Publicacions Matemàtiques, Vol. **44** (2000), 177–192

(LB^∞)-STRUCTURE OF SPACES OF GERMS OF HOLOMORPHIC FUNCTIONS

Nguyen Dinh Lan

Abstract

We study the structure of spaces of germs of holomorphic functions on compact sets in Fréchet spaces for (LB^∞) as well as for $(\bar{\Omega}, \tilde{\Omega})$.

Introduction

Let E be a Fréchet space and let K be a compact subset in E. By $\mathcal{H}(K)$ we denote the space of germs of holomorphic functions on K equipped with the inductive limit topology. Some linear topological invariants, in particular those of the (Ω)-type for the strong dual $[\mathcal{H}(K)]'$ of the space $\mathcal{H}(K)$, were investigated by several authors. For example, in the finite dimensional case, Zaharjuta proved that $[\mathcal{H}(K)]'$ has $(\bar{\Omega})$ if and only if K is L-regular [**17**]. This problem, in the infinite dimensional case, has been considered already by R. Meise, D. Vogt and many others. Meise and Vogt have shown in [**7**] that $[\mathcal{H}(K)]'$ has (Ω) for every compact subset K in a nuclear Fréchet space E as long as E has (Ω). Recently, this result has been extended to the general case where E is only Fréchet by Nguyen Van Khue and Phan Thien Danh [**10**]. For the invariants $(\bar{\Omega})$ and $(\tilde{\Omega})$, Meise and Vogt in [**8**] gave some necessary and sufficient conditions for the compact polydiscs $\bar{\mathbb{D}}$ in a nuclear Fréchet space having a Schauder basis such that $[\mathcal{H}(\bar{\mathbb{D}})]'$ has $(\bar{\Omega})$ and has $(\tilde{\Omega})$ respectively.

The aim of the present paper is to study the invariant (LB^∞) as well as $(\bar{\Omega})$ and $(\tilde{\Omega})$ of $[\mathcal{H}(K)]'$ in the case where K is a balanced convex compact subset of a nuclear Fréchet space E. It should be mentioned that this problem has been treated very recently by Le Mau Hai and Nguyen Van Khue [**6**] in the case where E is a Fréchet-Schwartz space having an absolute basis. Our main results are explained in Sections 2 and 3. Namely, in Section 2 by employing an important characterization of (LB^∞) for Fréchet spaces [**15**], we prove that if B is a balanced convex compact subset of a Fréchet space E having $(\tilde{\Omega}_B)$ then $[\mathcal{H}(B)]'$ has

(LB^{∞}) (Theorem 2.1). In Theorem 2.2, under the additional assumption that E has the bounded approximation property, we prove that B is not pluripolar if $[\mathcal{H}(B)]'$ has (LB^{∞}). Combining this result and a characterization of $(\tilde{\Omega}_B)$ in terms of the non-pluripolarity of B [**2**] we also obtain a converse to Theorem 2.1 in the special case mentioned above. In Section 3, we prove in Theorem 3.1 that if B is a balanced compact subset of a nuclear Fréchet space having a Schauder basis then $[\mathcal{H}(B)]'$ has either $(\bar{\Omega}_B)$ or $(\tilde{\Omega}_B)$ if and only if E has the same property.

Finally, we note that the invariants of (DN)-type for spaces of entire functions of bounded type on (DF)-spaces were considered by several authors (for example [**6**], [**10**], …).

1. Preliminaries

1.1. Some linear topological invariants. Let E be a Fréchet space with a fundamental system of semi-norms $\{\|\bullet\|_k\}$. For a subset B of E, put $\|u\|_B^* = \sup\{|u(x)| : x \in B\}$ for $u \in E'$.

Write $\|\bullet\|_k^*$ for $B = U_k = \{x \in E : \|x\|_k < 1\}$.

Using this notation we say E has the property

$$(\Omega) \Leftrightarrow \forall\, p\, \exists\, q\, \forall\, k\, \exists\, C, d > 0 \qquad \|\bullet\|_q^{*1+d} \le C\, \|\bullet\|_k^*\, \|\bullet\|_p^{*d}.$$

$$(\overline{\Omega}) \Leftrightarrow \forall\, p, d > 0\, \exists\, q\, \forall\, k > 0\, \exists\, C > 0 \quad \|\bullet\|_q^{*1+d} \le C\, \|\bullet\|_k^*\, \|\bullet\|_p^{*d}.$$

$$(\tilde{\Omega}) \Leftrightarrow \forall\, p\, \exists\, q, d > 0\, \forall\, k\, \exists\, C > 0 \qquad \|\bullet\|_q^{*1+d} \le C\, \|\bullet\|_k^*\, \|\bullet\|_p^{*d}.$$

$$\begin{aligned}(LB^{\infty}) \Leftrightarrow\ & \forall\, \rho_n \uparrow \infty\, \forall\, p\, \exists\, q \\ & \forall\, k\, \exists\, n_k, C > 0 \\ & \forall\, u \in E'\, \exists\, n_u \in [k; n_k] \qquad \|u\|_q^{*1+\rho_{n_u}} \le C\, \|u\|_{n_u}^*\, \|u\|_p^{*\rho_{n_u}}.\end{aligned}$$

The above properties were introduced and investigated by Vogt (see [**9**] or [**16**] for (Ω) and [**15**] for the others).

In [**15**] Vogt gave the following important characterization of (LB^{∞}) for Fréchet spaces.

