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A CARTAN-TYPE RESULT FOR INVARIANT DISTANCES AND ONE-DIMENSIONAL HOLOMORPHIC RETRACTS

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Abstract

We derive conditions under which a holomorphic mapping of a taut Riemann surface must be an automorphism. This is an analogue involving invariant distances of a result of H. Cartan. Using similar methods we prove an existence result for 1-dimensional holomorphic retracts in a taut complex manifold.

1. Introduction

In what follows, W denotes a connected Riemann surface, $\mathrm{Hol}(W,W)$ denotes the set of holomorphic self-maps of W and $\mathrm{Aut}(W)$ denotes the set of biholomorphic maps of W onto itself. A complex manifold M is *taut* if and only if for each complex manifold N, each sequence of holomorphic mappings from N to M contains a subsequence which either converges uniformly on compact subsets of N or is uniformly divergent to infinity (in the one point compactification of M) on compact subsets of N.

Definition 1.1. We call a distance function d on a Riemann surface W *invariant* if

$$d(f(w),f(z)) \leq d(w,z) \quad \forall\, w,z\in W, \quad \forall\, f\in \mathrm{Hol}(W,W).$$

A Hermitian metric h on W is called *invariant* provided

$$h(f_*(u),f_*(u)) \leq h(u,u) \quad \forall\, u\in \mathcal{O}_w W, \quad \forall\, w\in W, \quad \forall\, f\in \mathrm{Hol}(W,W).$$

We say that a distance function d on W is $\mathcal{C}^k$ $(k\geq 1)$ if d is the integrated distance function associated to a $\mathcal{C}^{k-1}$ Hermitian metric h on W.

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Remark 1. A standard example of an invariant metric is the square of the Kobayashi metric on a taut Riemann surface. For the unit disc in the complex plane, this metric is usually referred to as the Poincaré metric.

In his paper [**4**], J. P. Vigué proved the following result.

Theorem. *Let X and W be connected, taut complex manifolds with W 1-dimensional. Assume that h is an invariant Hermitian metric on W and choose $w \in W$, $x \in X$ and $v \in \mathcal{O}_x X \setminus \{0\}$. Then there exists a holomorphic retraction $\rho\colon X \to X$ such that $\rho(w) = x$, $\rho_*(\mathcal{O}_w W) = \mathbb{C}v$ and $\rho(X)$ is biholomorphic to W if and only if $E(x,v) = F(x,v)$ (where $E, F\colon \mathcal{O}X \to \mathbb{R}^+$ are invariant Finsler metrics which are defined in terms of w, W, h, x and X and which generalise the usual Carathéodory and Kobayashi metrics).*

Key to his proof of this is the following result of H. Cartan [**1**].

Cartan's Theorem. *Let W be a taut Riemann surface. If $f \in \mathrm{Hol}(W,W)$ fixes some point $w \in W$ and has unimodular derivative at w then $f \in \mathrm{Aut}(W)$.*

Note that the hypothesis of Cartan's result is equivalent to the requirement that f fixes w and that its derivative at w is unitary with respect to every (in particular every invariant) Hermitian metric on W. In a remark in [**4**], J. P. Vigué defined the invariant pseudodistances $c_{X,x}^{W,w}$ and $k_{X,x}^{W,w}$ in terms of an invariant distance on W (see Section 3 below). He would have liked to use these to investigate the existence of 1-dimensional holomorphic retracts through two given points of X. To do so, Vigué would have needed (but did not possess) an analogue of Cartan's result which uses an invariant distance in place of an invariant Hermitian metric. In Section 2 we prove such an analogue (Theorem 2.3) of Cartan's theorem under the assumption that the invariant distance arises from a continuous Hermitian metric on W. Then in the final section we use the invariant pseudodistances $c_{X,x}^{W,w}$ and $k_{X,x}^{W,w}$ (which generalise the usual Carathéodory distance and Kobayashi function on a complex manifold) and apply Theorem 2.3 to investigate the existence of holomorphic retractions of a complex manifold onto a 1-dimensional submanifold through two given points.

