lambda / q} \, dr = \infty,$$

for every non-degenerate interval $I \subset [0, 2\pi]$. We also prove a result of the same kind concerning functions $f$ such that $f' \in H^p$, $0 < p < 1$.

1. Introduction and statement of results

Let $\Delta$ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ the unit circle $\{\xi \in \mathbb{C} : |\xi| = 1\}$. For $0 < r < 1$ and $g$ analytic in $\Delta$ we set

$$M_p(r, g) = \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, g) = \max_{|z| = r} |g(z)|.$$


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For $0 < p \leq \infty$ the Hardy space $H^p$ consists of those functions $g$, analytic in $\Delta$, for which

$$\|g\|_{H^p} = \sup_{0 < r < 1} M_p(r, g) < \infty.$$ 

Hardy and Littlewood proved in [6] (see also [2, Th. 5.9]) the following.

**Theorem A.** If $0 < p < q \leq \infty$ and $f \in H^p$, then

$$(1.1) \quad M_q(r, f) = o \left( \frac{1}{(1-r)^{\frac{1}{p} - \frac{1}{q}}} \right), \quad \text{as } r \to 1.$$ 

Considering the function $f(z) = \frac{1}{(1-z)^{\frac{1}{p} - \frac{1}{q}}} + \varepsilon$ for small $\varepsilon > 0$, we easily see that the exponent $\frac{1}{p} - \frac{1}{q}$ is best possible. Duren and Taylor proved in [3] (see also [8]) that the Hardy-Littlewood estimate (1.1) is sharp in a stronger sense. Namely, they proved the following result.

**Theorem B.** Let $0 < p < q \leq \infty$, and let $\phi(r)$ be a positive and non-increasing function on $0 \leq r < 1$, with $\phi(r) \to 0$, as $r \to 1$. Then there exists a function $f \in H^p$ such that

$$M_q(r, f) \neq O \left( \frac{\phi(r)}{(1-r)^{\frac{1}{p} - \frac{1}{q}}} \right), \quad \text{as } r \to 1.$$ 

Although, as we have said, Theorem A is best possible in a strong sense, Hardy and Littlewood were able to sharpen it in one direction proving the following useful result (see [2, Th. 5.11]).

**Theorem C.** If $0 < p < q \leq \infty$, $f \in H^p$, $\lambda \geq p$, and $\alpha = \frac{1}{p} - \frac{1}{q}$, then

$$(1.2) \quad \int_0^1 (1-r)^{\lambda \alpha - 1} M_q(r, f)^\lambda \, dr < \infty.$$ 

The fact that (1.2) implies (1.1) is clear having in mind that $M_q(r, f)$ is an increasing function of $r$. Let us remark that Flett gave in [4] a proof of Theorem C based on the Marcinkiewicz interpolation theorem. Also, it is worth noticing that if we take $q < \infty$ and $\lambda = q$ then we obtain the following:

If $0 < p < q < \infty$ and $f \in H^p$, then

$$\int_0^{2\pi} \int_0^1 (1-r)^{\frac{2}{q} - 2} |f(re^{i\theta})|^q \, dr \, d\theta < \infty.$$ 

Our first result in this paper shows that Theorem C is sharp in a very strong sense.
**Theorem 1.** Let $0 < p < q < \infty$, $\lambda \geq p$, and $\alpha = \frac{1}{p} - \frac{1}{q}$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function with

$$\frac{\varphi(x)}{x^q} \to \infty, \quad \text{as } x \to \infty. \quad (1.3)$$

Then, there exists a function $f \in H^p$ such that

$$\int_0^1 \left(1 - r\right)^{\lambda\alpha - 1} \left(\int_I \varphi\left(|f(re^{i\theta})|\right) d\theta\right)^{\lambda/q} dr = \infty, \quad (1.4)$$

for every non-degenerate interval $I \subset [0, 2\pi]$.

In particular, if $0 < p < q < \infty$ and $\varphi : [0, \infty) \to [0, \infty)$ is as above, then there exists a function $f \in H^p$ such that

$$\int_0^1 \int_0^1 (1 - r)^{\lambda\alpha - 2} \varphi\left(|f(re^{i\theta})|\right) dr d\theta = \infty,$$

for every non-degenerate interval $I \subset [0, 2\pi]$.

According to a classical result of Privalov [2, Th. 3.11], a function $f$ analytic in $\Delta$ has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f' \in H^1$. In particular,

$$f' \in H^1 \Rightarrow f \in H^\infty. \quad (1.5)$$

This result has been shown to be sharp. Indeed, Yamashita proved in [9] that there exists a function $f$ analytic in $\Delta$ with $f' \in H^p$ for all $p \in (0, 1)$ but such that $f$ is not even a normal function, and the first author has recently proved in [5] that no restriction on the growth of $M_1(r, f')$ other than its boundedness is enough to conclude that $f$ is a normal function.