Vogt's Theorem ([**15**, Satz 5.2])**.** *For an arbitrary exponent sequence $\alpha = (\alpha_j)$ satisfying $\sup\limits_{j\ge 1} \frac{\alpha_{j+1}}{\alpha_j} < \infty$, the following assertions are equivalent*

(i) *E has (LB^{∞}).*

(ii) *Every continuous linear map from E into $\Lambda_\infty^\infty(\alpha)$ is bounded on a zero-neighbourhood, where*

$$\Lambda_\infty^\infty(\alpha) = \left\{(\xi_j) \subset \mathbb{C} : \|(\xi_j)\|_k := \sup |\xi_j| k^{\alpha_j} < \infty\, \forall\, k \ge 1\right\}.$$

1.2. Holomorphic functions. Let E, F be locally convex spaces and D an open subset in E. A function $f\colon D \longrightarrow F$ is called holomorphic if it is continuous and $u \circ f$ is Gâteaux holomorphic for $u \in F'$. By $\mathcal{H}(D,F)$ we denote the space of F-valued holomorphic functions on D, equipped with the compact-open topology. When F is omitted, it is understood to be the scalar field $\mathbb{C}$, e.g. $\mathcal{H}(D) = \mathcal{H}(D,\mathbb{C})$.

Finally for each compact set K in E, by $\mathcal{H}(K)$ we denote the space of holomorphic functions on K, equipped with the inductive topology, i.e.

$$\mathcal{H}(K) := \liminf_{U \supset K} \mathcal{H}^\infty(U)$$

where U ranges over all neighbourhoods of K and $\mathcal{H}^\infty(U)$ denotes the Banach space of bounded holomorphic functions on U.

For the details concerning the holomorphic functions and the germs of holomorphic functions on compact sets in a locally convex space, we refer to the book of Dineen [**1**].

2. The structure (LB^∞)

Theorem 2.1. *Let E be a nuclear Fréchet space and B a balanced convex compact subset in E. Assume that E has $(\tilde{\Omega}_B)$:*

$$(\tilde{\Omega}_B): \forall\, p\ \exists\, q, d, C > 0 \quad \|\bullet\|_q^{*1+d} \leq C\, \|\bullet\|_B^{*}\, \|\bullet\|_p^{*d}.$$

Then $[\mathcal{H}(B)]' \in (LB^\infty)$.

Note that in the definition of $(\tilde{\Omega}_B)$, by choosing q sufficiently large, we may assume that $C = 1$.

We need the following:

Lemma 2.2. *Let E and B be as in Theorem 2.1. Then B is a set of uniqueness.*

Here we say that the compact set B is a set of uniqueness if for every $f \in \mathcal{H}(B)$, $f_{|B} = 0$ implies $f = 0$.

Proof: First, since E has $(\tilde{\Omega}_B)$ by the hypothesis, it is easy to see that $\operatorname{span} B$ is dense in E. Now given $f \in \mathcal{H}(B)$ with $f_{|B} = 0$, consider the Taylor expansion of f at $0 \in B$ in a balanced convex neighbourhood W of B in E:

$$f(x) = \sum_{n \geq 0} P_n f(x), \quad x \in W,$$

where

$$P_n f(x) = \frac{1}{2\pi i} \int\limits_{|\lambda|=\delta_x>0} \frac{f(\lambda x)}{\lambda^{n+1}}\, d\lambda \text{ for } x \in E.$$

Since $P_n f$ are n-homogeneous polynomials and $P_n f_{|B} = 0$, it follows that $P_n f_{|\operatorname{span} B} = 0$. By the continuity of $P_n f$ and by $\overline{\operatorname{span} B} = E$, we have $P_n f = 0$ for $n \geq 0$. Thus $f = 0$ in W and hence B is a set of uniqueness. □

Proof of Theorem 2.1: Since $\mathcal{H}(\mathbb{C}) = \Lambda_\infty^\infty(\alpha)$ where $\alpha = (\alpha_j)$ with $\alpha_j = j$ for $j \geq 1$, by Vogt's theorem it suffices to show that every continuous linear map $T\colon [\mathcal{H}(B)]' \longrightarrow \mathcal{H}(\mathbb{C})$ is compact.

(i) Consider the function $f\colon B \longrightarrow \mathcal{H}(\mathbb{C})$ induced by T:

$$f(x)(\lambda) = T(\delta_x)(\lambda) \text{ for } x \in B,\ \lambda \in \mathbb{C},$$

where $\delta_x \in [\mathcal{H}(B)]'$ denotes the Dirac functional associated to $x \in B$:

$$\langle \varphi, \delta_x \rangle = \varphi(x) \text{ for } \varphi \in \mathcal{H}(B).$$

It follows that f is weakly holomorphic, i.e. $\mu \circ f \in \mathcal{H}(B)$ for $\mu \in [\mathcal{H}(\mathbb{C})]'$, because $T'(\mu) \in [\mathcal{H}(B)]'' \cong \mathcal{H}(B)$. By Grothendieck's factorization theorem [**9**], this yields that $f\colon B \longrightarrow \mathcal{H}^\infty(2\Delta)$, where Δ is the open unit disc in $\mathbb{C}$, is extended to a holomorphic function $\hat{f}$ on a neighbourhood W of B in E.

