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ALMOST EVERYWHERE CONVERGENCE AND BOUNDEDNESS OF CESÀRO-α ERGODIC AVERAGES IN $L_{p,q}$-SPACES

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Abstract

Let (X, μ) be a σ-finite measure space and let τ be an ergodic invertible measure preserving transformation. We study the a.e. convergence of the Cesàro-α ergodic averages associated with τ and the boundedness of the corresponding maximal operator in the setting of $L_{p,q}(w\,d\mu)$ spaces.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let T be a positive linear operator on some Lorentz space $L_{p,q}(\mu)$, $1 < p \leq \infty$ and $1 \leq q \leq \infty$ or $p = q = 1$ (see [**7**] for the definition of the $L_{p,q}$ spaces). For every $f \in L_{p,q}(\mu)$ and every $\alpha \in (0, 1]$, the Cesàro-α averages of the sequence $\{T^i f\}_{i=0}^{\infty}$ and the corresponding Cesàro-α maximal operator are defined by

$$R_{n,\alpha} f = \frac{1}{A_n^\alpha} \sum_{i=0}^{n} A_{n-i}^{\alpha-1} T^i f \quad \text{and} \quad M_\alpha f = \sup_{n \in \mathbb{N}} |R_{n,\alpha} f|,$$

where $A_n^\alpha = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}$ and $A_0^\alpha = 1$ (see [**20**] or [**6**] for the properties of the coefficients A_n^α).

Observe that $R_{n,1} f$ is the usual average $\frac{1}{n+1} \sum_{i=0}^{n} T^i f$. In this case, $\alpha = 1$, and assuming that T is a positive linear contraction on some $L^p(\mu)$, $p > 1$, M. Akcoglu [**1**] proved that the averages $R_{n,1} f$ converge a.e. for all $f \in L^p(\mu)$. R. Irmisch [**8**] generalized Akcoglu's theorem to the $R_{n,\alpha}$ averages, $0 < \alpha < 1$. His theorem is the following:

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Theorem A [8]. *Let α and p be such that $0 < \alpha \leq 1$ and $\alpha p > 1$. Let $T : L^p(\mu) \to L^p(\mu)$ be a positive linear contraction. Then there exists $C > 0$ such that*

$$\int_X |M_\alpha f|^p \, d\mu \leq C \int_X |f|^p \, d\mu$$

and $R_{n,\alpha} f$ converges a.e. for all $f \in L^p(\mu)$.

In the limit case $\alpha p = 1$, Y. Deniel [**5**] gave an example showing that the theorem does not hold for the functions f in $L^{1/\alpha}(\mu)$. This left open the question of knowing what can be said if $\alpha p = 1$. Broise, Deniel and Derriennic [**3**] obtained that if $\alpha p = 1$ then a restricted weak type inequality holds for operators defined by composition with a measure preserving transformation. As a consequence, the a.e. convergence of the averages $R_{n,\alpha} f$ is established for functions f in the Lorentz-space $L_{1/\alpha,1}(\mu)$, which is contained in $L^{1/\alpha}(\mu)$. More precisely, they obtained the following result.

Theorem B [3]. *Let $(X, \mathcal{M}, \mu)$ be a probability measure space and assume that $\tau : X \to X$ is a measure preserving transformation. Let $Tf = f \circ \tau$. Then the maximal operator M_α applies the Lorentz space $L_{1/\alpha,1}(\mu)$ into $L_{1/\alpha,\infty}(\mu)$. Furthermore, the sequence $R_{n,\alpha} f$ converges a.e. for all $f \in L_{1/\alpha,1}(\mu)$.*

Moreover, in the same paper, they proved that if $0 < \alpha < 1$, τ is ergodic and $f \notin L_{1/\alpha,1}(\mu)$, then there exists a function g with the same distribution function as f such that the averages $R_{n,\alpha} g$ do not converge a.e. Notice that if $\alpha = 1$ then $L_{1/\alpha,1}(\mu) = L^1(\mu)$ and therefore, in this case, Theorem B is nothing but the well known weak type $(1,1)$ inequality for the ergodic maximal operator and the a.e. convergence of the usual averages for functions in $L^1(\mu)$.

As we see, the Lorentz space $L_{1/\alpha,1}(\mu)$ plays a key role in the study of the convergence of the averages $R_{n,\alpha}$. Therefore, it is interesting to study the behaviour of these averages on the Lorentz spaces $L_{p,q}$. In this way we arrive to the goal of this paper: to characterize the boundedness of the ergodic maximal operator M_α on $L_{p,q}(\omega \, d\mu)$-spaces and to study the a.e. convergence of the averages $R_{n,\alpha} f$, $f \in L_{p,q}(\omega \, d\mu)$, associated with operators $Tf = f \circ \tau$ where τ is an ergodic invertible transformation, ω is a positive measurable function and the measure μ is preserved by τ. Also, we remark that we only consider measures ν of the form $d\nu = \omega \, d\mu$ because it is known [**11**] that, if τ is an invertible measurable and non singular transformation with respect to a finite measure ν, i.e., $\nu(E) = 0 \Rightarrow \nu(\tau^{-1} E) = 0$, and the averages $R_{n,1} f$ converge a.e. for every $f \in L^p(d\nu)$, then the measure ν is equivalent to a finite measure μ which is preserved by τ. Having into account that the a.e. convergence of the averages $R_{n,\alpha}f$, $0 < \alpha < 1$, implies the a.e. convergence of the averages $R_{n,1}f$ (see [**20**]), the above result remains valid for the case $0 < \alpha < 1$.

