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STRATIFICATIONS OF POLYNOMIAL SPACES

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Abstract

In the paper we construct some stratifications of the space of monic polynomials in real and complex cases. These stratifications depend on properties of roots of the polynomials on some given semialgebraic subset of $\mathbb{R}$ or $\mathbb{C}$. We prove differential triviality of these stratifications. In the real case the proof is based on properties of the action of the group of interval exchange transformations on the set of all monic polynomials of some given degree. Finally we compare stratifications corresponding to different semialgebraic subsets.

1. Introduction

In Theory of Singularities many authors use the following conception: local (or global) stability follows from transversality to some stratification of some space of polynomial (analytic or smooth) maps (see, for example, [**AVG**], [**dPW**]). Here we present some stratifications of spaces of monic polynomials on one variable connected to problems of stability of singularities of projection and singularities near boundary points of hypersurfaces.

In the proof of Tarski-Seidenberg Theorem (see, for example, [**BR**]) appears the following partition $\{Q_k\}$ of the space of monic polynomials: $f \in Q_k$ if f has exactly k roots. In fact the main step of the proof of Tarski-Seidenberg Theorem is to prove that Q_k are semialgebraic subsets. The partition $\{Q_k\}$ is not a stratification. In [**GWPL**] it is mentioned that if we consider not just a number of roots but multiplicities of these roots we obtain a Whitney regular stratification of the space of monic polynomials.

In [**G**] it is proved that if we fix multiplicities of the eigenvalues of $n\times n$ matrices and the structures of their Jordan blocks (so-called Segre symbol) we obtain a stratification of the space of $n\times n$ matrices. C. G. Gibson also proved that this stratification is Whitney regular.

In the present paper we consider some generalizations of the stratification considered in [**GWPL**]. Namely, let N be some closed semialgebraic subset of $\mathbb{C}$. Thus we have the following decomposition $N = \mathrm{Int}(N) \cup S_1 \cup S_2 \cup \mathrm{Sing}(\partial N)$, where $\mathrm{Int}(N)$ is the set of internal points of N; $S_1 \subset \partial N$ is a subset of the border ∂N contains only C^ρ-smooth points which have the internal points of N in every small neighbourhood; S_2 is a complement to S_1 in the set of C^ρ-smooth points of ∂N; as usual $\mathrm{Sing}(\partial N)$ is a subset of C^ρ-singular points of ∂N. We define a multiplicity symbol (see sections 2.1 and 3.1) which is determined by multiplicities of the roots of given polynomial on $\mathrm{Int}(N)$, S_1, S_2 and $\mathrm{Sing}(N)$.

The main result of the paper is the theorem that all stratifications given by these multiplicity symbols are differentially trivial and thus Whitney regular (Theorems 2.4 and 3.1). Observe that the stratifications under considerations are different and depend on N and ρ. In the case $N = \mathbb{C}$ we obtain the stratification considered in [**GWPL**].

The paper has the following structure. Part 2 is devoted to stratifications of the space of real monic polynomials. Since any closed semialgebraic subset of $\mathbb{R}$ is or $\mathbb{R}$ itself, or a finite union of closed segments, points and closed halflines we begin our consideration from the case of finite segment. The case of the union of segments and halflines can be treated in the same way. We define a multiplicity symbol for polynomials. This multiplicity symbol is connected to some fixed segment $[b, c]$ and characterizes the number of roots a polynomial has on this segment and their multiplicities. For each multiplicity symbol we define a stratum (the set of all polynomials with the same multiplicity symbol). We prove that the stratification by multiplicity symbols gives us a semialgebraic stratification. It is known [**BCR**] that each semialgebraic stratification can be finitely subdivided to obtain a semialgebraic stratification satisfying "*a*" and "*b*" axioms of Whitney. The interesting property of the stratification by multiplicity symbol is the following: it is not necessary to subdivide it because it is Whitney regular itself. Paragraphs 2.3 and 2.4 are devoted to proving this fact. Section 2.3 is devoted to Interval Exchange Transformations [**Ke**]. Interval Exchange Transformation is a very popular object in ergodic theory. Here we use it in real algebraic geometry. We prove that all the strata defined in 2.1 are invariant by an action of group of interval exchange transformations. The action of any nontrivial interval exchange transformation on the space of polynomials is not continuous, but if the roots of some polynomial are well situated the action of a given interval exchange transformation is a difeomorphism on a neighbourhood of the polynomial. This important property of this group action helps us a lot of prove in section 2.4 that the considering stratifications satisfy the boundary axiom. In fact the boundary axiom can be obtained as a corollary of Whitney regularity, but the semiorder relation itself has a nice combinatorial and geometrical nature, which gives us a more detailed description of the stratifications.

Part 3 is connected to the complex case, where N is a closed semi-algebraic subset of $\mathbb{C}$. To prove a differential triviality we consider so-called disk exchange transformations, which correspond to interval exchange transformations in real case. These transformations have not such nice properties as interval exchange transformations (they do not form a group), but they are useful to prove a differential triviality of the stratifications under consideration.

Part 4 is devoted to the comparing of different stratifications. In section 4.1 we compare real and complex stratifications for the same N. In section 4.2 we prove that two stratifications $\{P_{n,\mu(N_1)}\}$ and $\{P_{n,\mu(N_2)}\}$ (see the notations of part 3) are C^ρ-equivalent if the corresponding subsets N_1 and N_2 are C^ρ-equivalent.

Finally I'd like to mention that these stratifications are rather natural. If we consider a partition $\{Q_k(N)\}$ (the similar as in [**BR**]) given by the number of roots of some polynomial in N and apply the algorithm defined in [**BCR**] to obtain a Whitney regular stratification we will obtain exactly the stratification by multiplicity symbol.

2. Stratifications of real monic polynomials

2.1. Multiplicity symbol.

Let P_n be a space of all monic polynomials of degree n with real coefficients. P_n can be identified to the space $\mathbb{R}^n$ in the standard way:

$$f = u^n + a_{n-1}\, u^{n-1} + \cdots + a_1 u + a_0 \in P_n \leftrightarrow (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{R}^n.$$

Consider some (closed) segment $[b, c]$ in $\mathbb{R}$. Let's define the symbol $\mu = \{\mu_b, \mu_1, \ldots, \mu_r, \mu_c\}$ corresponding to $[b, c]$ such that all numbers μ_b, μ_c, μ_i $(1 \le i \le r)$ satisfy the following conditions:

1. $0 \le r \le n$.
   (Note that we consider also $r = 0$; in this case $\mu = \{\mu_b, \mu_c\}$).
2. $\mu_b, \mu_c \in \mathbb{N} \cup \{0\}$; $\mu_i \in \mathbb{N}$.
   ($\mathbb{N}$ is the set of natural numbers).
3. $\sum_{i=1}^{r} \mu_i + \mu_b + \mu_c \le n$.
4. $1 \le \mu_1 \le \cdots \le \mu_r$.

Let $\mathcal{M}$ be the set of all these symbols μ.

Denote by $P_{n,\mu}$ the set of polynomials $f \in P_n$ such that

1. f has roots of multiplicities μ_b and μ_c at the points b and c correspondently.
   (Note that $\mu_b = 0$ (or $\mu_c = 0$) means that $f(b) \neq 0$ (or $f(c) \neq 0$)).
2. f has exactly r (different) roots $v_1 \neq v_2 \neq \cdots \neq v_r$ of multiplicities $\mu_1, \mu_2, \ldots, \mu_r$ (correspondently) in the open interval (b, c).

In other words $P_{n,\mu}$ is defined by the intersection of the zero divisor of f with the segment $[b, c]$. Note that $P_{n,\mu}$ depends on the segment $[b, c]$.

Theorem 2.1. *The collection $\{P_{n,\mu}\}_{\mu \in \mathcal{M}}$ is a semialgebraic stratification of P_n.*

Proof: It is clear that $\bigcup_{\mu \in \mathcal{M}} P_{n,\mu} = P_n$ because each polynomial belongs to some $P_{n,\mu}$ (for some μ). To prove that all $P_{n,\mu}$ are smooth submanifolds and semialgebraic sets we need the following.

Lemma 2.1. *Let $\mu = \{0, \overbrace{1, 1, \ldots, 1}^{r}, 0\}$ (r can be equal to 0). Then $P_{n,\mu}$ is an open semialgebraic set in P_n.*

Proof of Lemma 2.1: Let us prove that $P_{n,\mu}$ is semialgebraic. Each polynomial $f \in P_{n,\mu}$ can be presented in the following form:

$$(1) \quad f = (u - v_1) \cdot \ldots \cdot (u - v_r)(u - \ell_1) \cdot \ldots \cdot (u - \ell_s) \cdot (u^2 + p_1 u + q_1) \cdot \ldots \cdot (u^2 + p_t u + q_t)$$

for some $s, t \in \mathbb{N} \cup \{0\}$ uniquely determined by f such that $r + s + 2t = n$.

For given pair (s, t) the set $P_{n,\mu}$ defines a semialgebraic set $U(s, t)$ in the space $\mathbb{R}^n$ of coefficients $(v_1, \ldots, v_r, \ell_1, \ldots, \ell_s, p_1, q_1, \ldots, p_t, q_t)$ by the following inequalities:

$$\begin{cases} b < v_i < c, & i = 1, 2, \ldots, r \text{ (it means: } v_i \in (b, c)) \\ v_i \neq v_j \text{ for } i \neq j; & i, j = 1, 2, \ldots, r \\ \ell_i < b \text{ or } \ell_i > c, & i = 1, 2, \ldots, s \text{ (it means: } \ell_i \notin [b, c]) \\ p_j^2 - 4q_j < 0, & j = 1, 2, \ldots, t. \end{cases}$$

Let $Q_{s,t} : \mathbb{R}^n \to P_n$ be a map obtained by the opening brackets in (1). (Remined that P_n is identified to $\mathbb{R}^n$). We get

$$P_{n,\mu} = \bigcup_{\substack{s,t \\ s+r+2t=n}} Q_{s,t}(U(s,t)).$$

Since $Q_{s,t}$ is an algebraic map we obtain (by Tarski-Seidenberg Theorem) that $P_{n,\mu}$ is a semialgebraic set.