Let $g\colon (B \times \mathbb{C}) \cup (W \times \bar{\Delta}) \longrightarrow \mathbb{C}$ given by

$$g(x, \lambda) = \begin{cases} f(x)(\lambda) & \text{for } x \in B,\ \lambda \in \mathbb{C} \\ \hat{f}(x)(\lambda) & \text{for } x \in W,\ \lambda \in \bar{\Delta}. \end{cases}$$

Obviously g is separately holomorphic in the sense of Sciak [**14**], this means that $g(x, \cdot)$ is holomorphic in $\lambda \in \mathbb{C}$ for every $x \in B$ and $g(\cdot, \lambda)$ is too in $x \in W$ for every $\lambda \in \bar{\Delta}$. We denote by $\mathcal{F}$ the family of all finite dimensional subspaces $P \neq 0$ of $E(B)$, where $E(B)$ is the Banach space spanned by B. For each $P \in \mathcal{F}$ consider $g_P = g_{|((B\cap P)\times\mathbb{C})\cup((W\cap P)\times\bar{\Delta})}$. Since $B \cap P$ is the unit ball in P and $\bar{\Delta}$ is not polar, by Nguyen Thanh Van-Zeriahi [**11**] g_P is uniquely extended to a holomorphic function $\tilde{g}_P$ on $(W \cap P) \times \mathbb{C}$. The uniqueness implies that the family $\{\tilde{g}_P : P \in \mathcal{F}\}$ defines a Gâteaux holomorphic function $\tilde{g}$ on $(W \cap E(B)) \times \mathbb{C}$. On the other hand, since $\tilde{g}$ is holomorphic on $(W \cap E(B)) \times \Delta$, Zorn's theorem [**1**] implies that $\tilde{g}$ is holomorphic on $(W \cap E(B)) \times \mathbb{C}$. Consider the holomorphic function $\hat{g}\colon (W \cap E(B)) \longrightarrow \mathcal{H}(\mathbb{C})$ associated to $\tilde{g}$. We prove that $\hat{g}$ can be extended to a bounded holomorphic function on a neighbourhood of B with values in $\mathcal{H}(\mathbb{C})$.

(ii) The following is a modification of Meise-Vogt [**8**] and of Le Mau Hai [**5**].

Let $\left\{\|\bullet\|_\gamma\right\}_{\gamma=1}^\infty$ and $\{\|\bullet\|_k\}_{k=1}^\infty$ be two fundamental systems of semi-norms of E and $\mathcal{H}(\mathbb{C})$ respectively. Since $\mathcal{H}(\mathbb{C})$ has (DN) we have

$$\exists\, p\, \forall\, q, d > 0\, \exists\, k, C > 0 \quad \|\bullet\|_q^{1+d} \leq C\, \|\bullet\|_k\, \|\bullet\|_p^d .$$

Note that by replacing k with some $k' > k$, we always may assume that $C = 1$. Choose α such that $U_\alpha \subset W$ and

$$M(\alpha, p) = \sup\left\{\|\hat{g}(x)\|_p : x \in U_\alpha \cap E(B)\right\} < \infty.$$

Let ω_α from E into E_α, the Banach space associated to $\|\bullet\|_\alpha$, be the canonical map and $A = \omega_\alpha|_{E(B)} : E(B) \longrightarrow E_\alpha$. Since E is nuclear, without loss of generality we may assume that $E(B)$ and E_α are Hilbert spaces.

Then, by [**12**, Proposition 8.6.6, p. 143], A can be written in the form

$$A(x) = \sum_{j\geq 1} \lambda_j \langle x, y_j\rangle z_j$$

where $\lambda_j > 0\ \forall\, j \geq 1$, $\lambda = (\lambda_j) \in s$, the space of rapidly decreasing sequences, (y_j) is a complete orthonormal system in $E(B)$ and (z_j) an orthonormal system in E_α.

Since

$$A\left(\frac{y_j}{\lambda_j}\right) = z_j \in \omega_\alpha(U_\alpha) \quad \forall\, j \geq 1,$$

we have

$$\frac{y_j}{\lambda_j} \in U_\alpha \quad \forall\, j \geq 1.$$

It follows that

$$\sum_{j=1}^m \left(\frac{\mu_j}{\lambda_j}\right) y_j \in U_\alpha, \quad \forall\, m \geq 1,$$

where $\mu_j = \frac{\delta}{j^k}$ and $\delta > 0$ is chosen such that

$$\left\{u \in E_\alpha : u = \sum_{j=1}^\infty \xi_j z_j \text{ and } |\xi_j| < \mu_j\ \forall\, j \geq 1\right\} \subset \omega_\alpha(U_\alpha)$$

and

$$\delta \sum_{j\geq 1}^\infty \frac{1}{j^k} \leq 1.$$

We set

$$\chi_k \in E'_\alpha \colon z \in E_\alpha \mapsto \langle z, z_k\rangle_\alpha, \text{ the scalar product in } E_\alpha.$$

Then

$$\|\chi_k\| = 1 \quad \forall\, k \geq 1$$

and

$$\begin{aligned} \forall\, k \geq 1 \quad \|A^*\chi_k\|^*_B &= \sup_{\|x\|\leq 1} |\chi_k A(x)| \\ &= \sup_{\|x\|\leq 1} |\langle A(x), z_k\rangle| \\ &= \sup_{\|x\|\leq 1} |\lambda_k\langle x, y_k\rangle| \\ &= \lambda_k \text{ (by the Bessel inequality: } |\langle x, y_k\rangle| \leq \|x\|). \end{aligned} \tag{1}$$

Now put

$$\varphi_k = \omega^*_\alpha \chi_k, \tag{2}$$

and choose β such that

$$\exists\, d, C > 0 \quad \|\bullet\|^{*1+d}_\beta \leq C\,\|\bullet\|^*_B\,\|\bullet\|^{*d}_\alpha\,. \tag{3}$$

For β sufficiently large, we can choose $C = 1$.