It is worth noting that the problem we are going to consider here was studied in [**12**] in the setting of Lebesgue spaces $L^p(\omega\, d\mu) = L_{p,p}(\omega\, d\mu)$ and for more general operators. Keeping in mind Brunel's theorem [**4**], one could expect that the boundedness of the maximal operator M_α in $L^p(\omega\, d\mu)$ is equivalent to the uniform boundedness in $L^p(\omega\, d\mu)$ of the averages, but this is not the case if $0 < \alpha < 1$, as it was shown in [**12**]. However, one can consider a countable family of a kind of Cesàro-α averages for which the equivalence holds. In this paper we prove that the equivalence holds also in the Lorentz spaces $L_{p,q}(\omega\, d\mu)$ and characterize in terms of the weight ω the boundedness of M_α in $L_{p,q}$-spaces. The results that we obtain in the case $\alpha = 1$ can be considered as particular cases of those ones obtained by P. Ortega in [**16**] for the averages $R_{n,1}f$, $f \in L_{p,q}(\omega\, d\mu)$.

In order to introduce the kind of Cesàro averages to be considered we need to state some definitions and results.

Definition 1.1 (Definition 4.1 in [12]). If B is a measurable subset and $x \in \cup_{j=0}^{\infty} \tau^{-j} B$ we define

$$n_B(x) = \inf\{k \geq 0 : \tau^k x \in B\}$$

and

$$L_B(x) = \begin{cases} \sup\{j \geq 1 : \tau^{-1}x, \dots, \tau^{-j}x \notin B\}, & \\ & \text{if } \{j \geq 1 : \tau^{-1}x, \dots, \tau^{-j}x \notin B\} \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $L_B(x)$ can take the value $+\infty$.

Definition 1.2 (Definition 4.2 in [12]). If B is a measurable subset we define the average $R_{B,\alpha}f$ as

$$R_{B,\alpha}f(x) = \begin{cases} (A^\alpha_{n_B(x)})^{-1} \sum_{i=0}^{n_B(x)} A^{\alpha-1}_{n_B(x)-i} T^i f(x), & \text{if } x \in \cup_{j=0}^{\infty} \tau^{-j} B \\ 0, & \text{otherwise.} \end{cases}$$

Notice that

$$\sup_{B \in \mathcal{F}} |R_{B,\alpha}f(x)| \leq M_\alpha f(x). \tag{1.1}$$

It has been proved in [**12**, Proposition 4.5] that the equality holds taking the supremum over a certain countable family $\mathcal{B}$ of sets. In what follows we introduce this family.

Definition 1.3 (Definition 2.8 in [9]). Let k be a natural number. The measurable set $B \subset X$ is said to be the base of an (ergodic) rectangle of length $k+1$ if $\tau^i B \cap \tau^j B = \emptyset$ whenever $i \neq j$, $0 \leq i, j \leq k$. In such a case the set $R = \cup_{i=0}^{k} \tau^i B$ will be called an (ergodic) rectangle with base B and length $k+1$.

The bases of ergodic rectangles have the following nice property:

Proposition 1.1 (Corollaries 2.12 and 2.13 in [9]). *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space which is nonatomic if $\mu(X) < \infty$ and let τ be an ergodic, invertible measure preserving transformation from X onto itself. Then for every nonnegative integer k there exists a countable family of bases of ergodic rectangles of length $k+1$, $\{B_n^{(k)} : n \in \mathbb{N}\}$, such that*

$$X = \cup_n B_n^{(k)}.$$

We shall denote by $\mathcal{B}$ to the family $\{\tau^k(B_n^{(k)}) : k,\, n \in \mathbb{N}\}$.

As a consequence of Proposition 1.1 we obtained in [**12**] that the family $\mathcal{B}$ is enough to obtain the equality in (1.1).

Proposition 1.2 (Proposition 4.5 in [12]). *With the above notations and assumptions we have that*

$$\sup_{B \in \mathcal{B}} |R_{B,\alpha} f(x)| = M_\alpha f(x) \quad \textit{for almost every } x \in X.$$

The paper is organized as follows: we state the main results in section 2 while we establish in section 3 some results for the Cesàro maximal operator in the integers which are necessary ingredients in the proof of the theorems. Finally, in section 4 we give the proofs of our results.

We finish this section with some notations that we shall use throughout the paper. The letter C will mean a positive constant not necessarily the same at each occurrence and if $1 < p < \infty$ then p' will stand for the conjugate exponent of p, i.e., $p + p' = pp'$. We shall also denote by $\mathcal{B}$ the family $\{\tau^k(B_n^{(k)}) : k,\, n \in \mathbb{N}\}$, where $\{B_n^k : n \in \mathbb{N}\}$ are fixed families given by Proposition 1.1. If u is a positive function on X the $L_{p,q}$ norm of f with respect to $u\,d\mu$ is denoted by $\|f\|_{p,q;u\,d\mu}$ or simply $\|f\|_{p,q;u}$. If a is a positive function on the integers then $\|f\|_{p,q;a}$ stands for the $L_{p,q}$ norm of the function f on the integers with respect to the measure ν given by $\nu(\{n\}) = a(n)$. Finally, if u is a function on X and $x \in X$ then u^x is the function on $\mathbb{Z}$ given by $u^x(i) = u(\tau^i x)$.