Let's prove that $P_{n,\mu}$ is an open set. By the definition of $P_{n,\mu}$ the graph f considered in the subset $(b,c) \times \mathbb{R}$ intersects the "zero section" $(b,c) \times \{0\}$ transversally. Hence there exists a neighbourhood of f in P_n satisfies the same property. It means that $P_{n,\mu}$ is an open subset. Lemma 2.1 is proved. ■

Lemma 2.2. *$P_{n,\mu}$ is a smooth submanifold of P_n and* $\operatorname{codim} P_{n,\mu} = \mu_b + \mu_c + \sum_{i=1}^{r} (\mu_i - 1)$.

We prove this lemma in several steps.

Claim. *Let μ satisfies the following condition:*

$$\sum_{i=1}^{r} \mu_i + \mu_b + \mu_c = n.$$

Then $P_{n,\mu}$ is a smooth submanifold of P_n.

Proof: Consider a space $\mathbb{R}^r$ and a subset $V_\mu \subset \mathbb{R}^r$ defined in the following way: $V_\mu = \{(v_1, \dots, v_r) \in \mathbb{R}^r$ such that

$$\text{(2)} \qquad \begin{cases} v_i \in (b,c), & i = 1, \dots, r \\ v_i \neq v_j, & \text{if } i \neq j \\ v_i < v_j, & \text{if } \mu_i = \mu_j \text{ and } i < j \quad \}. \end{cases}$$

Let $F_\mu : V_\mu \to P_n$ be the following map

$$\text{(3)} \quad F_\mu(v_1, \dots, v_r) = (u - v_1)^{\mu_1} \cdot (u - v_2)^{\mu_2} \cdot \dots \cdot (u - v_r)^{\mu_r} \cdot (u - b)^{\mu_b} \cdot (u - c)^{\mu_c}.$$

F_μ is a one-to-one map because each polynomial is uniquely defined by it's roots.

$F_\mu(V_\mu) = P_{n,\mu}$ because for each polynomial $f \in P_{n,\mu}$ we have the presentation (3).

Now we have to prove that F_μ is an immersion. We do it using the induction by n.

If $n = 1$ we have:

$P_{1,\{0,1\}}$ and $P_{1,\{1,0\}}$ are one points each,

$P_{1,\{0,1,0\}}$ is an open interval ($V_{\{0,1,0\}} = \{v \in (b,c)\}$, $F_{\{0,1,0\}}(v) = u - v$).

Now suppose that F_μ is an immersion for each $n < n_0$. Let's prove it for n_0.

Suppose that $r \neq 0$. For each polynomial $f \in P_{n_0,\mu}$ we have the following presentation:

$$f = (u - v_r)^{\mu_r} \cdot h, \tag{4}$$

where v_r is a root of f belonging to (b,c) and $h \in P_{n_0-\mu_r,\nu}$, $\nu = \{\mu_b, \mu_1, \dots, \mu_{r-1}, \mu_c\}$.

Let $G_{\mu_r} : (b,c) \times P_{n_0-\mu_r} \to P_{n_0}$ be the following map:

$$G_{\mu_r}(v_r, g) = (u - v_r)^{\mu_r} \cdot g \tag{5}$$

where $g \in P_{n_0-\mu_r}$.

We will prove that G_{μ_r} is a local immersion in a neighbourhood of a point (v_r, g) such that v_r is not a root of g. Thus we can consider the map F_μ as the following:

$$F_\mu(v_1, \dots, v_r) = G_{\mu_r}(v_r, F_\nu(v_1, \dots, v_{r-1}))\big|_{V_\mu}$$

and obtain that F_μ is an immersion (F_ν is an immersion by the induction hypothesis).

Let U_{v_r} and U_g be neighbourhoods of v_r and g such that for each $v \in U_{v_r}$ and for each $\tilde{g} \in U_g$ v is not a root of $\tilde{g}$. In local coordinates (putting $\tilde{u} = u - v_r$) we obtain

$$\tilde{g}(\tilde{u}) = \tilde{u}^{n_0-\mu_r} + \tilde{g}_{n_0-\mu_r-1}\tilde{u}^{n_0-\mu_r-1} + \dots + \tilde{g}_1\tilde{u} + \tilde{g}_0, \tag{6}$$

where $\tilde{g}_0 \neq 0$ because v_r is not a root of $\tilde{g}$.

Putting this presentation (6) to (5) we obtain

$$G_{\mu_r}(v, \tilde{g}) = \tilde{u}^{n_0} + q_{n_0-1}\tilde{u}^{n_0-1} + \dots + q_1\tilde{u} + q_0,$$

where

$$
(7)\qquad
\begin{cases}
q_{n_0-1} = \tilde{g}_{n_0-\mu_r-1} + \mu_r \cdot w \\
q_{n_0-2} = \tilde{g}_{n_0-\mu_r-2} + \mu_r \cdot w \cdot \tilde{g}_{n_0-\mu_r-1} + o(w) \\
\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
q_{\mu_r} = \tilde{g}_0 + \mu_r \cdot w \cdot \tilde{g}_1 + o(w) \\
q_{\mu_r-1} = \mu_r \cdot w \cdot \tilde{g}_0 + o(w) \\
q_i = o(w) \text{ for } i = 0, \dots, \mu_r - 2.
\end{cases}
$$

(We use the formula: $(\tilde{u}+w)^{\mu_r} = \tilde{u}^{\mu_r} + \mu_r \tilde{u}^{\mu_r - 1} w + o(w); w = v_r - v$.)

From (7) we can see that the Jacobian matrix of G_{μ_r} has an above triangular structure. Since $\tilde{g}_0 \neq 0$ it has a maximal rank. Thus the map G_{μ_r} is a local immersion in a neighbourhood of (v_r, g).

If $r = 0$ symbol μ has the following form: $\{\mu_b, \mu_c\}$ and $\mu_b + \mu_c = n_0$. Hence $P_{n_0,\mu}$ contains only one point $f = (u-b)^{\mu_b} \cdot (u-c)^{\mu_c}$.

So, we proved that $P_{n,\mu}$ is an immersed submanifold. To prove that $P_{n,\mu}$ is a submanifold we'll show that $F_\mu(\partial V_\mu) \cap P_{n,\mu} = \emptyset$. (Note, that V_μ is bounded.) F_μ can be extended to all $\mathbb{R}^r$ by the formula (3). The system (2) describes V_μ is an open semialgebraic set in $\mathbb{R}^r$. By the Second Main Structural Theorem (see [**BR**, p. 68]) each point $v \in \partial V_\mu$ satisfies the following condition:

$$
\left[\bigvee_{i=1}^{r} ((v_i = b) \vee (v_i = c))\right] \vee \left[\bigvee_{i \neq j} (v_i = v_j)\right].
$$

But by the definition of $P_{n,\mu}$ we get that $F_\mu(v) \notin P_{n,\mu}$ because the polynomial $F_\mu(v)$ has the different multiplicity symbol.

So, $P_{n,\mu}$ is a smooth submanifoled of P_n. The claim is proved. ■

Remark 2.1. Since V_μ is a semialgebraic set and F_μ is a semialgebraic map we obtain (by Tarski-Seidenberg Theorem) that $P_{n,\mu}$ is also a semialgebraic set.

Remark 2.2. Since F_μ is an immersion then

$$
\dim P_{n,\mu} = \dim V_\mu = r = n - \left(\sum_{i=1}^{r} (\mu_i - 1) + \mu_b + \mu_c\right).
$$

So, we obtained a formula for codimension of $P_{n,\mu}$ for this special case.

Proof of the lemma: Suppose that $\sum_{i=1}^{r} \mu_i + \mu_b + \mu_c = m < n$. For each polynomial $f \in P_{m,\mu}$ we have:

$$f = g \cdot p \tag{8}$$

where $g \in P_{m,\mu}$ and $p \in P_{n-m,\{0,0\}}$.

Let $\widetilde{F}_\mu : P_{m,\mu} \times P_{n-m,\{0,0\}} \to P_n$ be the map defined by the formula (8). The map is one-to-one because a polynomial is uniquely defined by it's roots and sets of roots of g and p do not intersect.

Let us prove that $\widetilde{F}_\mu$ is an immersion. The proof uses the same arguments as the proof of the claim. Consider the same set V_μ described by inequalities (2) and construct a map $G_\mu : V_\mu \times P_{n-m,\{0,0\}} \to P_n$ such that

$$G_\mu(v,p) = F_\mu(v) \cdot p, \tag{9}$$

where F_μ is a map constructed in the proof of the claim (the formula (3)). Since $F_\mu : V_\mu \to P_{m,\mu}$ is a diffeomorphism it is enough to prove that G_μ is an immersion.

Let us prove it by induction by m. If $m = 0$ then G_μ is just the identity map on $P_{n-m,\{0,0\}}$ and $P_{n-m,\{0,0\}}$ is an open set (by Lemma 2.1). Suppose that we proved the statement for $m < m_0$. Let's prove it for m_0. Consider the polynomial $f \in P_{n,\mu}$. Suppose that $r \neq 0$. Take the root v_r. We have the following presentation

$$f = (u - v_r)^{\mu_r} \cdot h, \tag{10}$$

where $h \in P_{n-\mu_r,\nu}$, $\nu = \{\mu_b, \mu_1, \dots, \mu_{r-1}, \mu_c\}$.

The continuation of the proof is the same as in the claim.

Let $r = 0$. It means that $\mu = \{\mu_b, \mu_c\}$. In this case $V_\mu = \emptyset$. Thus it is enough to prove that the maps $G_{\mu_b} : P_{n-\mu_b,\{0,0\}} \to P_n$ and $G_{\mu_c} : P_{n-m,\{0,0\}} \to P_{n-\mu_b}$ defined as follows

$$G_{\mu_b}(h) = (u - b)^{\mu_b} \cdot h, \quad G_{\mu_c}(g) = (u - c)^{\mu_c} \cdot g$$

are immersions.

We will show it for G_{μ_b}. For G_{μ_c} the proof is the same.