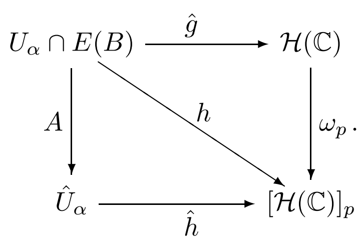
From (1)–(3) we have

$$\|\varphi_k\|^{*1+d}_\beta = \|\omega^*_\alpha\chi_k\|^{*1+d}_\beta \leq \|A^*\chi_k\|^*_B\,\|\chi_k\|^{*d}_\alpha \leq \lambda_k \quad \forall\, k \geq 1.$$

Hence

$$\|\varphi_k\|^*_\beta \leq (\lambda_k)^{\frac{1}{1+d}} \quad \forall\, k \geq 1.$$

Let $h = \omega_p \hat{g}$. Since $M(\alpha, p) < \infty$ and $A(U_\alpha \cap E(B))$ is dense in $\omega_\alpha(U_\alpha)$, h is holomorphically factorized through $A\colon U_\alpha \cap E(B) \longrightarrow \hat{U}_\alpha$ by $\hat{h}\colon \hat{U}_\alpha \longrightarrow [\mathcal{H}(\mathbb{C})]_p$, where $\hat{U}_\alpha$ is the unit ball in E_α. This may be illustrated in the following diagram.



For each $m = (m_1, m_2, \ldots, m_n, 0, 0, \ldots) \in M$, with

$$M = \left\{m = (m_j) \in \mathbb{N}^{\mathbb{N}} : m_j \neq 0 \text{ only for finitely many } j \in \mathbb{N}\right\},$$

we put

$$a_m = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|\rho_1|=\mu_1} \int\limits_{|\rho_2|=\mu_2} \cdots \int\limits_{|\rho_n|=\mu_n} \frac{\hat{h}(\rho_1 z_1 + \rho_2 z_2 + \cdots + \rho_n z_n)}{\rho^{m+1}}\, d\rho$$

where

$$\rho^{m+1} := \rho_1^{m_1+1}\rho_2^{m_2+1}\ldots\rho_n^{m_n+1},$$
$$d\rho := d\rho_1 d\rho_2 \ldots d\rho_n,$$

then

$$\|a_m\|_p \leq \frac{M(\alpha,p)}{\mu^m} \quad \forall\, m \in M.$$

From the relation

$$\sum_{j=1}^{k} \frac{\rho_j}{\lambda_j} y_j \in U_\alpha \cap E(B) \quad \forall\, k \geq 1,$$

we deduce that

$$\hat{h}\left(\sum_{j\geq 1} \rho_j z_j\right) = \hat{h}A\left(\sum_{j\geq 1} \frac{\rho_j}{\lambda_j} y_j\right) = \omega_p \hat{g}\left(\sum_{j\geq 1} \frac{\rho_j}{\lambda_j} y_j\right).$$

On the other hand, by Cauchy's theorem, we get

$$a_m = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|\rho_1|=\lambda_1\mu_1} \int\limits_{|\rho_2|=\lambda_2\mu_2} \cdots \int\limits_{|\rho_n|=\lambda_n\mu_n} \frac{\hat{h}(\rho_1 z_1 + \rho_2 z_2 + \cdots + \rho_n z_n)}{\rho^{m+1}}\, d\rho.$$

It follows that

$$a_m = \left(\frac{1}{2\pi i}\right)^n \int\limits_{|\rho_1|=\lambda_1\mu_1} \int\limits_{|\rho_2|=\lambda_2\mu_2} \cdots \int\limits_{|\rho_n|=\lambda_n\mu_n} \frac{\omega_p \hat{g}(\sum\limits_{j=1}^{n} \frac{\rho_j}{\lambda_j} y_j)}{\lambda^{m+1}\left(\frac{\rho}{\lambda}\right)^{m+1}}\, d\rho$$
$$= \omega_p \left(\underbrace{\frac{1}{\lambda^m}\left(\frac{1}{2\pi i}\right)^n \int\limits_{|\theta_1|=\mu_1} \int\limits_{|\theta_2|=\mu_2} \cdots \int\limits_{|\theta_n|=\mu_n} \frac{\hat{g}(\theta_1 y_1 + \theta_2 y_2 + \cdots + \theta_n y_n)}{\theta^{m+1}}\, d\theta}_{b_m}\right)$$

where

$$\theta_j = \frac{\rho_j}{\lambda_j} \quad \forall\, j \geq 1.$$

We have

$$\|b_m\|_q \leq \frac{N(q)}{\lambda^m \mu^m} \quad \forall\, m \in M, \quad \forall\, q \geq p,$$

where

$$N(q) = \sup\left\{ \left\|\hat{h}(x)\right\|_q : x = \sum_{j=1}^{\infty} \xi_j y_j \text{ and } |\xi_j| \leq \mu_j \,\forall\, j \geq 1 \right\} < \infty,$$

because the set

$$\left\{ x = \sum_{j=1}^{\infty} \xi_j y_j : \xi_j y_j \,\forall\, j \geq 1 \right\}$$

is compact in $E(B)$.