We refer to [1] and [7] for the theory of normal functions. On the other hand, Hardy and Littlewood obtained the following generalization of (1.5) (see [2, Th. 5.12]).

**Theorem D.** Let $f$ be a function which is analytic in $\Delta$. If $0 < p < 1$ and $f' \in H^p$ then $f \in H^q$, where $q = p/(1 - p)$.

Taking $f'(z) = (1 - z)^{e^{-\frac{1}{2}}}$ for small $\varepsilon > 0$ shows that for each value of $p \in (0, 1)$ the index $q$ is best possible. Our next result proves the sharpness of Theorem D in a much stronger sense.
Theorem 2. Let $0 < p < 1$ and $q = p/(1-p)$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function satisfying (1.3). Then, there exists a function $f$ analytic in $\Delta$ with $f' \in H^p$ such that

$$\int_I \varphi \left( |f(e^{i\theta})| \right) d\theta = \infty,$$

for every non-degenerate interval $I \subset [0, 2\pi]$.

Let us remark that if $p$ and $q$ are as in Theorem 2 and $f_0 \in H^p$, then, by Theorem D, $f \in H^q$ and, hence, $f$ has a finite non-tangential limit $f(e^{i\theta})$ for almost every $\theta$. Hence, the left hand side of (1.6) makes sense.

2. Proof of the results

The proofs of our results will be constructive. Let $\alpha$ and $\beta$ be two positive real numbers, and let $\{\delta_k\}_{k=1}^\infty$ be a sequence of real numbers with

$$0 < \delta_k < 2^{-k}, \quad \text{for all } k.$$

For $k = 1, 2, \ldots$, and $j = 1, 2, \ldots, 2^k$, define

$$\theta_j^k = \frac{2\pi(2j-1)}{2^k+1},$$

$$I_j^k = (\theta_j^k - \delta_k, \theta_j^k + \delta_k).$$

Notice that, for each $k$, the intervals $I_j^k, j = 1, 2, \ldots, 2^k$, are pairwise disjoint. Set

$$r_k = 1 - \delta_k, \quad k = 1, 2, \ldots.$$

For $k = 1, 2, \ldots$, define

$$f_k(z) = \sum_{j=1}^{2^k} \frac{\delta_j^\alpha}{(1 - r_k e^{-i\theta_j^k} z)^\beta}, \quad z \in \Delta.$$

Let us remark that the functions $f_k$ are in fact analytic in the closed unit disc $\overline{\Delta}$. Actually, the functions $f_k$ depend on $\alpha$, $\beta$ and the sequence $\{\delta_k\}$, however, we shall not indicate this dependence explicitly. We believe that this will not cause any confusion.
We shall make use of some lemmas to deal with the functions $f_k$. The proofs are elementary and some of them will be omitted. First of all, let us recall that

\begin{equation}
|1 - re^{i\theta}| \leq 2|\theta|, \quad 0 \leq r \leq 1, \quad 1 - r \leq |\theta| \leq \pi,
\end{equation}

\begin{equation}
|1 - re^{i\theta}| \geq \frac{|\theta|}{\pi}, \quad 0 \leq r \leq 1, \quad |\theta| \leq \pi,
\end{equation}

\begin{equation}
|1 - e^{i\theta}| \geq 2\frac{|\theta|}{\pi}, \quad |\theta| \leq \pi.
\end{equation}

Lemma 1. If $l \neq m$, then

$$|\theta^k_l - \theta^k_m| \geq \frac{\pi}{2k-1}.$$

Lemma 2. If $\theta \in I^k_j$, then

\begin{equation}
|e^{i\theta^k_j} - r_k e^{i\theta}| \leq 2\delta_k
\end{equation}

and

\begin{equation}
|e^{i\theta^k_j} - r_k e^{i\theta}| \geq \frac{1}{2k-1}, \quad \text{for all } l \neq j.
\end{equation}

Proof: Let $\theta \in I^k_j$, then $|\theta - \theta^k_j| < \delta_k$, which, with (2.6), implies

$$|e^{i\theta^k_j} - r_k e^{i\theta}| = |1 - r_k e^{i(\theta - \theta^k_j)}| \leq |1 - r_k e^{i\delta_k}| \leq 2\delta_k.$$