Let $h \in P_{n-\mu_b,\{0,0\}}$. Putting $\tilde{u} = u - b$ we obtain

$$h(\tilde{u}) = \tilde{u}^{n-\mu_b} + h_{n-\mu_b-1}\, \tilde{u}^{n-\mu_b-1} + \cdots + h_0$$

and

$$G_{\mu_b}(h)(\tilde{u}) = \tilde{u}^n + h_{n-\mu_b-1}\,\tilde{u}^{n-1} + \cdots + h_0\,\tilde{u}^{\mu_b}.$$

So, $P_{n,\mu} = G_{\mu_b} \circ G_{\mu_c}(P_{n-m,\{0,0\}})$ is an immersed submanifold.

Let us prove that $P_{n,\mu}$ is a submanifold. The map G_μ, defined by formula (9), can be extended to the set $\mathbb{R}^r \times P_{n-m}$. Let $(v,p) \in \partial(V_\mu \times P_{n-m,\{0,0\}})$. It means that either $v \in \partial V_\mu$ or $p \in \partial P_{n-m,\{0,0\}}$. The case $v \in \partial V_\mu$ was considered in the proof of the claim. Let $p \in \partial P_{n-m,\{0,0\}}$. It means that P has a root in $[b,c]$. Hence, $f = G_\mu(v,p)$ does not belong to $P_{n,\mu}$.

Since $P_{n-m,\{0,0\}}$ is unbounded it is also necessary to prove that if a sequence (v^k,p^k) tends to infinity for $k \to \infty$ then so for $G_\mu(v^k,p^k)$. But it is a partial case of Proposition 1.5.5 [**BR**].

Since G_μ is a homeomorphism to the image

$$\dim P_{n,\mu} = r + n - m = n - \left(\sum_{i=1}^{r}(\mu_i - 1) + \mu_b + \mu_c\right).$$

It proves the codimension formula. Lemma 2.2 is proved. ■

Remark 2.3. We also proved that $P_{n,\mu}$ is a semialgebraic set.

Let's define *a codimension of* μ as a codimension of the corresponding stratum $P_{n,\mu}$ if $P_{n,\mu} \neq \emptyset$: $c(\mu) = \operatorname{codim} P_{n,\mu}$. This definition is correct (does not depend on n), because by Lemma 2.2 we have:

$$c(\mu) = \sum_{i=1}^{r}(\mu_i - 1) + \mu_b + \mu_c.$$

Now we are going to define some semiorder relation $\mathcal{R}$ on the set of multiplicity symbols $\mathcal{M}$. We do it in the following way.

1. If $c(\mu^1) = c(\mu^2)$ then (μ^1,μ^2) and (μ^2,μ^1) do not belong to $\mathcal{R}$.
2. Let $c(\mu^2) = c(\mu^1) + 1$. Then $(\mu^1,\mu^2) \in \mathcal{R}$ if these symbols satisfy one of the following conditions:
   a) $r^2 = r^1 + 1$, $\mu_b^1 = \mu_b^2$, $\mu_c^1 = \mu_c^2$ and there exists μ_j^2 such that $\mu_i^2 = \mu_i^1$ for all $i \neq j$ (and hence $\mu_j^2 = 2$).
   b) $r^2 = r^1 - 1$, $\mu_b^1 = \mu_b^2$, $\mu_c^1 = \mu_c^2$ and there exist μ_s^1, μ_j^1 and μ_k^2 such that $\mu_i^2 = \mu_i^1$ for all $i \neq s,j,k$ (and hence $\mu_s^1 + \mu_j^1 = \mu_k^2$).
   c) $r^2 = r^1 - 1$, $\mu_b^1 = \mu_b^2$ and there exists μ_j^1 such that $\mu_i^2 = \mu_i^1$ for all $i \neq j$ (and hence $\mu_c^1 + \mu_j^1 = \mu_c^2$).

d) $r^2 = r^1 - 1$, $\mu_c^1 = \mu_c^2$ and there exists μ_j^1 such that $\mu_i^2 = \mu_i^1$ for all $i \neq j$ (and hence $\mu_b^1 + \mu_j^1 = \mu_b^2$).

e) $r^2 = r^1$, $\mu_i^1 = \mu_i^2$ for all i and $\mu_b^1 + 1 = \mu_b^2$.

f) $r^2 = r^1$, $\mu_i^1 = \mu_i^2$ for all i and $\mu_c^1 + 1 = \mu_c^2$.

(We denote by r^1 and r^2 the numbers of internal elements of μ^1 and μ^2).

We write $\mu \prec \nu$ if $(\mu, \nu) \in \mathcal{R}$.

3. Let $c(\mu) < c(\nu)$. Then $\mu \prec \nu$ if there exists a finite set of symbols $\mu = \mu^1, \mu^2, \dots, \mu^k = \nu$ such that $\mu^1 \prec \mu^2 \prec \mu^3 \prec \cdots \prec \mu^k = \nu$.

Proposition 2.1. *$\mathcal{R}$ is a semiorder relation on $\mathcal{M}$.*

The proof just follows from the definition of $\mathcal{R}$.

Proposition 2.2. *If $\mu \prec \nu$ then $P_{n,\nu} \subset C\ell(P_{n,\mu})$.*

Proof: It is enough to prove this statement just in the case $c(\nu) = c(\mu) + 1$. So, we have to check all possibilities for pair (μ, ν) (2.a)-2.e)).

2.a) Let $f \in P_{n,\nu}$. Then we have the presentation

$$f(u) = (u - v_j)^2 \cdot g(u),$$

where $g \in P_{n-2,\mu}$ and $v_j \in (b, c)$ is not a root of g.

Consider a family of polynomials

$$f_\epsilon(u) = [(u - v_j)^2 + \epsilon] \cdot g(u).$$

If $\epsilon \to 0$ then $f_\epsilon \to f$, but $f_\epsilon \in P_{n,\mu}$ for $\epsilon > 0$. It means that $f \in C\ell(P_{n,\mu})$.

2.b) Consider also $f \in P_{n,\nu}$. In this case we have:

$$f(u) = (u - v_k)^{\nu_k} \cdot g(u),$$

where $g \in P_{n-\nu_k,\overline{\nu}}$ and v_k is not a root of g.

($\overline{\nu}$ is equal to ν without a coordinate ν_k:

$$\overline{\nu} = (\nu_b, \dots, \nu_{k-1}, \nu_{k+1}, \dots, \nu_c)).$$

Consider a family

$$f_\epsilon(u) = (u - (v_k - \epsilon))^{\mu_s} \cdot (u - (v_k + \epsilon))^{\mu_j} \cdot g(u).$$

If $\epsilon \to 0$ then $f_\epsilon \to f$ and $f_\epsilon \in P_{n,\mu}$ for sufficiently small $\epsilon > 0$.

2.c) For $f \in P_{n,\nu}$ we can consider the following presentation

$$f(u) = (u - c)^{\mu_j} \cdot g(u),$$

where $g \in P_{n-\mu_j,\overline{\mu}}$ ($\overline{\mu}$ is equal to μ without a coordinate $\mu_j : \overline{\mu} = (\mu_b, \dots, \mu_{j-1}, \mu_{j+1}, \dots, \mu_c)$).

Consider $f_\epsilon(u) = (u - (c - \epsilon))^{\mu_j} \cdot g(u)$.

If $\epsilon \to 0$ then $f_\epsilon \to f$ and $f_\epsilon \in P_{n,\mu}$ for sufficiently small $\epsilon > 0$.

The case 2.d) can be considered by the same arguments.

2.e) Let $f \in P_{n,\nu}$. Consider the presentation

$$f(u) = (u - b) \cdot g(u),$$

where $g \in P_{n-1,\mu}$.

Let's define $f_\epsilon(u) = (u - (b - \epsilon)) \cdot g(u)$.

Clearly, $f_\epsilon \in P_{n,\mu}$ for $\epsilon > 0$ and $f_\epsilon \to f$ if $\epsilon \to 0$.

The case 2.f) can be considered in the same manner. ■

Proposition 2.3. *If $(\mu, \nu) \notin \mathcal{R}$ then $C\ell(P_{n,\mu}) \cap P_{n,\nu} = \emptyset$.*

Proof: Consider the map $G_\mu : V_\mu \times P_{n-m,\{0,0\}} \to P_n$ defined in the proof of Lemma 2 (the formula (9)). This map can be extended to $\mathbb{R}^r \times P_{n-m}$. We obtain

$$G : \mathbb{R}^r \times P_{n-m} \to P_n \quad \text{and} \quad G\big|_{V_\mu \times P_{n-m,\{0,0\}}} = G_\mu.$$

By [**BR**] we know that this map is proper. From the definition of the semiorder $\mathcal{R}$ we obtain that

$$G(C\ell(V_\mu \times P_{n-m,\{0,0\}})) = \left(\bigcup_{\eta \succ \mu} P_{n,\eta}\right) \cup P_{n,\mu}.$$

Since G is a proper map it is also a closed map. So

$$P_{n,\mu} \cup \left(\bigcup_{\eta \succ \mu} P_{n,\eta}\right)$$

is a closed set. Since $P_{n,\mu} \cap P_{n,\nu} = \emptyset$ and for each $\eta \succ \mu$, $P_{n,\eta} \cap P_{n,\nu} = \emptyset$. Hence, $C\ell(P_{n,\mu}) \cap P_{n,\nu} = \emptyset$. ■

So we verified all conditions for semialgebraic stratification. Theorem is proved. ■

2.2. Finite set of segments. Some generalizations of the stratifications.

Here we present some other stratifications of P_n different from the first one but connected to it.

Let $\{[b_i, c_i], i = 1, \dots, s\}$ be a finite set of segments such that $[b_i, c_i] \cap [b_j, c_j] = \emptyset$ for $i \neq j$. We can define corresponding multiplicity symbols in the following way.

A. Let $\alpha = \{\alpha_{b_1}, \dots, \alpha_{b_s}, \alpha_1, \dots, \alpha_r, \alpha_{c_1}, \dots, \alpha_{c_s}\}$, where r, α_{b_i}, α_{c_i} and α_j are integer numbers satisfying the following conditions:

1) $r \geq 0$,
2) $\alpha_{b_i} \geq 0$, $\alpha_{c_i} \geq 0$,
3) $\alpha_j > 0$ for $j = 1, \dots, r$ if $r \neq 0$,
4) $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_r$.