2. Automorphisms of Riemann surfaces

First we recall some standard notions from differential geometry. Let h be a continuous Hermitian metric on a connected Riemann surface W. Thus h determines a sesquilinear, positive definite inner product h_w

on each tangent space $\mathcal{O}_w W$ and h_w varies continuously with w. The associated norm on $\mathcal{O}_w W$ is denoted $|\cdot|_w$ (to simplify notation, the subscript w is often omitted). If $f \in \mathrm{Hol}(W, W)$ we denote its derivative at w by

$$f_{*w} \colon \mathcal{O}_w W \to \mathcal{O}_{f(w)} W.$$

The operator norm of f_{*w} (with respect to $|\cdot|_w$ and $|\cdot|_{f(w)}$) is denoted $||f_{*w}||$ (or by $||f_*||$ when w is clear from the context). As $\mathcal{O}_w W$ is one-dimensional it follows that

$$|f_{*w}(u)|_{f(w)} = ||f_{*w}|| \cdot |u|_w \quad \forall\, u \in \mathcal{O}_w W.$$

Continuity of h implies that the map $w \mapsto ||f_{*w}||$ is continuous. A piecewise $\mathcal{C}^1$ path in W is a mapping $\gamma \colon [a, b] \to W$ for which there exists a finite set of points $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_i, t_{i+1}]}$ is $\mathcal{C}^1$ and has nowhere vanishing tangent for each $i = 0, \ldots, n-1$. The length of such a path γ is defined by

$$l(\gamma) = \int_a^b h(\gamma'(t), \gamma'(t))^{\frac{1}{2}}\, dt = \int_a^b |\gamma'(t)|\, dt.$$

For any two points w and z in W the distance $d(w, z)$ is defined by

$$d(w, z) = \inf\{l(\gamma) : \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ path from } w \text{ to } z\}.$$

This function is clearly symmetric, positive and satisfies the triangle inequality. It is a standard result that $d(w, z) > 0$ when $w \neq z$ and that d generates the given topology on W (for example, see [**2**]). The open ball $B(w, r) \subset W$ with centre w and radius $r > 0$ is given by

$$B(w, r) = \{z \in W : d(w, z) < r\}.$$

If d is the distance arising from a hermitian metric h, it is easy to show that invariance of h implies invariance of d. In this first proposition we prove a converse result.

Proposition 2.1. *Let d be the integrated distance associated to a continuous Hermitian metric h on a Riemann surface W. If d is invariant then h must also be invariant.*

Proof: Assume that there exists $f \in \mathrm{Hol}(W, W)$ such that $||f_{*w}|| > 1$ for some $w \in W$. We will show that d cannot be invariant.

As $||f_*|| \neq 0$ at w, f maps some neighbourhood U of W biholomorphically onto an open neighbourhood of $f(w)$. Shrinking U if necessary and using continuity, we may assume that $||f_*|| \geq 1 + \epsilon$ on U for some $\epsilon > 0$.

Choose $\delta > 0$ such that $B(f(w), \delta) \subset f(U)$. Let $y \in B(f(w), \delta)$ and let γ be a path from $f(w)$ to y such that

$$d(f(w), y) \leq l(\gamma) < \delta.$$

As any path which leaves $B(f(w), \delta)$ will have length at least δ, the image of γ must be contained in $B(f(w), \delta)$. Thus we may write $\gamma = f\sigma$ where $\sigma = (f|_U)^{-1}\gamma$ lies in U and starts at w. Denote the endpoint $(f|_U)^{-1}(y)$ of σ by z. Then

$$\begin{aligned} l(\gamma) &= \int |\gamma'(t)|\, dt \\ &= \int |(f\sigma)'(t)|\, dt \\ &= \int ||f_{*\sigma(t)}|| \cdot |\sigma'(t)|\, dt \\ &\geq \int (1+\epsilon)|\sigma'(t)|\, dt \\ &\geq (1+\epsilon) d(w, z). \end{aligned}$$

Taking the infimum over paths joining $f(w)$ to $y = f(z)$, it follows that

$$d(f(w), f(z)) \geq (1+\epsilon) d(w, z).$$

Hence d cannot be invariant. □

Proposition 2.2. *Let d be an invariant $\mathcal{C}^1$ distance on a Riemann surface W. Let $f \in \mathrm{Hol}(W, W)$ and assume that there exists a sequence of paths $\gamma_n \colon [0, a_n] \to W$ which all start at w and for which*

$$\lim_{n\to\infty} l(\gamma_n) = \lim_{n\to\infty} l(f\gamma_n) = a > 0.$$

Then $||f_{*w}|| = 1$.