This is (2.9). Now, let $l \neq j$ and let $\varphi^k_l$ be an angle such that $e^{i\varphi^k_l} = e^{i\theta^k_l}$ and $|\theta - \varphi^k_l| \leq \pi$. Then, using (2.4), (2.8), Lemma 1 and (2.1), we obtain

\begin{align*}
|e^{i\theta^k_l} - r_k e^{i\theta}| &= |e^{i\theta^k_l} - r_k e^{i\theta}| \geq |e^{i\varphi^k_l} - e^{i\theta}| - |e^{i\theta} - r_k e^{i\theta}| \\
&= |e^{i\varphi^k_l} - e^{i\theta}| - \delta_k \geq 2\frac{|\varphi^k_l - \theta|}{\pi} - \delta_k \\
&\geq \frac{2}{\pi}(|\varphi^k_l - \theta^k_j| - |\theta^k_j - \theta|) - \delta_k \geq \frac{2}{\pi} \left(\frac{\pi}{2k-1} - \delta_k\right) - \delta_k \\
&\geq \frac{1}{2k-2} - 2\delta_k \geq \frac{1}{2k-1}.
\end{align*}

Hence, (2.10) holds. ■
Lemma 3. If $n < k$, then

$$|\theta^n_l - \theta^k_j| \geq \frac{\pi}{2^k}, \quad \text{for all } l, j.$$

Lemma 4. If $\theta \in I^n_j$, $n < k$ and $0 < r \leq 1$, then

$$|e^{i\theta^n_l} - r_n e^{i\theta}| \geq \frac{1}{2^{k+1}}, \quad \text{for all } l \in \{1, 2, \ldots, 2^n\}.$$

Proof: Let $\theta \in I^n_j$ and let $\varphi^n_l$ be defined as in the proof of Lemma 2. Then, using (2.7), Lemma 3, (2.3) and (2.1), we see that

$$|e^{i\varphi^n_l} - r_n e^{i\theta}| = |e^{i\varphi^n_l} - r_n e^{i\theta}| \geq \frac{|\varphi^n_l - \theta|}{\pi}$$

$$\geq \frac{1}{\pi}(|\varphi^n_l - \theta^n_j| - |\theta^n_j - \theta|)$$

$$\geq \frac{1}{\pi} \left( \frac{\pi}{2^k} - \delta_k \right) > \frac{1}{2^{k+1}}.$$

We shall see that a suitable choice of the numbers $\alpha$, $\beta$ and the sequence $\{\delta_k\}$ will allow us to construct functions $f$ analytic in $\Delta$ having the properties asserted in Theorems 1 and 2. Precisely, we can prove the following results.

**Theorem 3.** Let $0 < p < q < \infty$, $\lambda \geq p$, and $\alpha = \frac{1}{p} - \frac{1}{q}$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function satisfying (1.3). Then, there exist two positive numbers $\alpha$ and $\beta$, a sequence of real numbers $\{\delta_k\}_{k=1}^{\infty}$ which satisfies (2.1), and a sequence of positive numbers $\{c_k\}_{k=1}^{\infty}$, such that, if $f$ is the function defined by

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z), \quad z \in \Delta,$$

(2.11)

then $f \in H^p$ and (1.4) holds for every non-degenerate interval $I \subset [0, 2\pi]$. 
Theorem 4. Let $0 < p < 1$ and $q = p/(1-p)$. Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function satisfying (1.3). Then, there exist two positive numbers $\alpha$ and $\beta$, a sequence of real numbers $\{\delta_k\}_{k=1}^\infty$ which satisfies (2.1), and a sequence of positive numbers $\{c_k\}_{k=1}^\infty$, such that, if $f$ is the function defined by

$$f(z) = \sum_{k=1}^{\infty} c_k f_k(z), \quad z \in \Delta,$$

then $f$ is analytic in $\Delta$, $f' \in H^p$ and (1.6) holds for every non-degenerate interval $I \subset [0, 2\pi]$.

Clearly, Theorem 1 and Theorem 2 follow from Theorem 3 and Theorem 4 respectively.

Proof of Theorem 3: Let $\alpha$ be any positive number, and let

$$\beta = \alpha + \frac{1}{p}.$$ (2.13)

Let $\{\delta_k\}_{k=1}^\infty$ be a sequence of real numbers which satisfies (2.1) to be specified later. Set

$$c_k = 2^{-k(\frac{1}{p}+2)}, \quad k = 1, 2, \ldots.$$ (2.12)

Define the functions $f_k, k = 1, 2, \ldots$, as in (2.5), let $g_k = c_k f_k$ for all $k$ and let $f$ be defined as in (2.11). Hence,

$$f(z) = \sum_{k=1}^{\infty} g_k(z), \quad z \in \Delta.$$ (2.12)