B. Let $\beta = \{\beta_B, \beta_1, \dots, \beta_r\}$ be a set of integer numbers satisfying the following conditions:

1) $r \geq 0$,
2) $\beta_B \geq 0$,
3) $\beta_j > 0$ for $j = 1, \dots, r$ if $r \neq 0$,
4) $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_r$.

Let $\mathcal{A}$ be a set of multiplicity symbols defined in A and $\mathcal{B}$ be a set of multiplicity symbols defined in B. Let $\alpha \in \mathcal{A}$. Define $P_{n,\alpha}$ as a set of polynomials f from P_n satisfying the following conditions:

1. At the boundary points b_i and c_i, f has roots of multiplicities α_{b_i} and α_{c_i} correspondently.
2. f has exactly r different roots $v_1, \dots, v_r$ on the open set $\bigcup_{i=1}^{s} (b_i, c_i)$ and at each point v_j f has a root of multiplicity α_j.

Let $\beta \in \mathcal{B}$. Define $P_{n,\beta}$ as a set of polynomials f from P_n satisfying the following conditions:

1*. The sum of all multiplicities of all roots of f at all boundary points b_i, c_i is equal to β_B.
2*. is the same as 2.

Remark 2.4. If $s = 1$ the stratification $\{P_{n,\alpha}\}_{\alpha \in \mathcal{A}}$ is the same as the stratification $\{P_{n,\mu}\}_{\mu \in \mathcal{M}}$ defined in section 2.1.

Theorem 2.2. *For each finite set of segments* $\{[b_i, c_i], i = 1, \dots, s\}$ $\{P_{n,\alpha}\}_{\alpha\in\mathcal{A}}$ *and* $\{P_{n,\beta}\}_{\beta\in\mathcal{B}}$ *are semialgebraic stratifications of* P_n.

The proof is the same as the proof of Theorem 2.1.

2.3. Interval Exchange Transformations.

Let us recall a definition of Interval Exchange Transformation [**Ke**]. Consider some semiopen interval $[s, t) \subset \mathbb{R}$. Let $\{s_i\}(i = 0, \dots, m)$ such that $s = s_0 < s_1 < \dots < s_{m-1} < s_m = t$ be a finite sequence of numbers defining a finite partition of $[s, t)$ to semiopen intervals $\Delta_i = [s_i, s_{i+1})$. *Interval Exchange Transformation* is a map $T : [s, t) \to [s, t)$ satisfying the following conditions:

1. T is a bijection.
2. At each subinterval Δ_i T is a translation: $T(u)\big|_{\Delta_i} = u + \alpha_i$ $(\alpha_i \in \mathbb{R})$.

Set $T(u) = u$ for $u \notin [s, t)$.

Let $[b, c]$ be some fixed segment. Denote by GT the set of all interval exchange transformations non trivially defined on all $[s, t) \subset [b, c]$ such that $s \neq b$.

Proposition 2.4.

1. *GT is a group with composition as a group operation.*
2. *There exists a natural action of the group GT on the spaces P_n such that all the strata $P_{n,\mu}$ are invariant by this action.*

Proof: 1. It follows from the definition of Interval Exchange Transforamtions: associativity we obtain from bijectivity, a unity is the identity map and the inverse map for each interval exchange transformation is also an interval exchange transformation.

2. Let f be a polynomial of degree $n : f \in P_n$. Let T be some interval exchange transformation defined on some semiopen interval $[s, t) : T \in GT$. Let $v_1, \dots, v_r$ be roots of f belonging to $[s, t)$. We have the following presentation

$$f = (u - v_1)^{\mu_1} \cdot \ldots \cdot (u - v_r)^{\mu_r} \cdot g(u),$$

where g is a polynomial which has no roots on $[s, t)$. Set

$$F_T(f) = (u - T(v_1))^{\mu_1} \cdot \ldots \cdot (u - T(v_r))^{\mu_r} \cdot g(u).$$

It is easy to see that a map $\mathcal{F} : GT \times P_n \to P_n$ defined by $\mathcal{F}(T, f) = F_T(f)$ is a group action. All strata $P_{n,\mu}$ are invariant by this action (it follows from the definition of $P_{n,\mu}$). ■

Now we are going to define another action of GT on P_n which can be useful for our problems. Consider a set $R_{\epsilon,T} = (-\epsilon,\epsilon) \times [s,t)$. Let $\widetilde{T}_\epsilon : \mathbb{C} \to \mathbb{C}$ be the following map:

$$\widetilde{T}_\epsilon(z) = \begin{cases} T(x) + iy & \text{for } z = x + iy \in R_{\epsilon,T} \\ z & \text{for } z \notin R_{\epsilon,T}. \end{cases}$$

Let $\mathcal{F}_\epsilon : GT \times P_n \to P_n$ be a map defined as follows: $\mathcal{F}_\epsilon(T,f) = F_{T,\epsilon}(f)$, where $F_{T,\epsilon}(f) = (u - \widetilde{T}_\epsilon(v_1))^{\mu_1} \cdot \ldots \cdot (u - \widetilde{T}_\epsilon(v_k))^{\mu_k}$ for $f = (u - v_1)^{\mu_1} \cdot \ldots \cdot (u - v_k)^{\mu_k}$. Here $v_1, \ldots, v_k$ are all roots of f.

Proposition 2.5. *$\mathcal{F}_\epsilon$ is a group action. Strata $P_{n,\mu}$ are invariant by this action.*

It is clear that the map $F_{T,\epsilon}$ is not necessary continuous on P_n. But we can formulate some continuity property of this map, which will be useful for analysing the geometry of the strata $P_{n,\mu}$. We will use a notation $z(f)$ for the set of all roots of f.

Theorem 2.3. *Let $f \in P_n$ be a polynomial of degree n. Let T be an interval exchange transformation defined on $[s,t)$ such that $s > b$, $t < c$ and for each $v \in z(f) \cap [s,t)$ there exists a subinterval $\Delta_i \subset [s,t)$ such that $v \in \operatorname{Int}\Delta_i$.*

Then there exist $\epsilon > 0$ and a neighbourhood U_f such that $F_{T,\epsilon}\big|_{U_f}$ is a diffeomorphism to the image.

Proof:

Step 1: Let us consider the case that v be a unique root of f and the multiplicity of v is equal to n. In other words, $f = (u-v)^n$. Let Δ_i be a subinterval such that $v \in \operatorname{Int}\Delta_i$. It means that for some $\delta > 0$ $(v-\delta, v+\delta) \subset \Delta_i$. Let U_f be a neighbourhood of f and $\epsilon > 0$ such that for each $g \in U_f$ we have $z(g) \subset (-\epsilon,\epsilon) \times (v-\delta, v+\delta)$. We have a presentation

$$g = (u-v)^n + g_{n-1}(u-v)^{n-1} + \cdots + g_0.$$

By definition of $\widetilde{T}_\epsilon$ we obtain

$$F_{T,\epsilon}(g) = (u - T(v))^n + g_{n-1}(u - T(v))^{n-1} + \cdots + g_0.$$

Obvious, this map $F_{T,\epsilon}$ is diffeomorphism.

Step 2: Let T be a transformation such that there exists a unique root v of f such that $T(v) \notin z(f)$ and for other roots $v' \in [b,c]$ we have $T(v') = v'$. Then we obtain

$$f = (u-v)^{\mu_v} \cdot \tilde{h}(u).$$

Set $\tilde{f} = (u-v)^{\mu_v}$. Now take $\delta > 0$, $\epsilon > 0$ a neighbourhood $U_{\tilde{f}} \subset P_{\mu_v}$, a neighbourhood $U_{\tilde{h}} \subset P_{n-\mu_v}$ such that for each $q \in U_{\tilde{f}}$ and for each $h \in U_{\tilde{h}}$ we have: $z(q) \subset (-\epsilon,\epsilon)\times(v-\delta, v+\delta)$, $z(h) \cap ((-\epsilon,\epsilon)\times(v-\delta,v+\delta)) = \emptyset$ and T is continuous on $(v-\delta, v+\delta)$. Consider a neighbourhood $U_f \subset P_n$ defined as follows: $U_f = U_{\tilde{f}} \cdot U_{\tilde{h}}$. (The symbol $\cdot$ means a usual product of polynomials). Thus for each $g \in U_f$ we have

$$g = \left((u-v)^{\mu_v} + q_{\mu_v-1}(u-v)^{\mu_v-1} + \cdots + q_0\right) \cdot \left(u^{n-\mu_v} + h_{n-\mu_v-1}u^{n-\mu_v-1} + \cdots + h_0\right).$$

So,

$$F_{T,\epsilon}(g) = ((u-T(v))^{\mu_v} + \cdots + q_0) \cdot \left(u^{n-\mu_v} + h_{n-\mu_v-1}u^{n-\mu_v-1} + \cdots + h_0\right).$$

We see that $F_{T,\epsilon}\big|_{U_{\tilde{f}}}$ is a diffeomorphism, $F_{T,\epsilon}\big|_{U_{\tilde{h}}}$ is the identity map. By the straightforward calculations we obtain that the Jacobian matrix for the product operation on $U_{\tilde{f}} \times U_{\tilde{h}}$ at the point $(\tilde{f}, \tilde{h})$ is the resultant of $\tilde{f}$ and $\tilde{h}$ [**J**].

Analogically, the Jacobian matrix for the product operation on $U_{F_{T,\epsilon}(\tilde{f})} \times U_{\tilde{h}}$ at $(F_{T,\epsilon}(\tilde{f}), \tilde{h})$ is the resultant of $F_{T,\epsilon}(\tilde{f})$ and $\tilde{h}$. Since the pair $(\tilde{f}, \tilde{h})$ (and correspondently $(F_{T,\epsilon}(\tilde{f}), \tilde{h})$) has no common zeros on $\mathbb{C}$ these resultants are not equal to zero. So, the Jacobians are non degenerate and hence $F_{T,\epsilon}\big|_{U_f}$ is a diffeomorphism.