Since $\mathcal{H}(\mathbb{C})$ has (DN), for every $q \geq p$ and $\bar{d} = \frac{d}{\delta}$ there exists $k \geq q$ and $C > 0$ such that

$$\|\bullet\|_q^{1+\bar{d}} \leq C \|\bullet\|_k \|\bullet\|_p^{\bar{d}},$$

where $0 < \delta < 1$ is chosen such that

$$\varepsilon := t - \frac{1-t}{1+\bar{d}} > 0 \text{ with } t = \frac{1}{2(1+d)}.$$

Again we may assume $C = 1$. Then

$$\begin{aligned} S &:= \sum_{m \in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_\beta^{*m_j} \leq \sum_{m \in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} (\lambda_j)^{\frac{m_j}{1+d}} \\ &= \sum_{m \in M} r^m \|b_m\|_q \lambda^{2tm} = \sum_{m \in M} r^m \left[\lambda^m \|b_m\|_q\right]^t \lambda^{tm} \|b_m\|_q^{1-t} \\ &\leq N(q)^t N(k)^{\frac{1-t}{1+\bar{d}}} M(\alpha,p)^{\frac{(1-t)\bar{d}}{1+\bar{d}}} \sum_{m \in M} r^m \frac{\lambda^{m(t-\frac{1-t}{1+\bar{d}})}}{\mu^{m(t+\frac{1-t}{1+\bar{d}}+\frac{(1-t)\bar{d}}{1+\bar{d}})}} \\ &\leq N(q)^t N(k)^{\frac{1-t}{1+\bar{d}}} M(\alpha,p)^{\frac{(1-t)\bar{d}}{1+\bar{d}}} \sum_{m \in M} r^m \frac{\lambda^{m(t-\frac{1-t}{1+\bar{d}})}}{\mu^m}. \end{aligned}$$

Since $\lambda = (\lambda_j) \in s$, the sequence $\left(\frac{\lambda_j^\varepsilon}{\mu_j}\right)$ is in l^1 and hence for $R = \sum_{j \geq 1} \left(\frac{\lambda_j^\varepsilon}{\mu_j}\right)$ we have

$$2R > R > \frac{\lambda_j^\varepsilon}{\mu_j} \text{ for } j \geq 1.$$

This implies

$$0 < \sup\left\{\frac{\lambda_j^\varepsilon}{2R\mu_j} : j \geq 1\right\} < \frac{1}{2}.$$

We have

$$\begin{aligned} S &= \sum_{m\in M} r^m \|b_m\|_q \prod_{j=1}^{\infty} \|\varphi_j\|_\beta^{*m_j} \\ &\leq N(q)^t N(k)^{\frac{1-t}{1+\bar d}} M(\alpha,p)^{\frac{(1-t)\bar d}{1+\bar d}} \sum_{m\in M}\left(\frac{r\lambda^\varepsilon}{\mu}\right)^m \\ &= N(q)^t N(k)^{\frac{1-t}{1+\bar d}} M(\alpha,p)^{\frac{(1-t)\bar d}{1+\bar d}} \prod_{j=1}^{\infty}\frac{1}{1-\frac{r\lambda_j^\varepsilon}{\mu_j}} < \infty. \end{aligned}$$

Hence the form

$$x \mapsto \sum_{m\in M} b_m \prod_{j\geq 1} (\varphi_j(x))^{m_j}$$

defines a bounded holomorphic function $\hat h_1$ on δU_β with $\delta = \frac{1}{4R}$ such that $\hat h_1\big|_{\delta U_\beta\cap B} = \hat g\big|_{\delta U_\beta \cap B}$, i.e. $\hat h_1(z)(\lambda) = g(z,\lambda)$ for $z \in \delta U_\beta \cap B$ and $\lambda \in \bar\Delta$. Since $\overline{\operatorname{span} B} = E$, by considering the Taylor expansion of $\hat h_1(\cdot)(\lambda) - g(\cdot,\lambda)$ in $z \in \operatorname{span} B$ at $0 \in B$, we get $\hat h_1(z)(\lambda) = g(z,\lambda)$ for $z \in \delta U_\beta \cap B$ and $\lambda \in \bar\Delta$.

(iii) Consider the separately holomorphic function h_1 in the sense of Siciak [**14**] on $(\delta U_\beta \times \mathbb{C}) \cup (W \times \bar\Delta)$, induced by $\hat h_1$ and g. By the same argument as in (i), h_1 is holomorphically extended to a function $\bar h_1$ on $W\times\mathbb{C}$. Let $\hat{\bar h}_1 : W \longrightarrow \mathcal{H}(\mathbb{C})$ denote the holomorphic function associated to $\bar h_1$. Since B is convex, balanced and the equality $(\hat h_1 - \hat g)\big|_{\delta U_\beta\cap B} = 0$ holds, from the Taylor expansion of $(\hat h_1 - \hat g)|_B$ at $0 \in B$ it follows that $\hat{\bar h}_1|_B = \hat g|_B$.