Notice that

$$|g_k(z)| = |2^{-k(\frac{1}{p}+2)} f_k(z)| \leq \frac{2^{-k} \delta_k^p}{(1 - |z|)^{\beta}} \leq \frac{2^{-k}}{(1 - |z|)^{\beta}}$$

for all $z \in \Delta$ and, hence, the series $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on every compact subset of $\Delta$ and then it defines a function $f$ which is analytic in $\Delta$. Now, having in mind the elementary inequality

$$(a_1 + a_2 + \cdots + a_n)^p \leq n^p (a_1^p + a_2^p + \cdots + a_n^p),$$

$p > 0, a_i \geq 0$ for $i = 1, 2, \ldots, n$. 


and the fact that for each $\gamma > 1$ there exists a constant $c = c_\gamma > 0$ such that

$$
(2.14) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|^\gamma} \, d\theta \leq \frac{c}{(1 - r)^\gamma - 1}, \quad 0 < r < 1,
$$

and using (2.4) and (2.13), we obtain that

$$
\|g_k\|_{H^p} = \|g_k(e^{i\theta})\|_{L^p} = \frac{1}{2\pi} \int_0^{2\pi} |g_k(e^{i\theta})|^p \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} c_k^p 2^{kp} \delta_k^{\alpha_p} 2^k \frac{1}{|1 - r_k e^{-i\theta} e^{i\theta}|^{\beta_p}} \, d\theta
$$

$$
= (2^k c_k)^p \delta_k^{\alpha_p} 2^k \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - r_k e^{i\theta}|^{\beta_p}} \, d\theta
$$

$$
\leq 2^{-kp} \delta_k^{\alpha_p} \frac{c}{(1 - r_k)^{\beta_p - 1}} = 2^{-kp} c,
$$

where $c$ is the positive constant which appears in (2.14) with $\gamma = \beta_p > 1$.

Thus, we have proved that

$$
(2.15) \quad \|g_k\|_{H^p} \leq 2^{-k} c^{1/p}, \quad k = 1, 2, \ldots,
$$

which, clearly, implies that $f \in H^p$.

Next we turn to estimate the value of $|f(re^{i\theta})|$ when $\theta$ belongs to one of the intervals $I^k_j$ given in (2.3), and $0 < r < 1$, or at least when $\theta$ is in a suitable subset of $I^k_j$ and $r$ is close to 1, say $r_k < r < 1$.

Suppose that $\theta \in I^k_j$ and $0 < r < 1$. Then

$$
|f(re^{i\theta})| = \left| \sum_{n=1}^{\infty} g_n(re^{i\theta}) \right| \geq |g_k(re^{i\theta})| - \sum_{n=1}^{k-1} |g_n(re^{i\theta})| - \sum_{n=k+1}^{\infty} |g_n(re^{i\theta})|.
$$

(2.16)

We shall estimate each of these three terms separately.
First, for $\theta \in I^k_j$ and $0 < r < 1$,

$$|g_k(re^{i\theta})| = c_k \left| \sum_{l=1}^{2^k} \frac{\delta_k^\alpha}{|1 - r_ke^{-i\theta l}\ e^{i\theta}|^\beta} \right|.$$

(2.17)

For $r > r_k$, using (2.9) and (2.4), we see that

$$|1 - r_ke^{-i\theta l}\ e^{i\theta}| = |e^{i\theta l} - r_ke^{i\theta}| \leq |e^{i\theta l} - r_ke^{i\theta}| + |r_ke^{i\theta} - r_ke^{i\theta}|$$

$$\leq 2\delta_k + r_k(1 - r) \leq 2\delta_k + (1 - r_k) = 3\delta_k.$$

If $l \neq j$ and $r > r_k$, (2.10), (2.4) and (2.1) give

$$|1 - r_ke^{-i\theta l}\ e^{i\theta}| = |e^{i\theta l} - r_ke^{i\theta}| \geq |e^{i\theta l} - r_ke^{i\theta}| - |r_ke^{i\theta} - r_ke^{i\theta}|$$

$$\geq \frac{1}{2^{k-1}} - r_k(1 - r) \geq \frac{1}{2^{k-1}} - (1 - r_k) = \frac{1}{2^{k-1}} - \delta_k \geq \frac{1}{2^k}.$$

Then, (2.17) implies that

$$|g_k(re^{i\theta})| \geq c_k \left( \frac{\delta_k^\alpha}{(3\delta_k)^\beta} - 2^k \frac{\delta_k^\alpha}{(2^k)^\beta} \right).$$

(2.18)

$$= c_k \delta_k^\alpha \left( \frac{1}{(3\delta_k)^\beta} - 2^{k(1+\beta)} \right), \quad \theta \in I^k_j, r_k < r < 1.$$