Step 3: Let the pair (T,f) satisfies the following conditions: there are two different roots v^1 and v^2 of f such that $T(v^1) = v^2$ and $T(v^2) = v^1$ and for every other root $v \in [b,c]$ we have $T(v) = v$. In this case we obtain the following presentations: $f = (u-v)^{\mu_{v^1}}(u-v^1)^{\mu_{v^2}} \cdot \tilde{h}(u)$ and $F_{T,\epsilon}(f) = (u-v^2)^{\mu_{v^1}}(u-v^1)^{\mu_{v^2}} \cdot \tilde{h}(u)$. Set $f^1 = (u-v^1)^{\mu_{v^1}}$, $f^2 = (u-v^2)^{\mu_{v^2}}$. Take $\delta > 0$, $\epsilon > 0$ and neighbourhoods U_{f^1}, U_{f^2} and $U_{\tilde{h}}$ such that for each $q^1 \in U_{f^1}$, $q^2 \in U_{f^2}$ and $h \in U_{\tilde{h}}$ we have:

1. $z(q^1) \subset (-\epsilon,\epsilon) \times (v^1-\delta, v^1+\delta)$, $z(q^2) \subset (-\epsilon,\epsilon) \times (v^2-\delta, v^2+\delta)$.
2. $(v^2-\delta, v^2+\delta) \cap (v^1-\delta, v^1+\delta) = \emptyset$.
3. $z(h) \cap ((-\epsilon,\epsilon) \times ((v^2-\delta, v^2+\delta) \cup (v^1-\delta, v^1+\delta))) = \emptyset$.
4. T is continuous on $(v^1-\delta, v^1+\delta)$ and $(v^2-\delta, v^2+\delta)$.

Then applying the same arguments as in the Step 2 we obtain that $F_{T,\epsilon}$ is a diffeomorphism to the image in a neighbourhood of f.

Step 4:

Lemma 2.3. *Let the pair (T, f) satisfies the condition of the theorem. Then there exist $\delta > 0$, a finite set of points $\{w_k\}$ and a special interval exchange transformation $S : [s,t) \to [s,t)$ such that*

1) *There exist $\epsilon > 0$ and a neighbourhood U_f such that $F_{S,\epsilon}\big|_{U_f} = F_{T,\epsilon}\big|_{U_f}$.*
2) *$[w_k - \delta, w_k + \delta] \cap [w_j - \delta, w_j + \delta] = \emptyset$ for $k \neq j$.*
3) *Let $\Omega = \bigcup\limits_k [w_k - \delta, w_k + \delta)$. Then $S\big|_\Omega$ is a permutation of intervals $\{[w_k - \delta, w_k + \delta)\}$ and $S\big|_{[s,t)-\Omega} = \mathrm{id}$.*

Proof: Set

$$\{w_k\} = (z(f) \cap [s,t)) \cup T(z(f) \cap [s,t)).$$

Take $\delta > 0$ such that it satisfies the property 2) and for every $v \in z(f) \cap [s,t)$ we have $[v - \delta, v + \delta] \subset \mathrm{Int}\, \Delta_i$ (Δ_i is the corresponding to v subinterval (see the condition of the theorem)).

Set

$$\Omega' = \bigcup_{v \in z(f) \cap [s,t)} [v - \delta, v + \delta) \quad \text{and} \quad S\big|_{\Omega'} = T\big|_{\Omega'}.$$

Let us extend the map S to $\Omega - \Omega'$ such that $S\big|_\Omega$ will be a permutation of intervals.

And finally, let us extend S to $[s,t) - \Omega$ as the identity map. A neighbourhood U_f of f and $\epsilon > 0$ we can find in the same way as in the Steps 2 and 3. ■

Step 5: Now we can restrict our consideration to permutations of intervals. For each permutation S of intervals we have: $S = S^1 \circ S^2 \circ \cdots \circ S^m$, where $\{S^i\}_{i=1}^m$ are permutations of intervals corresponding to the standard generators of the permutation group S_p ($p = \#\{\omega_i\}$). Each generator is an interchanging of a pair of intervals, which were considered in the Steps 2 and 3. It completes the proof of the theorem. ■

2.4. Differential triviality of the stratifications defined by the multiplicity symbol.

Let A be a stratified set and $\{K_i\}$ be a stratification. We say that this stratification is *differentially trivial* if for each stratum K_i and for each two points $x_1, x_2 \in K_i$ there exist neighbourhoods of these points $U_{x_1} \subset A$ and $U_{x_2} \subset A$ and a diffeomorphism $F : U_{x_1} \to U_{x_2}$ such that $F(U_{x_1} \cap K_j) = U_{x_2} \cap K_j$ for every j such that $K_i \subset C\ell(K_j)$.

Theorem 2.4. *The stratification $\{P_{n,\mu}\}_{\mu\in\mathcal{M}}$ is differentially trivial.*

Proof: Let $f_1, f_2 \in P_{n,\mu}$. Let $m = \sum_{i=1}^{r} \mu_i + \mu_b + \mu_c$. Then we have: $f_1 = \tilde{f}_1 \cdot g_1$, $\tilde{f}_2 \cdot g_2$, where $\tilde{f}_1, \tilde{f}_2 \in P_{m,\mu}$ and $g_1, g_2 \in P_{n-m,\{0,0\}}$.

Lemma 2.4. *There exist neighbourhoods $U_{\tilde{f}_1}$ of $\tilde{f}_1$ and $U_{\tilde{f}_2}$ of $\tilde{f}_2$ in P_m and a diffeomorphism $H : U_{\tilde{f}_1} \to U_{\tilde{f}_2}$ such that for each $\nu \prec \mu$ $H(P_{m,\nu} \cap U_{\tilde{f}_1}) = P_{m,\nu} \cap U_{\tilde{f}_2}$ and $H(P_{m,\mu} \cap U_{\tilde{f}_1}) = P_{m,\mu} \cap U_{\tilde{f}_2}$.*

Proof of the lemma: Let $\mu = \{\mu_b, \mu_1, \dots, \mu_r, \mu_c\}$, $v_i \in (b, c)$ and $\omega_i \in (b, c)$ $(i = 1, \dots, r)$ be roots of $\tilde{f}_1$ and $\tilde{f}_2$ (correspondently) corresponding to μ_i. Take $\delta > 0$ such that for all $i = 1, \dots, r$ we have:

1) $[v_i - \delta, v_i + \delta] \cap [v_j - \delta, v_j + \delta] = \emptyset$ if $i \neq j$.
2) $[v_i - \delta, v_i + \delta] \cap [b, b + \delta] = \emptyset$.
3) $[v_i - \delta, v_i + \delta] \cap [c - \delta, c] = \emptyset$.

Let for this δ the same properties hold for ω_i and

4) $[v_i - \delta, v_i + \delta] \cap [w_j - \delta, w_j + \delta] = \emptyset$ if $w_j \neq v_i$.

Denote by $\Omega_1 = \bigcup_{i=1}^{r} [v_i - \delta, v_i + \delta)$, $\Omega_2 = \bigcup_{i=1}^{r} [w_i - \delta, w_i + \delta)$ and $\Omega = \Omega_1 \cup \Omega_2$. Let us define a permutation S of intervals on $[b, c]$ in the following way.

Step 1: Define S on $\Omega_1 : S(u) = u - v_i + w_i$ for $u \in [v_i - \delta, v_i + \delta)$.

Step 2: Define S on $\Omega_2 - \Omega_1$ such that $S\big|_{\Omega}$ be a permutation of intervals.

Step 3: Define $S(u) = u$ for $u \in [b, c] - \Omega$.

It is clear that $F_{S,\epsilon}(\tilde{f}_1) = \tilde{f}_2$ for every $\epsilon > 0$. By Proposition 2.5 we have $F_{S,\epsilon}(P_{m,\nu}) = P_{m,\nu}$ for every multiplicity symbol ν. By Theorem 2.3 there exist neighbourhoods $U_{\tilde{f}_1}$ and $U_{\tilde{f}_2}$ such that $H = F_{S,\epsilon}$ is a diffeomorphism of $U_{\tilde{f}_1}$ onto $U_{\tilde{f}_2}$. The lemma is proved. ■

Remark 2.5. The neighbourhoods $U_{\tilde{f}_1}$ and $U_{\tilde{f}_2}$ can be choosen such that for each $h \in U_{\tilde{f}_1} \cup U_{\tilde{f}_2}$ for sufficiently small $\delta' > 0$, $z(h) \subset (-\epsilon, \epsilon) \times (b - \delta', c + \delta')$.

Now take ϵ, δ' and neighbourhoods U_{g_1} and U_{g_2} such that:

1. For each $g \in U_{g_1} \cup U_{g_2}$ $z(g) \cap ((-\epsilon, \epsilon) \times (b - \delta', c + \delta')) = \emptyset$.
2. The map $L(g) = g - g_1 + g_2$ maps U_{g_1} onto U_{g_2}.

Let $U_{f_1} = U_{\tilde{f}_1} \cdot U_{g_1}$ and $U_{f_2} = U_{\tilde{f}_2} \cdot U_{g_2}$. Define a map $\psi : U_{f_1} \to U_{f_2}$ in the following way. For each $p \in U_{f_1}$ we have a unique presentation $p = \tilde{p} \cdot g$ such that $\tilde{p} \in U_{\tilde{f}_1}$, $g \in U_{g_1}$ (because the resultant of $\tilde{f}_1$ and g_1 is not equal to zero). Set $\psi(p) = H(\tilde{p}) \cdot L(g)$. Since H and L are local diffeomorphisms and the resultant of $\tilde{f}_2$ and g_2 is nondegenerate ψ is a diffeomorphism. By lemma it has required properties. Theorem 2.4 is proved. ■

Since the stratification is differentially trivial and semialgebraic it is Whitney regular (see, for example, [**GWPL**]). Hence we have

Theorem 2.5. *The stratification* $\{P_{n,\mu}\}_{\mu \in \mathcal{M}}$ *defined in section 1 is Whitney regular (satisfies the axioms "a" and "b" of Whitney).*

Theorem 2.6. *The stratifications defined in section 2.2 are Whitney regular.*

The proof is the same as the proof of Theorem 2.4.