(iv) Applying a similar argument as in (ii) to each point of W, it follows that $\hat{\bar h}_1$ is locally bounded. Thus, by shrinking W, without loss of generality, we may assume that $\hat{\bar h}_1(W)$ is bounded. Define the continuous linear map $S\colon [\mathcal{H}^\infty(W)]' \longrightarrow \mathcal{H}(\mathbb{C})$ as

$$S(\mu)(\lambda) = \mu(\hat{\bar h}_1(\bullet)(\lambda)) \text{ for } \mu \in [\mathcal{H}^\infty(W)]' \text{ and } \lambda \in \mathbb{C}.$$

We have

$$\begin{aligned} T\left(\sum_{j=1}^{m}\alpha_j\delta_{x_j}\right)(\lambda) &= \sum_{j=1}^{m}\alpha_j T(\delta_{x_j})(\lambda) = \sum_{j=1}^{m}\alpha_j f(x_j)(\lambda) \\ &= \sum_{j=1}^{m}\alpha_j \hat{g}(x_j)(\lambda) = \sum_{j=1}^{m}\alpha_j \hat{\bar{h}}_1(x_j)(\lambda) \\ &= \sum_{j=1}^{m}\alpha_j S(\delta_{x_j})(\lambda) = S\left(\sum_{j=1}^{m}\alpha_j\delta_{x_j}\right) \end{aligned}$$

for $x_1, \dots, x_m \in B$ and $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{C}$.

On the other hand, since B is of uniqueness and $\mathcal{H}(B)$ is reflexive, it follows that $S = T$. Hence T is compact. □

For the formulation of the second theorem we recall the following [**2**], [**3**]:

An upper-semicontinuous function $\varphi\colon E \longrightarrow [-\infty; +\infty)$ is called plurisubharmonic if φ is subharmonic on every complex line in E. A subset $B \subset E$ is said to be pluripolar if there exists a plurisubharmonic function φ on E such that $\varphi \not\equiv -\infty$ and $\varphi_{|B} = -\infty$.

Theorem 2.3. *Let E be a nuclear Fréchet space with the bounded approximation property and B a balanced convex compact subset in E. Then the following assertions are equivalent:*

a) *E has $(\tilde{\Omega}_B)$.*
b) *$[\mathcal{H}(B)]'$ has (LB^∞).*
c) *B is not pluripolar.*

For the proof of Theorem 2.3 we need the following lemma which was proved independently in [**6**].

Lemma 2.4. *Let K be a compact subset of a Fréchet space E such that $[\mathcal{H}(K)]' \in (LB^\infty)$. Then K is a compact set of uniqueness.*

Proof: Given $f \in \mathcal{H}(K)$ with $f_{|K} = 0$. Choose a decreasing neighbourhood basis $\{V_k\}$ of K such that $\varepsilon_k := \sup\{|f(z)| : z \in V_k\} < 1$ for $k \geq 1$.

Since $f_{|K} = 0$ and K is compact, it follows that $\varepsilon_k \searrow 0$. By applying the (LB^∞) property of $[\mathcal{H}(K)]'$ to the sequence $\rho_k = \sqrt{-\log \varepsilon_k} \nearrow +\infty$ and to $p = 1$, we can find $q \geq 1$, $N_1 \geq 1$ and $C > 0$ such that

$$\forall\, n\ \exists\, k_n \in [1; N_1] \quad \|f^n\|_q^{1+\rho_{k_n}} \leq C\, \|f^n\|_{k_n} \|f^n\|_1^{\rho_{k_n}}.$$

This inequality gives

$$\|f\|_q \leq C^{\frac{1}{n}}\, \|f\|_{k_n}^{\frac{1}{1+\rho_{k_n}}}\, \|f\|^{\frac{\rho_{k_n}}{1+\rho_{k_n}}} \text{ for } n \geq 1.$$

Take $1 \leq k \leq N_1$ such that

$$\sharp\{n : k_n = k\} = +\infty.$$

Then

$$\|f\|_q \leq \|f\|_k^{\frac{1}{1+\rho_k}}\, \|f\|_1^{\frac{\rho_k}{1+\rho_k}}$$
$$= (\varepsilon_k)^{\frac{1}{1+\sqrt{-\log \varepsilon_k}}} (\varepsilon_1)^{\frac{\sqrt{-\log \varepsilon_k}}{1+\sqrt{-\log \varepsilon_k}}} \to 0 \text{ as } k \to +\infty.$$

Hence $f = 0$. □

Proof of Theorem 2.2: a) $\Rightarrow$b): By Theorem 2.1.

c) $\Rightarrow$ a): By Theorem 7 in [**2**].

It remains to show that b) $\Rightarrow$ c).

Let B be pluripolar. Choose a plurisubharmonic function $\varphi \neq -\infty$ on E such that $\varphi_{|B} = -\infty$. Consider the Hartogs domain Ω_φ given by

$$\Omega_\varphi = \left\{(x, \lambda) : |\lambda| < e^{-\varphi(x)}\right\}.$$

Then Ω_φ is pseudoconvex. Since E has the bounded approximation property, there exists $f \in \mathcal{H}(\Omega_\varphi)$ such that Ω_φ is the domain of existence of f (by [**13**]). Write the Hartogs expansion of f,

$$f(x, \lambda) = \sum_{n \geq 0} h_n(x) \lambda^n \text{ for } (x, \lambda) \in \Omega_\varphi,$$

where

$$h_n(x) = \frac{1}{2\pi i} \int\limits_{|\lambda| = e^{-\delta\varphi(x)}} \frac{f(x, \lambda)}{\lambda^{n+1}}\, d\lambda \text{ for } n \geq 0,\ (\delta > 1).$$

It is easy to see that h_n are holomorphic on E, because of the upper-semicontinuity of φ.

Let $g\colon B \longrightarrow \mathcal{H}(\mathbb{C})$ given by $g(x)(\lambda) = f(x, \lambda)$ for $x \in B$, $\lambda \in \mathbb{C}$.