Let us take the numbers $\delta_k$ so small that

(2.19) $2^{k(1+\beta)} < \frac{1}{2} \frac{1}{(3\delta_k)^\beta}, \quad k = 1, 2, \ldots,$

then, (2.18) and (2.13) give

(2.20) $|g_k(re^{i\theta})| \geq \frac{1}{2} c_k \delta_k^{-1/p}, \quad \theta \in I^k_j, r_k < r < 1.$
Now we look at the second term of (2.16). Again, let $\theta \in I^k_j$ and $0 < r < 1$. For all $n < k$, we have, using Lemma 4, that

$$|g_n(re^{i\theta})| \leq c_n \sum_{l=1}^{2^n} \frac{\delta_n^{\alpha}}{|1 - r_ne^{-i\theta}re^{i\theta}|^\beta}$$

$$= c_n\delta_n^{\alpha} \sum_{l=1}^{2^n} \frac{1}{|e^{i\theta} - rre^{i\theta}|^\beta}$$

$$\leq c_n\delta_n^{\alpha} 2^n (2^{k+1})^\beta = 2^{-n(\frac{k}{2}+1)}\delta_n^{\alpha} 2^{(k+1)\beta} \leq 2^{-n2^{(k+1)\beta}},$$

which shows that

$$\sum_{n=1}^{k-1} |g_n(re^{i\theta})| \leq 2^{(k+1)\beta} \sum_{n=1}^{k-1} 2^{-n} \leq 2^{(k+1)\beta} \sum_{n=1}^{\infty} 2^{-n} = 2^{(k+1)\beta}.$$

So we have found that

$$\sum_{n=1}^{k-1} |g_n(re^{i\theta})| \leq 2^{(k+1)\beta}, \quad \theta \in I^k_j, \; 0 < r < 1.$$  

(2.21)

Let us take the $\delta_k$'s such that

$$\delta_n^{\alpha/\beta} < \frac{\pi}{2^k}, \quad \text{for all } k.$$  

(2.22)

For $n = 1, 2, \ldots$, define

$$J^n_l = (\theta^n_l - \delta_n^{\alpha/\beta}, \theta^n_l + \delta_n^{\alpha/\beta}), \quad l = 1, 2, \ldots, 2^n.$$  

(2.23)

Notice that (2.22) implies that, for each $n$, the intervals $J^n_l$ ($l = 1, 2, \ldots, 2^n$) are pairwise disjoint. Then, using (2.7), we easily obtain the following.

**Lemma 5.** Let $n > k$. If $\theta \in I^k_j \setminus \bigcup_{l=1}^{2^n} J^n_l$ and $0 < r < 1$, then

$$|e^{i\theta} - rne^{i\theta}| \geq \frac{1}{n} \delta_n^{\alpha/\beta}, \quad \text{for all } l \in \{1, 2, \ldots, 2^n\}.$$
Now we are able to estimate the third term of (2.16). Take \( \theta \) and \( r \) as in Lemma 5. We have

\[
|g_n(re^{i\theta})| \leq c_n \sum_{l=1}^{2^n} \frac{\delta_n^\alpha}{\left|1 - r_n e^{-i\theta_l} re^{i\theta}\right|^{\beta}}
\]

\[
= c_n \delta_n^\alpha \sum_{l=1}^{2^n} \frac{1}{\left|e^{i\theta_l} - r_n e^{i\theta}\right|^{\beta}}
\]

\[
\leq c_n \delta_n^\alpha 2^n \left(\frac{\pi}{\delta_n^{\beta/\beta}}\right)^\beta = \pi^\beta 2^{-n(\frac{1}{p} + 1)} \leq \pi^\beta 2^{-n}.
\]

Thus for \( \theta \in I_k \setminus \bigcup_{n=k+1}^{\infty} \bigcup_{l=1}^{2^n} J_l^n \) and \( 0 < r < 1 \),

\[
\sum_{n=k+1}^{\infty} \sum_{l=1}^{2^n} |g_n(re^{i\theta})| \leq \sum_{n=k+1}^{\infty} \pi^\beta 2^{-n} \leq \pi^\beta \sum_{n=1}^{\infty} 2^{-n} = \pi^\beta.
\]

For \( k = 1, 2, \ldots \), let

(2.24) \quad E_k^j = I_k \setminus \bigcup_{n=k+1}^{\infty} \bigcup_{l=1}^{2^n} J_l^n, \quad j = 1, 2, \ldots, 2^k.

So we have proved

(2.25) \quad \sum_{n=k+1}^{\infty} |g_n(re^{i\theta})| \leq \pi^\beta, \quad \theta \in E_k^j, \quad 0 < r < 1.

We conclude from (2.16), (2.20), (2.21) and (2.25), that

(2.26) \quad |f(re^{i\theta})| \geq \frac{1}{2 \cdot 3^\beta} c_k \delta_k^{-1/p} - 2^{(k+1)\beta} - \pi^\beta, \quad \theta \in E_k^j, \quad r_k < r < 1.