3. Stratifications of complex monic polynomials

3.1. Multiplicity symbol.

Let $P_n(\mathbb{C})$ be a space of all monic polynomials of degree n with complex coefficients. Let $N \subset \mathbb{C}$ be a semialgebraic closed subset. We have $N = \mathrm{Int}(N) \cup \mathrm{Smooth}(\partial N) \cup \mathrm{Sing}(\partial N)$, where $\mathrm{Smooth}(\partial N)$ is a set of C^ρ-smooth points of ∂N (we can suppose the order of differentiability $0 \le \rho \le \infty$), $\mathrm{Sing}(\partial N) \overset{\mathrm{def}}{=} \partial N - \mathrm{Smooth}(\partial N)$. Since N is semialgebraic we have $\#\,\mathrm{Sing}(\partial N) < \infty$ for any ρ. The set $\mathrm{Smooth}(\partial N)$ can be obtained as a union of two connected components S_1 and S_2 defined in the following way: $S_1 = \{x \in \mathrm{Smooth}(\partial N)$ and there exists $\epsilon > 0$ such that $B(x, \epsilon) \cap \mathrm{Int}(N) \ne \emptyset\}$, $S_2 = \mathrm{Smooth}(\partial N) - S_1$ ($B(x, \epsilon)$ means a ball with the center x and radius ϵ).

Now let us define a *multiplicity symbol* $\mu(N) = \{\mu_1, \dots, \mu_{r_1}, \eta_1, \dots, \eta_{r_2}, \zeta_1, \dots, \zeta_{r_3}, \theta_1, \dots, \theta_{r_4}\}$ as a collection of natural numbers corresponding to the set N satisfy the following properties:

1. $r_1 \geq 0$, $r_2 \geq 0$, $r_3 \geq 0$, $r_4 = \#\operatorname{Sing}(N)$.
2. If $\operatorname{Int}(N) = \emptyset$ then $r_1 = r_2 = 0$.
3. If $S_1 = \emptyset$ then $r_2 = 0$.
4. If $S_2 = \emptyset$ then $r_3 = 0$.
5. $\sum_{i=1}^{r_1} \mu_i + \sum_{j=1}^{r_2} \eta_j + \sum_{s=1}^{r_3} \zeta_s + \sum_{k=1}^{r_4} \theta_k \leq n$.
6. $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{r_1}$,
   $0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_{r_2}$,
   $0 < \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_{r_3}$.

Denote by $\mathcal{M}_n(N)$ the set of all multiplicity symbols for N and n fixed.

Let $P_{n,\mu(N)}(\mathbb{C})$ be the set of polynomials $f \in P_n(\mathbb{C})$ such that

1. f has exactly r_1 roots $v_1 \neq v_2 \neq \cdots \neq v_{r_1}$ on $\operatorname{Int}(N)$ with multiplicities $\mu_1, \ldots, \mu_{r_1}$ (correspondently).
2. f has exactly r_2 roots $\tilde{v}_1 \neq \tilde{v}_2 \neq \cdots \neq \tilde{v}_{r_2}$ on S_1 with multiplicities $\eta_1, \ldots, \eta_{r_2}$.
3. f has exactly r_3 roots $v'_1 \neq v'_2 \neq \cdots \neq v'_{r_3}$ on S_2 with multiplicities $\zeta_1, \ldots, \zeta_{r_3}$.
4. f has a root at the point $y_k \in \operatorname{Sing}(N)$ with a multiplicity θ_k. (In the case $f(y_k) \neq 0$ set $\theta_k = 0$).

The main goal of this part is the following result.

Theorem 3.1. *The collection* $\{P_{n,\mu(N)}(\mathbb{C})\}_{\mu(N)\in\mathcal{M}_n(N)}$ *is a* C^ρ*-differentially trivial (and thus Whitney regular) stratification of* $P_n(\mathbb{C})$.

The proof contains several steps.

Definition 3.1. A zero-symbol $0(N)$ is a multiplicity symbol such that the first three parts μ, η and ζ are empty ($r_1 = r_2 = r_3 = 0$) and all θ-s are equal to zero.

Lemma 3.1. $P_{n,0(N)}$ *is an open semialgebraic subset of* $P_n(\mathbb{C})$.

Proof: Each polynomial $f \in P_n(\mathbb{C})$ can be presented in the following form:

$$f = (u - v_1)(u - v_2) \cdot \ldots \cdot (u - v_n), \tag{11}$$

$v_1, v_2, \ldots, v_n \in \mathbb{C}$ are the roots of f.

Let $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ be a "root vector". Consider a subset $V_{0(N)} \subset \mathbb{C}^n$ defined as follows

$$V_{0(N)} = \{v \in \mathbb{C}^n; v = (v_1, \dots, v_n), v_i \notin N \text{ for } i = 1, \dots, n\}.$$

Let $Q : \mathbb{C}^n \to P_n(\mathbb{C})$ be a map obtained by opening brackets in (11). We have $P_{n,0(N)}(\mathbb{C}) = Q(V_{0(N)})$. Thus $P_{n,0(N)}(\mathbb{C})$ is semialgebraic (by Tarski-Seidenberg Theorem). Since N is a closed subset of $\mathbb{C}$ $P_{n,0(N)}$ is open (by continuaty of roots [**BR**]). ■

Lemma 3.2. *For each $\mu(N) \in \mathcal{M}_n(N)$ the set $P_{n,\mu(N)}(\mathbb{C})$ is a semialgebraic subset and an immersed submanifold of $P_n(\mathbb{C})$.*

To prove the lemma we need one proposition which is in fact the statement of the lemma in some particular case.

Proposition 3.1. *Let $\mu(N) = \{(\mu_i), (\eta_j), (\zeta_s), (\theta_k)\} \in \mathcal{M}_n(N)$ and $\sum_{i=1}^{r_1} \mu_i + \sum_{j=1}^{r_2} \eta_j + \sum_{s=1}^{r_3} \zeta_s + \sum_{k=1}^{r_4} \theta_k = n$.*

Then $P_{n,\mu(N)}(\mathbb{C})$ is a semialgebraic subset and an immersed submanifold of $P_n(\mathbb{C})$.

Proof: Let $V_{\mu(N)} = \{v = (v_1, \dots, v_{r_1}, \tilde{v}_1, \dots, \tilde{v}_{r_2}, \dots, v'_1, \dots, v'_{r_3}\}$ be a subset of $\mathbb{C}^r$ $(r = r_1 + r_2 + r_3)$ obtained as a set of solutions of the following system of equations and inequalities

$$\begin{cases} v_i \in \operatorname{Int}(N), \\ \tilde{v}_j \in S_1, \\ v'_s \in S_2, \\ v_{i_1} \neq v_{i_2}, \tilde{v}_{j_1} \neq \tilde{v}_{j_2}, v'_{s_1} = v'_{s_2} \quad \text{for } i_1 \neq i_2, j_2 \neq j_2, s_1 \neq s_2. \end{cases}$$

By the definition $V_{\mu(N)}$ is a C^ρ semialgebraic submanifold of $\mathbb{C}^r$.

Let $F_{\mu(N)} : \mathbb{C}^r \to P_n(\mathbb{C})$ be a map defined in the following way

$$\begin{aligned} &F_{\mu(N)}(v_1, \dots, v_{r_1}, \tilde{v}_1, \dots, \tilde{v}_{r_2}, v'_1, \dots, v'_{r_3}) \\ &= (u - v_1)^{\mu_1} \cdot \dots \cdot (u - v_{r_1})^{\mu_{r_1}} (u - \tilde{v}_1)^{\eta_1} \cdot \dots \cdot (u - \tilde{v}_2)^{\eta_{r_2}} (u - v'_1)^{\zeta_1} \\ &\qquad \cdot \dots \cdot (u - v'_{r_3})^{\zeta_{r_3}} (u - y_1)^{\theta_1} \cdot \dots \cdot (u - y_{r_4})^{\theta_{r_4}} \end{aligned}$$

where $\operatorname{Sing}(\partial N) = \{y_1, \dots, y_{r_4}\}$. Observe that if $\theta_k = 0$ we have $(u - y_k)^{\theta_k} = 1$.

Clearly the map $F_{\mu(N)}$ is an algebraic map. The proof of the fact that $F_{\mu(N)}$ is a local immersion is the same as in Lemma 2.2. ■

Proof of Lemma 3.2: Now consider a general case:

$$\sum_{i=1}^{r_1}\mu_i+\sum_{j=1}^{r_2}\eta_j+\sum_{s=1}^{r_3}\zeta_s+\sum_{k=1}^{r_4}\theta_k\le n.$$

Each polynomial $f\in P_{n,\mu(N)}(\mathbb{C})$ can be presented in the following form:

$$f=g\cdot h,$$

where $g\in P_{m,\mu(N)}(\mathbb{C})$ such that $m=\sum_{i=1}^{r_1}\mu_i+\sum_{j=1}^{r_2}\eta_j+\sum_{s=1}^{r_2}\zeta_s+\sum_{k=1}^{r_4}\theta_k$ and $h\in P_{n-m,0(N)}(\mathbb{C})$.

We define a map $G_{\mu(N)}:V_{\mu(N)}\times P_{n-m,0(N)}(\mathbb{C})\to P_{n,\mu(N)}(\mathbb{C})$ in the following way:

$$G_{\mu(N)}(v,h)=F_{\mu(N)}(v)\cdot h.$$

By the same argument as in Lemma 2.2 we obtain that $G_{\mu(N)}$ is a local immersion. By Tarski-Seidenberg Theorem $P_{n,\mu(N)}(\mathbb{C})$ is a semialgebraic set. ■

Remark 3.1. In contrast to the real case the maps $F_{\mu(N)}$ and $G_{\mu(N)}$ are not necessary one-to-one. One can show that they are covering maps.

Remark 3.2. By the immediate calculations we obtain

$$\operatorname{Codim}P_{n,\mu(N)}=2\sum_{i=1}^{r_1}(\mu_i-1)+\sum_{j=1}^{r_2}(2\eta_j-1)+\sum_{s=1}^{r_3}(2\zeta_s-1)+2\sum_{k=1}^{r_4}\theta_k.$$

In the next sections we are going to prove the differential triviality of the partition $\{P_{n,\mu(N)}(\mathbb{C})\}$.