2. Main results

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space which is non atomic if $\mu(X) < \infty$. Let $\tau : X \to X$ be an invertible ergodic measure preserving transformation and let M_α and $R_{n,\alpha}$ be the Cesàro-α ergodic maximal operator and the Cesàro-α averages, respectively, associated with the operator T defined by composition with τ, i.e., $Tf = f \circ \tau$.

In Theorem 2.1 we characterize the boundedness of M_α from $L_{p,q}(v\,d\mu)$ to $L_{p,\infty}(u\,d\mu)$, $1 \le q \le p < \infty$, where u and v are positive measurable functions, and we prove that if this boundedness holds then the averages $R_{n,\alpha}f$ converge a.e. for all $f \in L_{p,q}(v\,d\mu)$. As in the case $p = q$ which was studied in [**12**], the boundedness of M_α is not only equivalent to the uniform boundedness from $L_{p,q}(v\,d\mu)$ to $L_{p,\infty}(u\,d\mu)$, $1 \le q \le p < \infty$, of the countable family of Cesàro-α averages $\{R_{B,\alpha} : B \in \mathcal{B}\}$ introduced in section 1, but also to a condition on the pair (u, v) that we introduce in the next definition.

Definition 2.1. A pair (u, v) of positive measurable functions on X verifies the condition $A^+_{p,q;\alpha}(\tau)$ (or belongs to the class $A^+_{p,q;\alpha}(\tau)$), $0 < \alpha \le 1$, $1 < p < \infty$ and $1 \le q \le \infty$ or $1 = p = q$, if there exists a positive constant C such that for a.e. $x \in X$

$$\|\chi_{[0,r]}\|_{p,q;u^x} \|\chi_{[r,k]}(v^x)^{-1} A^{\alpha-1}_{k-\cdot}\|_{p',q';v^x} \le C A^\alpha_k, \tag{2.1}$$

for all natural numbers r and k with $0 \le r \le k$, where $A^{\alpha-1}_{k-\cdot}$ stands for the function defined on the integer numbers by $A^{\alpha-1}_{k-\cdot}(i) = A^{\alpha-1}_{k-i}\chi_{(-\infty,k]}(i)$. (Keep in mind that the $L_{p,q}$ norms used in this definition are on $\mathbb{Z}$.)

The condition that we give in Definition 2.1 is equivalent to the ones obtained changing $[0, r]$ and $[r, k]$, with r and k natural numbers, by $[j, j + r]$ and $[j + r, j + k]$, respectively, j being every integer number.

If $u = v$, we shall only say that u satisfies or verifies $A^+_{p,q;\alpha}(\tau)$.

Finally, we observe that in the case $p = q$ and $u = v = \omega$ the $A^+_{p,q;\alpha}(\tau)$ condition coincides with the condition (3) of Theorem 4.6 in [**12**] and in the case $\alpha = 1$ it is just the condition $A^+_{p,q}(\tau)$ introduced by P. Ortega in [**16**].

Now we can state Theorem 2.1 which characterizes the weak type boundedness on certain Lorentz-spaces.

Theorem 2.1. *Let $(X, \mathcal{F}, \mu)$, τ and $\mathcal{B}$ be as above. Let $0 < \alpha \leq 1$, $1 \leq q \leq p < \infty$ and u and v be positive measurable functions. The following statements are equivalent:*

(i) *There exists a positive constant C such that*

$$\|M_\alpha f\|_{p,\infty;u} \leq C\|f\|_{p,q;v},$$

for every function f on $L_{p,q}(v\,d\mu)$.

(ii) *There exists a positive constant C such that*

$$\sup_{B\in\mathcal{B}} \|R_{B,\alpha} f\|_{p,\infty;u} \leq C\|f\|_{p,q;v},$$

for every function f on $L_{p,q}(v\,d\mu)$.

(iii) *The pair (u, v) satisfies the condition $A^+_{p,q;\alpha}(\tau)$.*

Furthermore, if any of the above conditions is satisfied, then, for every $f \in L_{p,q}(v\,d\mu)$, the averages $R_{n,\alpha} f$ converge almost everywhere.

In the single weight case we have the following theorem which gives the equivalence between the strong type and the weak type boundedness of M_α.

Theorem 2.2. *Let $(X, \mathcal{F}, \mu)$, τ, $\mathcal{B}$ and α be as in Theorem 2.1. Let $1 < p < \infty$, $1 < q < \infty$ and ω be a positive measurable function. The following statements are equivalent:*

(i) *There exists a positive constant C such that*

$$\|M_\alpha f\|_{p,\infty;\omega} \leq C\|f\|_{p,q;\omega},$$

for every function f on $L_{p,q}(\omega\,d\mu)$.