Then g is weakly holomorphic. Indeed, given $\mu \in [\mathcal{H}(\mathbb{C})]'$, take $r > 0$ such that μ can be considered as a continuous linear functional on $\mathcal{H}^\infty(r\Delta)$. Since $B \times \mathbb{C} \subset \Omega_\varphi$, we can find a neighbourhood V of B in E such that $V \times r\Delta \subset \Omega_\varphi$. Hence f induces a holomorphic extension of $\mu \circ g$ to V. On the other hand, since B is a set of uniqueness, the form $\mu, \mapsto \widehat{\mu \circ g}$, the unique holomorphic extension of $\mu \circ g$ for $\mu \in [\mathcal{H}(\mathbb{C})]'$, defines a linear map $T\colon [\mathcal{H}(\mathbb{C})]' \longrightarrow \mathcal{H}(B)$. Again since B is a set of uniqueness, T has a closed graph. The closed graph Grothendieck theorem [**4**] yields that T is continuous. By Vogt [**15**] we can find a neighbourhood W of $0 \in [\mathcal{H}(\mathbb{C})]'$ such that $T(W)$ is bounded in $\mathcal{H}(B)$. By the regularity of $\mathcal{H}(B)$ [**1**] there exists a neighbourhood V of B in E such that $T(W)$ is contained and bounded in $\mathcal{H}^\infty(V)$. This implies that g is extended to a holomorphic function $\hat{g}\colon V \longrightarrow \mathcal{H}(\mathbb{C})$. Obviously $\tilde{g} = f$ on non-empty open subset of Ω_φ, where $\tilde{g}(x, \lambda) = \hat{g}(x)(\lambda)$ for $x \in V$, $\lambda \in \mathbb{C}$. By the hypothesis Ω_φ is the domain of existence of f, thus we have $V \times \mathbb{C} \subset \Omega_\varphi$. Hence $\varphi_{|V} = -\infty$ which is impossible. $\square$

3. The structure $(\bar{\Omega}, \tilde{\Omega})$

Theorem 3.1. *Let E be a nuclear Fréchet space with a basis and B a balanced compact subset in E. Then $[\mathcal{H}(B)]'$ has either $(\tilde{\Omega}_B)$ or $(\bar{\Omega}_B)$ if and only if E has the same property.*

Proof: *Necessity.* Since the forms $f \mapsto f'(0)$ and $u \mapsto [u]$, where $[u]$ denotes the element of $\mathcal{H}(B)$ induced by $u \in E'$, define the continuous linear maps $P\colon \mathcal{H}(B) \longrightarrow E'$ and $Q\colon E' \longrightarrow \mathcal{H}(B)$ satisfying $P \circ Q = \mathrm{id}$, it follows that E' can be considered as a subspace of $\mathcal{H}(B)$. Hence $E \cong E''$ which is a quotient space of $[\mathcal{H}(B)]'$. This proves the necessity of the theorem.

Sufficiency. It suffices to prove the case $E \in (\tilde{\Omega}_B)$.

Let (e_j) be a basis of E and (e_j^*) its dual basis in E'. Since E is nuclear, without loss of generality we may assume that

$$\sum_{j \geq 1} \left\|e_j^*\right\|_{q+1}^* \left\|e_j\right\|_q < \frac{1}{e^2} \text{ for } q \geq 1.$$

Write each $f \in \mathcal{H}^{\infty}(B+U_q)$ in the form

$$f(x+u) = \sum_{n\geq 0} P_n f(x)(u) = \sum_{n\geq 0} P_n f(x)\left(\sum_{j\geq 1} e_j^*(u)e_j\right)$$
$$= \sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1} e_{j_1}^*(u)\dots e_{j_n}^*(u)\, P_n f(x)(e_{j_1},\dots,e_{j_n})$$

where

$$P_n f(x)(u) = \frac{1}{2\pi i}\int\limits_{|\lambda|=1} \frac{f(x+\lambda u)}{\lambda^{n+1}}\,d\lambda, \quad u \in U_q \text{ and } x \in B.$$

The above equality is correct, because

$$\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1} \left\|e_{j_1}^*\right\|_{q+1}^* \dots \left\|e_{j_n}^*\right\|_{q+1}^* \|e_{j_1}\|_q \dots \|e_{j_n}\|_q$$
$$\times \left| P_n f(x)\left(\frac{e_{j_1}}{\|e_{j_1}\|_q},\dots,\frac{e_{j_n}}{\|e_{j_n}\|_q}\right)\right|$$
$$\leq \|f\|_{B+U_q} \sum_{n\geq 0}\frac{n^n}{n!}\left(\sum_{j\geq 1}\left\|e_j^*\right\|_{q+1}^* \|e_j\|_q\right)^n$$
$$\leq \|f\|_{B+U_q} \sum_{n\geq 0}\left(\frac{n}{e^2}\right)^n \frac{1}{n!} < \infty.$$

From the above inequalities, it follows also that $\mathcal{H}(B) \cong \lim\operatorname{ind}_q \mathcal{H}_q$, where

$$\mathcal{H}_q = \left\{ f \in \mathcal{H}^{\infty}(B+U_q) : |||f|||_q < \infty \right\}$$

with

$$|||f|||_q := \sup_{x\in B}\left\{\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1}\left\|e_{j_1}^*\right\|_q^* \dots \left\|e_{j_n}^*\right\|_q^* |P_n f(x)(e_{j_1},\dots,e_{j_n})|\right\}.$$