Take the \( \delta_k \)'s so small that

(2.27) \quad 2^{(k+1)\beta} + \pi^\beta < \frac{1}{4 \cdot 3^\beta} c_k \delta_k^{-1/p}.

Then (2.26) gives

(2.28) \quad |f(re^{i\theta})| \geq \frac{1}{4 \cdot 3^\beta} c_k \delta_k^{-1/p}, \quad \theta \in E_k^j, \quad r_k < r < 1.
From (1.3) it is clear that
\[
\frac{\varphi(\lambda_0 x)}{x^q} \to \infty, \quad \text{as } x \to \infty,
\]
for every constant \( \lambda_0 > 0 \). Taking
\[
\lambda_k = \frac{1}{4 \cdot 3^k} c_k, \quad k = 1, 2, \ldots,
\]
for each \( k \), we have
\[
\left( \frac{\varphi(\lambda_k x)}{x^q} \right)^{\lambda/q} \to \infty, \quad \text{as } x \to \infty,
\]
and hence there exists \( \varepsilon_k > 0 \) such that
\[
\varepsilon^{\lambda/p} \varphi \left( \lambda_k \varepsilon^{-1/p} \right)^{\lambda/q} > k, \quad 0 < \varepsilon \leq \varepsilon_k.
\]
Let us choose the numbers \( \delta_k \) satisfying
\[
(2.29) \quad 0 < \delta_k \leq \varepsilon_k, \quad k = 1, 2, \ldots.
\]
Then it follows that
\[
(2.30) \quad \delta_k^{\lambda/p} \varphi \left( \frac{1}{4 \cdot 3^k} c_k \delta_k^{-1/p} \right)^{\lambda/q} > k, \quad k = 1, 2, \ldots.
\]
Furthermore, we take the numbers \( \delta_k \) so small that
\[
(2.31) \quad \sum_{n=k+1}^{\infty} 2^{n+1} \delta_n^{\alpha/\beta} \leq \delta_k, \quad k = 1, 2, \ldots,
\]
which implies
\[
(2.32) \quad |E_k^j| \geq \delta_k, \quad j = 1, 2, \ldots, 2^k,
\]
for all \( k = 1, 2, \ldots \), where \( |E_j^k| \) denotes the Lebesgue measure of the set \( E_j^k \).
Now, if \( k \) is any positive integer, and \( j \in \{1, 2, \ldots, 2^k\} \), using (2.28), the fact that \( \varphi \) is increasing, (2.30) and (2.32), we obtain

\[
\int_0^1 (1 - r)^{\lambda - 1} \left( \int_{I_j^k} \varphi \left( |f(re^{i\theta})| \right) d\theta \right)^{\lambda/q} dr \\
\geq \int_{r_k}^1 (1 - r)^{\lambda - 1} \left( \int_{E_j^k} \varphi \left( \frac{1}{4 \cdot 3^j} c_k \delta_k^{-1/p} \right) d\theta \right)^{\lambda/q} dr \\
= \varphi \left( \frac{1}{4 \cdot 3^j} c_k \delta_k^{-1/p} \right)^{\lambda/q} |E_j^k|^{\lambda/q} \int_{r_k}^1 (1 - r)^{\lambda - 1} dr \\
\geq k \delta_k^{-\lambda/p} \delta_k^{\lambda/q} \left( 1 - r_k \right)^{\lambda \alpha} = k \delta_k^{\lambda \alpha} \delta_k^{\lambda \alpha} = \frac{1}{\lambda \alpha} \cdot k.
\]

Thus, we have seen that

\[
(2.33) \quad \int_0^1 (1 - r)^{\lambda - 1} \left( \int_{I_j^k} \varphi \left( |f(re^{i\theta})| \right) d\theta \right)^{\lambda/q} dr \geq \frac{1}{\lambda \alpha} \cdot k, \quad j = 1, 2, \ldots, 2^k, \ k = 1, 2, \ldots.
\]

Now, if \( I \subset [0, 2\pi] \) is a non-degenerate interval, then it is clear that there exists \( k_0 \) such that for every \( k \geq k_0 \) there exists \( J_k \in \{1, 2, \ldots, 2^k\} \) with \( I_{J_k} \subset I \). Then, using (2.33), we see that

\[
\int_0^1 (1 - r)^{\lambda - 1} \left( \int_I \varphi \left( |f(re^{i\theta})| \right) d\theta \right)^{\lambda/q} dr \\
\geq \lim_{k \to \infty} \int_0^1 (1 - r)^{\lambda - 1} \left( \int_{I_{J_k}} \varphi \left( |f(re^{i\theta})| \right) d\theta \right)^{\lambda/q} dr = \infty.
\]