(ii) *There exists a positive constant C such that*

$$\sup_{B\in\mathcal{B}} \|R_{B,\alpha} f\|_{p,\infty;\omega} \leq C\|f\|_{p,q;\omega},$$

for every function f on $L_{p,q}(\omega\,d\mu)$.

(iii) *There exists a positive constant C such that*

$$\|M_\alpha f\|_{p,q;\omega} \leq C\|f\|_{p,q;\omega},$$

for every function f on $L_{p,q}(\omega\,d\mu)$.

(iv) *There exists a positive constant C such that*

$$\sup_{B\in\mathcal{B}} \|R_{B,\alpha} f\|_{p,q;\omega} \leq C\|f\|_{p,q;\omega},$$

for every function f on $L_{p,q}(\omega\, d\mu)$.

(v) *ω satisfies the condition $A^+_{p,q;\alpha}(\tau)$.*

(vi) *ω satisfies the condition $A^+_{p;\alpha}(\tau)$, i.e., ω satisfies the condition $A^+_{p,p;\alpha}(\tau)$.*

As we said above, we need some notation and several previous results for the proof of these theorems. The next section is devoted to state them.

3. Previous results

The results that we are going to need are those ones which characterize the boundedness of the maximal operator m^+_α associated with the Cesàro-α averages of functions on the set of the integer numbers.

Definition 3.1. Let $0 < \alpha \leq 1$. If a is a real-valued function on $\mathbb{Z}$, we define the Cesàro-α maximal function $m^+_\alpha a$ by

$$m^+_\alpha a(i) = \sup_{n\geq 0} \frac{1}{A^\alpha_n} \left| \sum_{j=0}^{n} A^{\alpha-1}_{n-j} a(i+j) \right|, \quad i \in \mathbb{Z}. \tag{3.1}$$

Definition 3.2. A pair (u,v) of positive functions on $\mathbb{Z}$ verifies the condition $A^+_{p,q;\alpha}(\mathbb{Z})$ (or belongs to the class $A^+_{p,q;\alpha}(\mathbb{Z})$), $0 < \alpha \leq 1$, $1 < p < \infty$ and $1 \leq q \leq \infty$ or $1 = p = q$, if there exists a positive constant C such that

$$\|\chi_{[r,s]}\|_{p,q;u} \|\chi_{[s,k]} v^{-1} A^{\alpha-1}_{k-\cdot}\|_{p',q';v} \leq C A^\alpha_k, \tag{3.2}$$

for every integer numbers r, s and k with $r \leq s \leq k$.

Lemma 3.1 ([18]). *Let u and v be two positive functions on $\mathbb{Z}$. Let $1 \leq q \leq p < \infty$ and $0 < \alpha \leq 1$. Then the following statements are equivalent:*

(i) *There exists a positive constant C such that*

$$\|m^+_\alpha a\|_{p,\infty;u} \leq C\|a\|_{p,q;v},$$

for every function a defined on $\mathbb{Z}$.

(ii) *The pair (u,v) satisfies the condition $A^+_{p,q;\alpha}(\mathbb{Z})$.*

In the case of equal weights we have:

Lemma 3.2 ([18]). *Let ω be a positive function defined on $\mathbb{Z}$. Let $1 < p < \infty$, $1 < q < \infty$ and $0 < \alpha \leq 1$. Then the following statements are equivalent:*

(i) *There exists a positive constant C such that*

$$\|m_\alpha^+ a\|_{p,\infty;\omega} \leq C\|a\|_{p,q;\omega},$$

for every function a defined on $\mathbb{Z}$.

(ii) *There exists a positive constant C such that*

$$\|m_\alpha^+ a\|_{p,q;\omega} \leq C\|a\|_{p,q;\omega},$$

for every function a defined on $\mathbb{Z}$.

(iii) *ω satisfies the condition $A^+_{p,q;\alpha}(\mathbb{Z})$, i.e., the pair (ω,ω) satisfies the condition $A^+_{p,q;\alpha}(\mathbb{Z})$.*

(iv) *ω satisfies the condition $A^+_{p;\alpha}(\mathbb{Z})$, i.e., ω satisfies $A^+_{p,p;\alpha}(\mathbb{Z})$.*

Lemma 3.1 and Lemma 3.2 are particular cases of Theorems 2.7 and 2.16 in [**18**], respectively.

The next result states a relationship between the classes $A^+_{p;\alpha}(\mathbb{Z})$ and the classical ones $A^+_p(\mathbb{Z}) = A^+_{p;1}(\mathbb{Z})$; it gives also the analogue in our setting of the implication $\omega \in A^+_p(\mathbb{Z}) \Rightarrow \omega \in A^+_{p-\varepsilon}(\mathbb{Z})$ (see [**15**], [**19**], [**13**] and [**10**]).

Lemma 3.3 (Lemma 2.4 in [12]). *Let ω be a positive function on $\mathbb{Z}$. Let $0 < \alpha \leq 1$ and $p > 1$. We have:*

(1) *If $\omega \in A^+_{p;\alpha}(\mathbb{Z})$ with a constant C, then there exists $\varepsilon > 0$, which depends only on C, such that $\omega \in A^+_{p-\varepsilon,\alpha}(\mathbb{Z})$. Furthermore, ω is also an $A^+_p(\mathbb{Z})$-weight with the same constant C.*

(2) *If $\alpha p > 1$ and $\omega \in A^+_{\alpha p}(\mathbb{Z})$, then ω is also in $A^+_{p;\alpha}(\mathbb{Z})$.*

For the proof, just look at the corresponding proof in [**14**] and write it in the setting of the integer numbers.

4. Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1: The proof of this theorem follows the idea of the proof of Theorem 3 in [**16**]. For the proof of (ii) $\Rightarrow$ (iii) we shall need the following lemma which is based on Lemma 10 in [**16**]. Its proof is similar to Ortega's lemma and will be omitted.

Lemma 4.1. *Let $0 < \alpha \leq 1$, $s, k \in \mathbb{N}$ with $s \leq k$ and let B be a measurable set. For every $x \in B$ and $n \in \mathbb{Z}$, let $H_n^x = \{i \in [s,k] : v^{-1}(\tau^i x) A_{k-i}^{\alpha-1} > 3^n\}$. Let $\mathcal{G}$ be the collection of all decreasing sequences in $\mathbb{Z} \cup \{-\infty\}$ with at most 2^{k-s+1} different terms and at least one term in $\mathbb{Z}$. If $\gamma = \{\gamma_n\} \in \mathcal{G}$, let G_γ^B be the set defined as follows*

$$G_\gamma^B = \left\{ x \in B : H_n^x = \emptyset \text{ if } \gamma_n = -\infty \text{ and } \right.$$

$$\left. 2^{\gamma_n} < \sum_{j \in H_n^x} v(\tau^j x) \leq 2^{\gamma_n + 1} \text{ if } \gamma_n \neq -\infty \right\}.$$

Then $\{G_\gamma^B\}_{\gamma \in \mathcal{G}}$ is a countable family of pairwise disjoint sets and $\cup_{\gamma \in \mathcal{G}} G_\gamma^B = B$.

The implication (i) $\Rightarrow$ (ii) is clear.

Proof of (ii) $\Rightarrow$ (iii): Let $r, k \in \mathbb{N}$ with $r \leq k$ and let $\{B_n^{(k)}\}_{n \in \mathbb{N}}$ be the sequence of bases of ergodic rectangles of length $k+1$ associated with X and k by Proposition 1.1. For fixed $B_i = B_i^{(k)}$, let $\{G_\gamma^{B_i}\}_{\gamma \in \mathcal{G}}$ be the decomposition of B_i for $s = r$ given by Lemma 4.1 and, for every $(n_0, n_1, \dots, n_k) \in \mathbb{Z}^{k+1}$ and every $\gamma \in \mathcal{G}$, let us consider the set

$$H_{n_0, n_1, \dots, n_k}^\gamma = \{x \in G_\gamma^{B_i} : 2^{n_j} < v(\tau^j x) \leq 2^{n_j + 1}, \quad j = 0, 1, \dots, k\}.$$

It is clear that the sets $H_{n_0, n_1, \dots, n_k}^\gamma$ are measurable and pairwise disjoint sets with union equals $G_\gamma^{B_i}$.

We will prove that for fixed $(n_0, n_1, \dots, n_k)$ and γ, and almost every $x \in H_{n_0, n_1, \dots, n_k}^\gamma$, the pair (u, v) satisfies condition (2.1) with a constant C independent of $H_{n_0, n_1, \dots, n_k}^\gamma$. Then, since $\cup_{(n_0, n_1, \dots, n_k)} H_{n_0, n_1, \dots, n_k}^\gamma = G_\gamma^{B_i}$, $\cup_\gamma G_\gamma^{B_i} = B_i$ and $\cup_i B_i = X$, we shall have that condition (2.1) is satisfied for almost every $x \in X$ with a constant C independent of x and, therefore, the pair (u, v) satisfies the condition $A_{p,q;\alpha}(\tau)$.

We shall start proving that given $H_{n_0, n_1, \dots, n_k}^\gamma$, then for every measurable subset E of $H_{n_0, n_1, \dots, n_k}^\gamma$ we have

$$\left\| \chi_{\cup_{j=0}^r \tau^j E} \right\|_{p,q;u} \left\| \sum_{j=r}^k \chi_{\tau^j E} v^{-1} A_{k-j}^{\alpha-1} \right\|_{p',q';v} \leq C A_k^\alpha \mu(E), \tag{4.1}$$

with a constant C independent of E and $H_{n_0, n_1, \dots, n_k}^\gamma$.

The proof of (4.1) is based on (ii) and uses the following inequality

$$\left\| \sum_{j=r}^{k} \chi_{\tau^j E} v^{-1} A_{k-j}^{\alpha-1} \right\|_{p',q';v} \leq C \mu(E)^{1/p'} \left\| \chi_{[r,k]} \omega^{-1} A_{k-\cdot}^{\alpha-1} \right\|_{p',q';\omega}, \tag{4.2}$$

where ω is the function defined on $\mathbb{Z}$ by $\omega(j) = 2^{n_j} \chi_{[r,k]}(j)$ and the constant C is independent of E and $H^{\gamma}_{n_0,n_1,\dots,n_k}$.