Applying $(\tilde{\Omega}_B)$ with $C = 1$ we have

$$\begin{aligned}
&|||f|||_q^{1+d}\\
&= \sup_{x\in B}\left\{\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1}\left\|e_{j_1}^*\right\|_q^*\cdots\left\|e_{j_n}^*\right\|_q^*|P_nf(x)(e_{j_1},\dots,e_{j_n})|\right\}^{1+d}\\
&\leq \sup_{x\in B}\left\{\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1}\left\|e_{j_1}^*\right\|_B^{*\frac{1}{1+d}}\cdots\left\|e_{j_n}^*\right\|_B^{*\frac{1}{1+d}}|P_nf(x)(e_{j_1},\dots,e_{j_n})|^{\frac{1}{1+d}}\right.\\
&\quad\left.\times\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1}\left\|e_{j_1}^*\right\|_p^{*\frac{d}{1+d}}\cdots\left\|e_{j_n}^*\right\|_p^{*\frac{d}{1+d}}|P_nf(x)(e_{j_1},\dots,e_{j_n})|^{\frac{d}{1+d}}\right\}^{1+d}\\
&\leq \sup_{x\in B}\left\{\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1}\left\|e_{j_1}^*\right\|_B^*\cdots\left\|e_{j_n}^*\right\|_B^*|P_nf(x)(e_{j_1},\dots,e_{j_n})|\right\}\\
&\quad\times\sup_{x\in B}\left\{\sum_{n\geq 0}\sum_{j_1,j_2,\dots,j_n\geq 1}\left\|e_{j_1}^*\right\|_p^*\cdots\left\|e_{j_n}^*\right\|_p^*|P_nf(x)(e_{j_1},\dots,e_{j_n})|\right\}^d\\
&= |||f|||_B\,|||f|||_p^d
\end{aligned}$$

for $f \in \mathcal{H}^\infty(B + U_q)$.

Hence $[\mathcal{H}(B)]' \in (\tilde{\Omega}_B)$ because $\mathcal{H}(B)$ is reflexive. □

Acknowledgement. The author would like to thank Prof. Nguyen Van Khue (Pedagogical Institute 1, Hanoi) for helpful suggestions during the preparation of the present paper and also to thank the referee for her/his suggestions providing the author with important references.

References

[1] S. DINEEN, *Complex analysis in locally convex spaces*, North-Holland Mathematics Studies **57**, North-Holland Publishing Co., Amsterdam, 1981.

[2] S. DINEEN, R. MEISE AND D. VOGT, Characterization of nuclear Fréchet spaces in which every bounded set is polar, *Bull. Soc. Math. France* **112(1)** (1984), 41–68.

[3] S. Dineen, R. Meise and D. Vogt, Polar subsets of locally convex spaces, in: *"Aspects of mathematics and its applications"*, North-Holland Math. Library **34**, North-Holland, Amsterdam, 1986, pp. 295–319.

[4] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Mem. Amer. Math. Soc.* **1955(16)** (1955), 140.

[5] Le Mau Hai, Weak extension of Fréchet-valued holomorphic functions on compact sets and linear topological invariants, *Acta Math. Vietnam.* **21(2)** (1996), 183–199.

[6] Le Mau Hai and N. Van Khue, Some characterizations of the properties (DN) and ($\tilde{\Omega}$), *Math. Scand.* (to appear).

[7] R. Meise and D. Vogt, Structure of spaces of holomorphic functions on infinite-dimensional polydiscs, *Studia Math.* **75(3)** (1983), 235–252.

[8] R. Meise and D. Vogt, Holomorphic functions of uniformly bounded type on nuclear Fréchet spaces, *Studia Math.* **83(2)** (1986), 147–166.

[9] R. Meise and D. Vogt, *"Introduction to functional analysis"*, Oxford Graduate Texts in Mathematics **2**, The Clarendon Press Oxford University Press, New York, 1997.

[10] N. Van Khue and P. Thien Danh, Structure of spaces of germs of holomorphic functions, *Publ. Mat.* **41(2)** (1997), 467–480.

[11] T. V. Nguyen and A. Zériahi, Familles de polynômes presque partout bornées, *Bull. Sci. Math. (2)* **107(1)** (1983), 81–91.

[12] A. Pietsch, *"Nuclear locally convex spaces"*, Ergebnisse der Mathematik und ihrer Grenzgebiete **66**, Springer-Verlag, New York, 1972.

[13] M. Schottenloher, The Levi problem for domains spread over locally convex spaces with a finite dimensional Schauder decomposition, *Ann. Inst. Fourier (Grenoble)* **26(4)** (1976), 207–237.

[14] J. Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^n$, *Ann. Polon. Math.* **39** (1981), 175–211.

[15] D. Vogt, Frécheträume, zwischen denen jede stetige lineare abbildung beschränkt ist, *J. Reine Angew. Math.* **345** (1983), 182–200.

[16] D. Vogt, On two classes of (F)-spaces, *Arch. Math. (Basel)* **45(3)** (1985), 255–266.

[17] V. P. Zaharjuta, Isomorphism of spaces of analytic functions, *Dokl. Akad. Nauk SSSR* **255(1)** (1980), 11–14.

Department of Mathematics
HoChiMinh City University of Education
280 An Duong Vuong, District 5
HoChiMinh City
Vietnam
E-mail address: ngdlanesed@hcm.vnn.vn

Primera versió rebuda el 8 de març de 1999,
darrera versió rebuda el 24 de març de 2000.