Hence, Theorem 3 is proved taking \( \alpha, \beta \) and the sequence \( \{c_k\} \) as above, and the \( \delta_k \)'s satisfying (2.1), (2.19), (2.22), (2.27), (2.29) and (2.31), which is clearly possible. \( \blacksquare \)
Proof of Theorem 4: Let $\alpha$ be any positive number, and let

$$\beta = \alpha + \frac{1}{p} - 1.$$  \hspace{1cm} (2.34)

Suppose that $\{\delta_k\}_{k=1}^\infty$ is a sequence of real numbers which satisfies (2.1). Set

$$c_k = 2^{-2k/p}, \quad k = 1, 2, \ldots,$$

define the functions $f_k, k = 1, 2, \ldots,$ as in (2.5), and let $g_k = c_kf_k$ for all $k$. Then,

$$g_k'(z) = c_k \sum_{j=1}^{2^k} \frac{\delta_k^\alpha \beta r^k e^{-i\theta_j}}{(1 - r^k e^{-i\theta_j} z)^{\beta+1}},$$

and

$$|g_k'(z)| \leq c_k \beta \sum_{j=1}^{2^k} \frac{\delta_k^\alpha}{|1 - r^k e^{-i\theta_j} z|^{\beta+1}}.$$

Now, using the elementary inequality

$$(a_1 + a_2 + \cdots + a_n)^p \leq a_1^p + a_2^p + \cdots + a_n^p, \quad a_i \geq 0 \text{ for } i = 1, 2, \ldots, n,$$

which holds since $0 < p < 1$, (2.14) with $\gamma = (\beta + 1)p > 1$, (2.4) and (2.34), we have

$$\|g_k'(e^{i\theta})\|^p = \frac{1}{2\pi} \int_0^{2\pi} |g_k'(e^{i\theta})|^p d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{2^k} \frac{1}{|1 - r^k e^{-i\theta_j} e^{i\theta}|^{(\beta+1)p}} \right) d\theta$$

$$= c_k^p \beta^p \delta_k^{\alpha p} \sum_{j=1}^{2^k} \frac{1}{|1 - r^k e^{-i\theta_j} e^{i\theta}|^{(\beta+1)p}} d\theta$$

$$\leq c_k^p \beta^p \delta_k^{\alpha p} \sum_{j=1}^{2^k} \frac{1}{|1 - r^k e^{-i\theta_j} e^{i\theta}|^{(\beta+1)p}} d\theta$$

$$\leq 2^{-k} \beta^p \delta_k^{\alpha p} \frac{c}{(1 - r_k)^{(\beta+1)p-1}} = 2^{-k} \beta^p c.$$

So we have obtained that

$$\|g_k'(e^{i\theta})\|^p \leq 2^{-k} \beta^p c, \quad k = 1, 2, \ldots.$$  \hspace{1cm} (2.35)
Let us define \( f \) by (2.12). It is clear that \( f \) is analytic in \( \Delta \). Since

\[
f'(z) = \sum_{k=1}^{\infty} g'_k(z),
\]

using (2.35), we deduce that

\[
\|f'\|_{H^p}^p \leq \sum_{k=1}^{\infty} \|g'_k\|_{H^p}^p \leq \sum_{k=1}^{\infty} 2^{-k} \beta^p c = \beta^p c < \infty,
\]

and, hence, \( f' \in H^p \).

We shall argue as in the proof of Theorem 3. If \( \theta \in I_j^k \), we have

\[
|f(e^{i\theta})| = \left| \sum_{n=1}^{\infty} g_n(e^{i\theta}) \right|
\]

(2.36)

\[
\geq |g_k(e^{i\theta})| - \sum_{n=1}^{k-1} |g_n(e^{i\theta})| - \sum_{n=k+1}^{\infty} |g_n(e^{i\theta})|.
\]

First, we apply Lemma 2 to get

\[
|g_k(e^{i\theta})| = c_k \left| \sum_{l=1}^{2^k} \frac{\delta_k^l}{|1 - r_k e^{-i\theta_l} e^{i\theta}|^\beta} \right|
\]

\[
\geq c_k \left( \sum_{l=1}^{2^k} \frac{\delta_k^l}{|1 - r_k e^{-i\theta_l} e^{i\theta}|^\beta} - \sum_{l=1}^{2^k} \frac{\delta_k^l}{|1 - r_k e^{-i\theta_l} e^{i\theta}|^\beta} \right)
\]

\[
\geq c_k \delta_k^l \left( \frac{1}{(2\delta_k)^\beta} - 2^{k(2(k-1))} \right).
\]

Notice that the sequence \( \{\delta_k\} \) may be supposed to satisfy

\[2^{k(2(k-1))} < \frac{1}{2 (2\delta_k)^\beta}, \quad k = 1, 2, \ldots \]