Proof of 4.2: Using the definitions of $\| \quad \|_{p',q';v}$ and $H^{\gamma}_{n_0,n_1,\dots,n_k}$, we have that, for $q' < \infty$,

$$\begin{aligned}
&\left\| \sum_{j=r}^{k} \chi_{\tau^j E} v^{-1} A_{k-j}^{\alpha-1} \right\|_{p',q';v} \\
&= \left[q' \int_0^\infty \left(\int_{\{x \in X : \sum_{j=r}^{k} \chi_{\tau^j E}(x) v^{-1}(x) A_{k-j}^{\alpha-1} > y\}} v(x)\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{1/q'} \\
&= \left[q' \int_0^\infty \left(\sum_{j=r}^{k} \int_{\{x \in \tau^j E : v^{-1}(x) A_{k-j}^{\alpha-1} > y\}} v(x)\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{1/q'} \\
&= \left[q' \int_0^\infty \left(\int_E \sum_{\{j \in [r,k] : (v^x)^{-1}(j) A_{k-j}^{\alpha-1} > y\}} v^x(j)\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{1/q'} \\
&\leq \left[q' \int_0^\infty \left(\int_E \sum_{\{j \in [r,k] : 2^{-n_j} A_{k-j}^{\alpha-1} > y\}} 2^{n_j+1}\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{1/q'} \\
&= 2^{1/p'} \mu(E)^{1/p'} \left[q' \int_0^\infty \left(\sum_{\{j \in [r,k] : 2^{-n_j} A_{k-j}^{\alpha-1} > y\}} 2^{n_j} \right)^{q'/p'} y^{q'-1}\, dy \right]^{1/q'} \\
&= 2^{1/p'} \mu(E)^{1/p'} \left\| \chi_{[r,k]} \omega^{-1} A_{k-\cdot}^{\alpha-1} \right\|_{p',q';\omega}
\end{aligned}$$

and, for $q' = \infty$,

$$\Big\| \sum_{j=r}^{k} \chi_{\tau^j E} v^{-1} A_{k-j}^{\alpha-1} \Big\|_{p',\infty;v}$$

$$= \sup_{y>0} y \left(\int_E \sum_{\{j\in[r,k]:(v^x)^{-1}(j)A_{k-j}^{\alpha-1}>y\}} v^x(j)\, d\mu(x) \right)^{1/p'}$$

$$\le \sup_{y>0} y \left(\int_E \sum_{\{j\in[r,k]:2^{-n_j}A_{k-j}^{\alpha-1}>y\}} 2^{n_j+1}\, d\mu(x) \right)^{1/p'}$$

$$= 2^{1/p'} \mu(E)^{1/p'} \left\| \chi_{[r,k]} \omega^{-1} A_{k-.}^{\alpha-1} \right\|_{p',\infty;\omega}.$$

This finishes the proof of (4.2). ■

Now, using an argument of duality, we obtain that there exists $\omega' \ge 0$ with $\|\omega'\|_{p,q;\omega} = 1$ such that

$$\|\chi_{[r,k]}\omega^{-1}A_{k-.}^{\alpha-1}\|_{p',q';\omega} \le C \sum_{j=-\infty}^{+\infty} \chi_{[r,k]}(j)\omega^{-1}(j)A_{k-j}^{\alpha-1}\omega'(j)\omega(j) \tag{4.3}$$

$$= C \sum_{j=r}^{k} \omega'(j) A_{k-j}^{\alpha-1}.$$

It follows from (4.2) and (4.3) that

$$\Big\| \sum_{j=r}^{k} \chi_{\tau^j E} v^{-1} A_{k-j}^{\alpha-1} \Big\|_{p',q';v} \le C\mu(E)^{1/p'} \sum_{j=r}^{k} \omega'(j) A_{k-j}^{\alpha-1}. \tag{4.4}$$

Let f be the function defined on X by

$$f(x) = \sum_{j=r}^{k} \omega'(j) \chi_{\tau^j E}(x). \tag{4.5}$$

For this function, the inequality

$$\|f\|_{p,q;v} \le C\mu(E)^{1/p} \tag{4.6}$$

can be proved as in [**16**]. We have also that

$$\cup_{j=0}^{r} \tau^j E \subset \left\{ x \in X : R_{B,\alpha} f(x) > \frac{1}{A_k^\alpha} \sum_{j=r}^{k} \omega'(j) A_{k-j}^{\alpha-1} \right\}, \tag{4.7}$$

where $B = \tau^k(B_i)$ (the proof is given below).

Using (4.7) and the weak type boundedness of the operator $R_{B,\alpha}$ with a constant C independent of B, we obtain

$$\begin{aligned} \int_{\cup_{j=0}^{r}\tau^j E} u \, d\mu &\le C \frac{(A_k^\alpha)^p}{\left(\sum_{j=r}^{k} \omega'(j) A_{k-j}^{\alpha-1}\right)^p} \|f\|_{p,q;v}^p \\ &\le C \frac{(A_k^\alpha)^p}{\left(\sum_{j=r}^{k} \omega'(j) A_{k-j}^{\alpha-1}\right)^p} \mu(E), \end{aligned} \tag{4.8}$$

where we have used (4.6) for the last inequality. Finally, (4.8) together with (4.4) give (4.1).