3.2. Disk-exchange transformations.

Let $N\subset\mathbb{C}$ be a closed semialgebraic subset. Let $\{x_1,\dots,x_p\}$ be a finite subset of $\operatorname{Int}(N)$. Consider $\epsilon>0$ such that

1. $B(x_i,\epsilon)\subset\operatorname{Int}(N)$ for all $i=1,\dots,p$.
2. $B(x_{i_1},\epsilon)\cap B(x_{i_2},\epsilon)=\emptyset$ for $i_1\ne i_2$.

Let $\{\tilde{x}_1,\dots,\tilde{x}_{t_1}\}$, $\{x'_1,\dots,x'_{i_2}\}$ be finite subsets of S_1 and S_2 correspondently and δ_1, δ_2 be numbers such that

3. Each pair $(B(\tilde{x}_j,\delta_1,B(\tilde{x}_j,\delta_1)\cap\partial N)$ is C^ρ-diffeomorphic to the pair $(B(0,1),B(0,1)\cap\mathbb{R})$.

Each pair $(B(x'_s,\delta_2), B(x'_s,\delta_2)\cap\partial N)$ is C^ρ-diffeomorphic to the pair $(B(0,1), B(0,1)\cap\mathbb{R})$.

4. $B(\tilde{x}_j,\delta_1)\cap B(x'_s,\delta_2)=\emptyset$ for all $j=1,\dots,t_1$, $s=1,\dots,t_2$.
5. $B(\tilde{x}_{j_1},\delta_1)\cap B(\tilde{x}_{j_2},\delta_1)=\emptyset$ for $j_1\neq j_2$.
   $B(x'_{s_1},\delta_2)\cap B(x'_{s_2},\delta_2)=\emptyset$ for $s_1\neq s_2$.
6. $B(x_i,\epsilon)\cap B(\tilde{x}_j,\delta_1)=\emptyset$
   $B(x_i,\epsilon)\cap B(x'_s,\delta_2)=\emptyset$ for all $i=1,\dots,p$, $j=1,\dots,t_1$, $s=1,\dots,t_2$.

Let

$$B=\left(\bigcup_{i=1}^{p}B(x_i,\epsilon)\right)\cup\left(\bigcup_{j=1}^{t_1}B(\tilde{x}_j,\delta_1)\right)\cup\left(\bigcup_{s=1}^{t_2}B(x'_s,\delta_2)\right).$$

Definition 3.2. A map $T:\mathbb{C}\to\mathbb{C}$ is called *a disk-exchange transformation associated to* N if it satisfies the following conditions:

1. T is a bijection.
2. $T\big|_{\mathbb{C}-B}=\text{id}$.
3. For each i_1 there exists i_2 such that $T\big|_{B(x_{i_2},\epsilon)}$ is a C^ρ-diffeomorphism of $B(x_{i_1},\epsilon)$ onto $B(x_{i_2},\epsilon)$ such that $T(x_{i_1})=x_{i_2}$.
4. For each j_1 (or s_1) there exists j_2 (or s_2, correspondently), such that $T\big|_{B(\tilde{x}_{j_1},\delta_1)}$ (or $T\big|_{B(x'_{s_1},\delta_2)}$) is a C^ρ-diffeomorphism onto $B(\tilde{x}_{j_2}\delta_1)$ (or $B(x'_{s_2},\delta_2)$) such that

$$\begin{aligned}&T(\tilde{x}_{j_1})=\tilde{x}_{j_2}\quad(\text{or } T(x'_{s_1})=x'_{s_2})\quad\text{and}\\&T(B(\tilde{x}_{j_1},\delta_1)\cap N)=B(\tilde{x}_{j_2},\delta_1)\cap N\\&\quad(\text{or } T(B(x'_{s_1},\delta_2)\cap N)=B(x'_{s_2},\delta_2)\cap N \text{ correspondently}).\end{aligned}$$

The points x_i, $\tilde{x}_j$, x'_s we call *centers* of the disk-exchange transformation.

Let us define a map $T_*:P_n(\mathbb{C})\to P_n(\mathbb{C})$ associated to a given disk-exchange transformation T. Let $f\in P_n(\mathbb{C})$ be presented in the following standard form:

$$f=(u-v_1)^{\alpha_1}\cdot\ldots\cdot(u-v_k)^{\alpha_k}.$$

Set $T_*(f)=(u-T(v_1))^{\alpha_1}\cdot\ldots\cdot(u-T(v_k))^{\alpha_k}$.

Proposition 3.2. *Let $f \in P_{n,\mu(N)}(\mathbb{C})$. Then for each disk-exchange transformation T associated to N we have $T_*(f) \in P_{n,\mu(N)}(\mathbb{C})$.*

This proposition follows from the fact that N, ∂N and $\mathbb{C} - N$ are invariant under the map T.

Remark 3.3. The disk-exchange transformations associated to N do not form a group for arbitrary chosen N.

Now we prove the theorem analogous to the Theorem 2.3.

Theorem 3.2. *Let $\mu(N) \in \mathcal{M}_m(N)$ and*

$$\sum_{i=1}^{r_1} \mu_i + \sum_{j=1}^{r_2} \eta_j + \sum_{s=1}^{r_3} \zeta_s + \sum_{k=1}^{r_4} \theta_k = m.$$

Let $f_1, f_2 \in P_{m,\mu(N)}(\mathbb{C})$.

Then

1) *There exists a disk-exchange transformation $T : \mathbb{C} \to \mathbb{C}$ such that $T_*(f_1) = f_2$.*
2) *There exist neighbourhoods U_{f_1} and U_{f_2} in $P_m(\mathbb{C})$ such that $T_* : U_{f_1} \to U_{f_2}$ is a C^ρ-diffeomorphism.*

Proof: 1) Let Z_1 be a zero set of f_1 and Z_2 be a zero set of f_2. We have $Z_q = Z_1^1 \cup Z_q^2 \cup Z_q^3 \cup Z_q^4$ ($q = 1$ or 2), where

$$Z_q^1 = Z_q \cap \mathrm{Int}(N), \quad Z_q^2 = Z_q \cap S_1, \quad Z_q^3 = Z_q \cap S_2, \quad Z_q^4 = Z_q \cap \mathrm{Sing}(\partial N).$$

Let

$$Z^1 = Z_1^1 \cup Z_2^1, \quad Z^2 = Z_1^2 \cup Z_2^2, \quad Z^3 = Z_1^3 \cup Z_2^3, \quad Z^4 = Z_1^4 \cup Z_2^4.$$

(Observe that by the definition of the multiplicity symbol we have $Z^4 = Z_1^4 = Z_2^4$).

Let $Z = Z_1 \cup Z_2$. Let $P : Z \to Z$ be a permutation of points satisfies the following conditions:

1. $P\big|_{Z^4} = \mathrm{id}$.
2. $P(Z_1^\ell) = Z_2^\ell$ ($\ell = 1, 2, 3, 4$).
3. If $v_i \in \mathrm{Int}(N)$ is a root of f_1 with multiplicity μ_i then $P(v_i)$ is a root of f_2 with the same multiplicity μ_i.
4. If $\tilde{v}_j \in S_1$ is a root of f_1 with multiplicity η_j then $P(\tilde{v}_j)$ is a root of f_2 with multiplicity η_j.
5. If $v'_s \in S_2$ is a root of f_1 with multiplicity ζ_s then $P(v'_s)$ is a root of f_2 with multiplicity ζ_s.

Let $Z^1 = \{x_1, \ldots, x_p\}$, $Z^2 = \{\tilde{x}_1, \ldots, \tilde{x}_{t_1}\}$, $Z^3 = \{x'_1, \ldots, x'_{r_2}\}$. Let ϵ, δ_1 and δ_2 be numbers such that conditions 1-6 from the definition of the disk-exchange transformation are satisfied. We define a disk-exchange transformation $T : \mathbb{C} \to \mathbb{C}$ with centers x_i, $\tilde{x}_j$ and x'_s by $T(x_i) = P(x_i)$, $T(\tilde{x}_j) = P(\tilde{x}_j)$ and $T(x'_s) = P(x'_s)$.

Clearly $T_*(f_1) = f_2$.

2) *Step 1:* Let us consider some particular case. Let $v^1_{i_0}, v^2_{i_0} \in \mathrm{Int}(N)$ be roots of f_1 and f_2 (correspondently) with multiplicity μ_{i_0}. Suppose that other roots of f_1 and f_2 are the same; $v^1_i = v^2_i$ $(i \neq i_0)$, $\tilde{v}^1_j = \tilde{v}^2_j$, $v'^1_s = v'^2_s$ (for all j, s). The disk-exchange transformation T constructed in the part 1) is just an interchanging of the two small disks with centers at $v^1_{i_0}$ and $v^2_{i_0}$ and radius ϵ. We have

$$f_1 = (u - v^1_{i_0})^{\mu_{i_0}} \cdot h \tag{12}$$

$$f_2 = (u - v^2_{i_0})^{\mu_{i_0}} \cdot h, \tag{13}$$

where $h \in P_{m-\mu_{i_0}}(\mathbb{C})$.

Let U^1_0 be a small neighbourhood of $(u-v^1_{i_0})^{\mu_{i_0}}$ in $P_{\mu_{i_0}}(\mathbb{C})$ and U_h be a small neighbourhood of h in $P_{m-\mu_{i_0}}(\mathbb{C})$ such that the map $\phi_1 : U^1_0 \times U_h \to P_m(\mathbb{C})$ obtained by "opening brackets" in (12) is a diffeomorphism to the image. (ϕ_1 is a local diffeomorphism because the Jacobian $J\phi_1$ at the point $((u - v^1_{i_0})^{\mu_{i_0}}, h)$ is a resultant of these polynomials [**J**] and they have not common roots). Analogically let U^2_0 be a small neighbourhood of $(u - v^2_{i_0})^{\mu_{i_0}}$ in $P_{\mu_0}(\mathbb{C})$ such that $\phi_2 : U^2_0 \times U_h \to P_m(\mathbb{C})$ obtained by "opening brackets" in (13) is a diffeomorphism to the image. Let $U_{f_1} = \phi_1(U^1_0 \times U_h)$ and $U_{f_2} = \phi_2(U^2_0 \times U_h)$. Let $\widetilde{T} : U^1_0 \to U^2_0$ be the map defined in the same way as T_* but in the space $P_{\mu_0}(\mathbb{C})$. Observe that $\widetilde{T}$ is a diffeomorphism. Thus in the coordinate system defined by ϕ_1 and ϕ_2 the map T_* can be written in the following way: $T_* = (\widetilde{T}, \mathrm{id})$. Since $\widetilde{T}$ is a diffeomorphism T_* is also diffeomorphism.