Then,

\[
|g_k(e^{i\theta})| \geq c_k \delta_k^l \frac{1}{2 (2\delta_k)^\beta} = \frac{1}{2 \cdot 2^\beta} c_k \delta_k^{\alpha - \beta} = \frac{1}{2^{\beta+1}} c_k \delta_k^{-1/q}.
\]
So, we have proved that
\begin{equation}
|g_k(e^{i\theta})| \geq \frac{1}{2^{j+1}} c_k \delta_k^{1/q}, \quad \theta \in I_j^k.
\end{equation}

Next, take $\theta \in I_j^k$ and $n < k$. Using Lemma 4, we deduce that
\[
|g_n(e^{i\theta})| \leq c_n \sum_{l=1}^{2^n} \frac{\delta_n^\alpha}{|1 - r_n e^{-i\theta} e^{i\theta}|^\beta}.
\]
\[
= c_n \delta_n^\alpha \sum_{l=1}^{2^n} \frac{1}{|e^{i\theta} - r_n e^{i\theta}|^\beta}.
\]
\[
\leq c_n \delta_n^\alpha 2^n (2^{k+1})^\beta = 2^n (1 - \frac{2}{\pi}) \delta_n^\alpha 2^{(k+1)\beta} \leq 2^{-n} 2^{(k+1)\beta}.
\]
Hence,
\begin{equation}
\sum_{n=1}^{k-1} |g_n(e^{i\theta})| \leq 2^{(k+1)\beta}, \quad \theta \in I_j^k.
\end{equation}

For every positive integer $n$, define $J_n^l$, $l = 1, 2, \ldots, 2^n$, by (2.23), and suppose, as in the proof of Theorem 3, that (2.22) is satisfied. Notice that Lemma 5 holds for every $r \in (0, 1)$, and so it also does for $r = 1$. Finally, define the sets $E_j^k$, for $k = 1, 2, \ldots$, by (2.24). Then, the same argument used in the proof of Theorem 3 shows that
\begin{equation}
\sum_{n=k+1}^{\infty} |g_n(e^{i\theta})| \leq \pi^\beta, \quad \theta \in E_j^k.
\end{equation}

It follows from (2.36), (2.37), (2.38) and (2.39), that
\[
|f(e^{i\theta})| \geq \frac{1}{2^{j+1}} c_k \delta_k^{1/q} - 2^{(k+1)\beta} - \frac{\pi^\beta}{c_k}, \quad \theta \in E_j^k,
\]
and, taking the numbers $\delta_k$ sufficiently small, we have
\begin{equation}
|f(e^{i\theta})| \geq \frac{1}{2^{j+2}} c_k \delta_k^{1/q} - \frac{\pi^\beta}{c_k}, \quad \theta \in E_j^k.
\end{equation}

For $k = 1, 2, \ldots$, let
\[
\lambda_k = \frac{1}{2^{j+2}} c_k.
\]
and notice that (1.3) implies
\[ \frac{\varphi(\lambda_k x)}{x^d} \to \infty, \quad \text{as } x \to \infty, \]
and so there exists \( \varepsilon_k > 0 \) such that
\[ \varepsilon \varphi \left( \lambda_k \varepsilon^{-1/q} \right) > k, \quad 0 < \varepsilon \leq \varepsilon_k. \]
We may assume that the numbers \( \delta_k \) also satisfy
\[ 0 < \delta_k \leq \varepsilon_k, \quad k = 1, 2, \ldots. \]
Therefore,
\[ \delta_k \varphi \left( \frac{1}{2^{d+2} c_k \delta_k^{-1/q}} \right) > k, \quad k = 1, 2, \ldots. \] (2.41)

Also, as in the proof of Theorem 3, we can take the numbers \( \delta_k \) small enough so that (2.31) holds, and then
\[ |E_j^k| \geq \delta_k, \quad j = 1, 2, \ldots, 2^k, \] (2.42)
for all \( k = 1, 2, \ldots. \)

From (2.40), the fact that \( \varphi \) is increasing, (2.41) and (2.42), we conclude that, for each set \( E_j^k \), we have
\[
\int_{E_j^k} \varphi \left( |f(e^{i\theta})| \right) \, d\theta \geq \int_{E_j^k} \varphi \left( \frac{1}{2^{d+2} c_k \delta_k^{-1/q}} \right) \, d\theta \\
= \varphi \left( \frac{1}{2^{d+2} c_k \delta_k^{-1/q}} \right) |E_j^k| \\
\geq k \delta_k^{-1} \delta_k = k.
\]

An argument similar to that used at the end of the proof of Theorem 3 shows that this implies that (1.6) holds for every non-degenerate interval \( I \subset [0, 2\pi] \). This finishes the proof. \( \blacksquare \)
References


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