In order to prove (4.7) we observe that if $B = \tau^k(B_i)$, then $n_B(x) = k - j$ for $x \in \tau^j E$, $0 \le j \le r$. Therefore, for fixed j, $0 \le j \le r$, and for each $x = \tau^j y$ with $y \in E$, we have

$$\begin{aligned} R_{B,\alpha} f(x) &= \frac{1}{A_{k-j}^\alpha} \sum_{s=0}^{k-j} A_{k-j-s}^{\alpha-1} f(\tau^{j+s} y) \\ &= \frac{1}{A_{k-j}^\alpha} \sum_{s=j}^{k} A_{k-s}^{\alpha-1} f(\tau^{s} y) \\ &\ge \frac{1}{A_{k-j}^\alpha} \sum_{s=r}^{k} A_{k-s}^{\alpha-1} f(\tau^{s} y) \\ &= \frac{1}{A_{k-j}^\alpha} \sum_{s=r}^{k} A_{k-s}^{\alpha-1} \omega'(s) \\ &\ge \frac{1}{A_{k}^\alpha} \sum_{s=r}^{k} \omega'(s) A_{k-s}^{\alpha-1}. \end{aligned}$$

Consequently, (4.7) holds and, hence, the proof of (4.1) is finished. ■

Notice that (4.1) can be written in the following way

(4.9)

$$\left[\int_E \sum_{j=0}^{r} u(\tau^j x)\, d\mu(x)\right]^{1/p} \left[q' \int_0^\infty \left(\int_E \sum_{j\in H_y^x} v(\tau^j x)\, d\mu(x)\right)^{q'/p'} y^{q'-1}\, dy\right]^{1/q'} \le C A_k^\alpha \mu(E),$$

where H_y^x is defined by $H_y^x = \{j \in [r,k] : v^{-1}(\tau^j x) A_{k-j}^{\alpha-1} > y\}$. Now, using (4.9), we shall obtain

(4.10)

$$\left(\int_E \sum_{j=0}^{r} u(\tau^j x)\, d\mu(x)\right)^{p'/p} \int_E \left[q' \int_0^\infty \left(\sum_{j\in H_y^x} v(\tau^j x)\right)^{q'/p'} y^{q'-1}\, dy\right]^{p'/q'} d\mu(x) \le C\,(A_k^\alpha)^{p'}\, \mu(E)^{p'}$$

and this will finish the proof. ∎

Keeping in mind (4.9), it is clear that (4.10) will follow from

$$\int_E \left[q' \int_0^\infty \left(\sum_{j\in H_y^x} v(\tau^j x)\right)^{q'/p'} y^{q'-1}\, dy\right]^{p'/q'} d\mu(x) \le C \left[q' \int_0^\infty \left(\int_E \sum_{j\in H_y^x} v(\tau^j x)\, d\mu(x)\right)^{q'/p'} y^{q'-1}\, dy\right]^{1/q'}. \tag{4.11}$$

Consequently, it remains to prove (4.11). This is what we do now.

Proof of (4.11): We have

$$\int_E \left[q' \int_0^\infty \left(\sum_{j\in H_y^x} v(\tau^j x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'} d\mu(x)$$

$$\leq \int_E \left[q' \sum_{n=-\infty}^{\infty} \int_{3^n}^{3^{n+1}} \left(\sum_{\{j\in[r,k]: v^{-1}(\tau^j x) A_{k-j}^{\alpha-1} > 3^n\}} v(\tau^j x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'} d\mu(x)$$

$$= C \int_E \left[q' \sum_{n=-\infty}^{\infty} \int_{3^{n-1}}^{3^n} \left(\sum_{\{j\in[r,k]: v^{-1}(\tau^j x) A_{k-j}^{\alpha-1} > 3^n\}} v(\tau^j x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'} d\mu(x).$$

Then, since E is a subset of $G_\gamma^{B_i}$, we obtain

$$\int_E \left[q' \int_0^\infty \left(\sum_{j\in H_y^x} v(\tau^j x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'} d\mu(x)$$

$$\leq C \int_E \left[q' \sum_{n=-\infty}^{\infty} \int_{3^{n-1}}^{3^n} 2^{(\gamma_n+1)q'/p'} y^{q'-1}\, dy \right]^{p'/q'} d\mu(x)$$

$$= C \left[q' \sum_{n=-\infty}^{\infty} \int_{3^{n-1}}^{3^n} \left(\int_E 2^{\gamma_n}\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'}$$

$$\leq C \left[q' \sum_{n=-\infty}^{\infty} \int_{3^{n-1}}^{3^n} \left(\int_E \sum_{\{j\in[r,k]: v^{-1}(\tau^j x) A_{k-j}^{\alpha-1} > 3^n\}} v(\tau^j x)\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'}$$

$$\leq C \left[q' \int_0^\infty \left(\int_E \sum_{j\in H_y^x} v(\tau^j x)\, d\mu(x) \right)^{q'/p'} y^{q'-1}\, dy \right]^{p'/q'}.$$

Consequently, (4.11) is proved and, hence, the proof of (ii) $\Rightarrow$ (iii) is finished. ■