Step 2: Let $\tilde{v}^1_{j_0}$, $\tilde{v}^2_{j_0}$ be roots of f_1 and f_2 (correspondently) with multiplicity η_{j_0}. Suppose also (as in Step 1) that other roots of f_1 and f_2 are the same. The proof is absolutely the same as in Step 1.

Step 3: The same arguments for f_1 and f_2 with roots $v'^1_{s_0}$, $v'^2_{s_0}$ with multiplicity ζ_{s_0} and the same other roots.

Step 4: For every pair $f_1, f_2 \in P_{m,\mu(N)}(\mathbb{C})$ a disk-exchange transformation T such that $T_*(1) = f_2$ can be obtained as a composition of maps considered in Steps 1, 2, 3.

The theorem is proved. ■

3.3. Differential triviality.

Theorem 3.3. *The family* $\{P_{n,\mu(N)}(\mathbb{C})\}_{\mu(N)\in\mathcal{M}_n(N)}$ *is* C^ρ*-differentially trivial.*

Proof: Let $f_1, f_2 \in P_{n,\mu(N)}(\mathbb{C})$ we have:

$$f_1 = \tilde{f}_1 \cdot g_1, \tag{14}$$

$$f_2 = \tilde{f}_2 \cdot g_2, \tag{15}$$

where $\tilde{f}_1, \tilde{f}_2 \in P_{m,\mu(N)}(\mathbb{C})$ with $m = \Sigma\mu_i + \Sigma\eta_j + \Sigma\zeta_s + \Sigma\theta_k$, $g_1, g_2 \in P_{n-m,0(N)}(\mathbb{C})$.

Let T be a disk-exchange transformation connecting $\tilde{f}_1$ and $\tilde{f}_2$ constructed in Theorem 3.2. Let $U_{\tilde{f}_1}$ and $U_{\tilde{f}_2}$ be small neighbourhoods such that $T_* : U_{\tilde{f}_1} \to U_{\tilde{f}_2}$ is a C^ρ-diffeomorphism. Let U_{g_1} and U_{g_2} be two small open balls in $P_{n-m}(\mathbb{C})$ belonging to $P_{n-m,0(N)}(\mathbb{C})$. (These two balls exist because $P_{n-m,0(N)}(\mathbb{C})$ is an open subset of $P_{n-m}(\mathbb{C})$ by Lemma 3.1). Let $H : U_{g_1} \to U_{g_2}$ be a diffeomorphism (can be a translation map). Let $\psi_1 : U_{g_1} \times U_{\tilde{f}_1} \to P_n(\mathbb{C})$ and $\psi_2 : U_{g_2} \times U_{\tilde{f}_2} \to P_n(\mathbb{C})$ be maps defined by "opening brackets" in (14) and (15). ψ_1 and ψ_2 are local diffeomorphisms by the same argument as in Theorem 3.2. Set $U_{f_1} = \psi_1(U_{g_1} \times U_{\tilde{f}_1})$ and $U_{f_2} = \psi_2(U_{g_2} \times U_{\tilde{f}_2})$. Thus the map

$$G = \psi_2 \circ (H, T_*) \circ \psi_1^{-1} : U_{f_1} \to U_{f_2}$$

defines a required diffeomorphism. ■

End of the proof of Theorem 3.1: So, we obtained that the family $\{P_{n,\mu(N)}(\mathbb{C})\}_{\mu(N)\in\mathcal{M}_n(N)}$ satisfies the following conditions:

1. $\displaystyle\bigcup_{\mu(N)\in\mathcal{M}_n(n)} P_{n,\mu(N)}(\mathbb{C}) = P_n(\mathbb{C})$.
2. $P_{n,\mu(N)}(\mathbb{C})$ are immersed submanifolds and semialgebraic subsets of $P_n(\mathbb{C})$.
3. $P_{n,\mu^1(N)}(\mathbb{C}) \cap P_{n,\mu^2(N)}(\mathbb{C}) = \emptyset$ for $\mu^1(N) \neq \mu^2(N)$.
4. The family is C^ρ-differentially trivial.

Thus (by [**GPWL**]) we can conclude that $\{P_{n,\mu(N)}(\mathbb{C})\}_{\mu(N)\in\mathcal{M}_n(N)}$ is a Whitney regular stratification of $P_n(\mathbb{C})$. The theorem is proved. ■

4. Comparing of the stratifications

4.1. Comparing of real and complex cases.

Now consider $N = [a,b] \subset \mathbb{C}$. Thus we have: $\mathrm{Int}(N) = \emptyset$, $S_1 = \emptyset$, $S_2 = (a,b)$ and $\mathrm{Sing}(\partial N) = \{a,b\}$. It means that the multiplicity symbol $\mu^{\mathbf{C}}([a,b])$ defined in the part 3 (complex case) has the following form:

$$\mu^{\mathbf{C}}([a,b]) = \{\zeta_1, \ldots, \zeta_{r_3}, \theta_1, \theta_2\} \quad (\text{here } r_1 = r_2 = 0).$$

From the other side the multiplicity symbol $\mu^{\mathbf{R}}([a,b])$ corresponding to $[a,b]$ defined in the part 2 (real case) has the form: $\mu^{\mathbf{R}}([a,b]) = \{\mu_a, \mu_1, \ldots, \mu_r, \mu_b\}$.

Denote by $P_{n,\mu^{\mathbf{R}}([a,b])}(\mathbb{R})$ the stratum corresponding to the multiplicity symbol $\mu^{\mathbf{R}}([a,b])$ in real case, $P_{n,\mu^{\mathbf{C}}(N)}(\mathbb{C})$ the stratum corresponding to $\mu^{\mathbf{C}}([a,b])$ in complex case. Let $\mathcal{M}_n^{\mathbf{R}}([a,b])$ be a set of all multiplicity symbols $\mu^{\mathbf{R}}([a,b])$ and $\mathcal{M}_n^{\mathbf{C}}([a,b])$ be a set of all multiplicity symbols $\mu^{\mathbf{C}}([a,b])$.

Define a map $L : \mathcal{M}_n^{\mathbf{R}}([a,b]) \to \mathcal{M}_n^{\mathbf{C}}([a,b])$ by $L(\{\mu_a, \mu_1, \ldots, \mu_r, \mu_b\}) = \{\mu_1, \ldots, \mu_r, \mu_a, \mu_b\}$. Clearly L is a bijection. We have that $P_n(\mathbb{R})$ is a subset of $P_n(\mathbb{C})$.

Theorem 4.1. $P_{n,\mu^{\mathbf{R}}([a,b])}(\mathbb{R}) = P_{n,L(\mu^{\mathbf{R}}([a,b]))}(\mathbb{C}) \cap P_n(\mathbb{R})$. *This statement follows directly from the definitions of stratifications.*

The same result is true for a finite number of intervals.

4.2. Comparing of two complex stratifications.

Let $\{X_i\}$ and $\{Y_j\}$ be two stratifications of $\mathbb{R}^n$. We say that these *stratifications are C^ρ-equivalent* if there exists a C^ρ-diffeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ such that for each i there exists j such that $H(X_i) = Y_j$.

Remark 4.1. It is clear that if $\{X_i\}$ is C^ρ-equivalent to $\{Y_j\}$ that these stratifications has the same number of stratum and for each i the sets X_i and $Y_j = H(X_i)$ are C^ρ-diffeomorphic.

Let N_1 and N_2 are two semialgebraic subsets of $\mathbb{C}$. We say that N_1 *and N_2 are C^ρ-equivalent* if there exists a C^ρ-diffeomorphism $L : \mathbb{C} \to \mathbb{C}$ such that $L(N_1) = N_2$.

Let $f = (u - v_1)^{m_1}(u - v_2)^{m_2} \cdot \ldots \cdot (u - v_k)^{m_k} \in P_n(\mathbb{C})$. Define $L_*(f)$ as the following:

$$L_*(f) = (u - L(v_1))^{m_1}(u - L(v_2))^{m_2} \cdot \ldots \cdot (u - L(v_k))^{m_k}.$$

We obtain a map $L_* : P_n(\mathbb{C}) \to P_n(\mathbb{C})$.

Theorem 4.2.

1) *L_* is a C^ρ-diffeomorphism.*
2) *The stratifications*

$$\{P_{n,\mu(N_1)}(\mathbb{C})\}_{\mu(N_1)\in\mathcal{M}_n(N_1)} \text{ and } \{P_{n,\mu(N_2)}(\mathbb{C})\}_{\mu(N_2)\in\mathcal{M}_n(N_2)}$$

are C^ρ-equivalent.

3) *The equivalence of these stratifications is given by L_*.*

Proof: 1) This part follows from the properties of L_*: 1. L_* is a local diffeomorphism (the proof is the same as in Lemma 2.2). 2. L_* is a one-to-one map. 3. L_* is a proper map (see [**BR**]).

The proof of 2) and 3) is the same as the proof of Theorem 3.1 of the part 3.

Remark 4.2. If N_1 and N_2 have a different number of C^ρ-singular points then the stratifications $\{P_{n,\mu(N_1)}(\mathbb{C})\}$ and $\{P_{n,\mu(N_2)}(\mathbb{C})\}$ have different number of strata and (by Remark 4.1) can not be C^ρ-equivalent. It means that different semialgebraic subsets define strictly different stratifications of $P_n(\mathbb{C})$.

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