Proof of (iii) ⇒ (i): Let $L \in \mathbb{N}$ and let $M_{\alpha,L}$ be the truncated maximal operator defined by

$$M_{\alpha,L}f(x) = \sup_{m\le L} \frac{1}{A_m^\alpha} \sum_{i=0}^{m} A_{m-i}^{\alpha-1}|f(\tau^i x)|, \quad x \in X.$$

Let f be a nonnegative measurable function defined on X and let L and N be natural numbers and λ a positive real number. Let $O_\lambda = \{x \in X : M_{\alpha,L}f(x) > \lambda\}$. Then,

$$\begin{aligned} \int_{O_\lambda} u(x)\,d\mu(x) &= \int_X \frac{1}{N+1} \sum_{j=0}^{N} \chi_{\tau^{-j}O_\lambda}(x) u(\tau^j x)\,d\mu(x) \\ &= \int_X \frac{1}{N+1} \sum_{\{j\in[0,N]:M_{\alpha,L}f(\tau^j x)>\lambda\}} u^x(j)\,d\mu(x) \qquad (4.12)\\ &\le \int_X \frac{1}{N+1} \sum_{\{j\in[0,N]:m_\alpha^+(f^x\chi_{[0,N+L]})(j)>\lambda\}} u^x(j)\,d\mu(x), \end{aligned}$$

where m_α^+ is the maximal operator associated with the Cesàro averages of order α of functions on $\mathbb{Z}$.

The condition $A_{p,q;\alpha}^+(\tau)$ is the same as saying that the pairs (u^x, v^x) satisfy the condition $A_{p,q;\alpha}^+(\mathbb{Z})$ for almost every $x \in X$ with a constant independent of x. Hence,

$$\sum_{\{j\in\mathbb{Z}:m_\alpha^+(f^x\chi_{[0,N+L]})(j)>\lambda\}} u^x(j) \le \frac{C}{\lambda^p} \|f^x\chi_{[0,N+L]}\|_{p,q;v^x}^p. \tag{4.13}$$

Then (4.13) and (4.12) imply

$$\begin{aligned} u(O_\lambda) &= \int_{O_\lambda} u(x)\,d\mu(x) \qquad (4.14)\\ &\le \frac{C}{\lambda^p} \int_X \frac{1}{N+1} \|f^x\chi_{[0,N+L]}\|_{p,q;v^x}^p\,d\mu(x). \end{aligned}$$

Now, using the definitions of the $L_{p,q}$-norm and Minkowski's integral inequality, we obtain

$$\begin{aligned} &u(O_\lambda) \qquad (4.15)\\ &\le \frac{C}{\lambda^p} \frac{1}{N+1} \left(q\int_0^\infty \left(\int_X \left[\sum_{\{j\in[0,N+L]:f^x(j)>y\}} v^x(j) \right] d\mu(x) \right)^{q/p} y^{q-1}\,dy \right)^{p/q}. \end{aligned}$$

Since τ preserves the measure μ, the right-hand side of (4.15) equals to

$$\frac{C}{\lambda^p}\frac{1}{N+1}\left(q\int_0^\infty\left((N+L+1)\int_X v(x)\chi_{\{z:f(z)>y\}}(x)\,d\mu\right)^{q/p}y^{q-1}\,dy\right)^{p/q}$$
$$=\frac{C}{\lambda^p}\frac{N+L+1}{N+1}\|f\|_{p,q;v}^p.$$

Consequently, we have

$$u(O_\lambda)\leq C\lambda^{-p}(N+L+1)(N+1)^{-1}\|f\|_{p,q;v}^p.$$

Letting N tend to ∞ and then letting L tend to ∞, we obtain

$$u\left(\{x:M_\alpha^+f(x)>\lambda\}\right)\leq\frac{C}{\lambda^p}\|f\|_{p,q;v}^p,$$

that is,

$$\|M_\alpha^+f\|_{p,\infty;u}\leq C\|f\|_{p,q;v}.$$

This finishes the proof of (iii) $\Rightarrow$ (i). ■

Remark 1. We remark that the implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) in Theorem 2.1 hold also for $1<p<\infty$ and $1<q<\infty$.

Finally, assume that one of the conditions (i), (ii) and (iii) holds. In order to prove that the averages $R_{n,\alpha}f$ converge a.e. for all $f\in L_{p,q}(v\,d\mu)$, it is enough to establish the convergence for a dense subset of $L_{p,q}(v\,d\mu)$. We can take the subset $D=L^r(\mu)\cap L_{p,q}(v\,d\mu)$, $r>1/\alpha$. The set D is clearly dense and, in virtue of Irmish's theorem, the averages $R_{n,\alpha}f$ converge a.e. for all $f\in D$.

Proof of Theorem 2.2: The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear. The equivalence between (v) and (vi) is contained in Lemma 3.2. The implication (ii) $\Rightarrow$ (v) is contained in Theorem 2.1 (see the above remark). Finally, the fact that $\omega\in A_{p,\alpha}^+(\tau)$ implies $\omega\in A_{p-\varepsilon,\alpha}^+(\tau)$ for some $\varepsilon>0$ with $p-\varepsilon>1$ (Lemma 3.3, (1)), Theorem 2.1 ((iii) $\Rightarrow$ (i)) and Marcinkiewicz's interpolation theorem allow us to prove that (vi) $\Rightarrow$ (iii). ■

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