Proof: Invariance of d implies that $||f_*|| \leq 1$ everywhere. Assume that $||f_{*w}|| = r < 1$. As d is $\mathcal{C}^1$, $w \mapsto ||f_{*w}||$ is continuous and hence there exists $\epsilon > 0$ such that $||f_*|| < \frac{1+r}{2}$ on $B(w, \epsilon)$. For each n define $t_n \in (0, a_n]$ by

$$t_n = \begin{cases} a_n & \text{if } \gamma_n([0, a_n]) \subset B(w, \epsilon) \\ \sup\{t : \gamma_n([0, t]) \subset B(w, \epsilon)\} & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} l\left(f\gamma_n|_{[0,t_n]}\right) &= \int_0^{t_n} |(f\gamma_n)'(t)|\, dt \\ &= \int_0^{t_n} ||f_{*\gamma_n(t)}|| \cdot |\gamma_n'(t)|\, dt \\ &< \frac{1+r}{2} \int_0^{t_n} |\gamma_n'(t)|\, dt \\ &= \frac{1+r}{2} l\left(\gamma_n|_{[0,t_n]}\right) \end{aligned}$$

and hence

$$\begin{aligned} l(f\gamma_n) &= l\left(f\gamma_n|_{[0,t_n]}\right) + l\left(f\gamma_n|_{[t_n,a_n]}\right) \\ &< \frac{1+r}{2} l\left(\gamma_n|_{[0,t_n]}\right) + l\left(\gamma_n|_{[t_n,a_n]}\right) \\ &= l(\gamma_n) - \frac{1-r}{2} l\left(\gamma_n|_{[0,t_n]}\right) \\ &\le l(\gamma_n) - \frac{1-r}{2} \min(\epsilon, l(\gamma_n)) \quad \forall\, n. \end{aligned}$$

Thus

$$\lim_{n\to\infty} l(f\gamma_n) \le a - \frac{1-r}{2} \min(\epsilon, a) < a \quad \text{since } a > 0.$$

As this contradicts the hypothesis that $l(f\gamma_n)$ converges to a, our assumption that $||f_{*w}|| < 1$ must have been false. □

We combine these two propositions to prove the following theorem.

Theorem 2.3. *Let d be a $\mathcal{C}^1$ invariant distance on a taut Riemann surface W. Assume that $f \in \mathrm{Hol}(W, W)$ and that there are distinct points w and z in W satisfying*

$$f(w) = w \text{ and } d(w, z) = d(w, f(z)).$$

Then $f \in \mathrm{Aut}(W)$.

Proof: Let γ_n be a sequence of paths from w to z whose lengths converge to $d(w, z)$. Taking the limit as $n \to \infty$ in the inequality

$$l(\gamma_n) \ge l(f\gamma_n) \ge d(f(w), f(z)) = d(w, z)$$

we deduce that

$$\lim_{n\to\infty} l(\gamma_n) = \lim_{n\to\infty} l(f\gamma_n) = d(w, z) > 0.$$

Proposition 2.2 now implies that $||f_{*w}|| = 1$. Since w is fixed by f and $\mathcal{O}_w(W)$ is one dimensional, f_{*w} must be given by multiplication by a unimodular complex number. Cartan's theorem now implies that $f \in \mathrm{Aut}(W)$. □

Corollary 2.4. *Let W be a taut Riemann surface and suppose $f \in \mathrm{Hol}(W, W)$ fixes two distinct points of W. Then $f \in \mathrm{Aut}(W)$.*

Proof: The map f preserves the Kobayashi distance between the two fixed points. As the Kobayashi distance is $\mathcal{C}^\infty$ (for any taut Riemann surface) the preceding theorem implies that $f \in \mathrm{Aut}(W)$. □

Corollary 2.5. *If $f \in \mathrm{Hol}(W, W)$ fixes two distinct points $w, z \in W$ which can be joined by a unique path $\gamma\colon [0, a] \to W$ satisfying*

$$d(w, z) = l(\gamma) \text{ and } l(\gamma|_{[0,t]}) = t \quad \forall\, t$$

then f is the identity map.

Proof: The path $f\gamma$ also joins w to z. For any $t \in [0, a]$ we have

$$\begin{aligned} d(w, z) &= l(\gamma|_{[0,t]}) + l(\gamma|_{[t,a]}) \\ &\geq l(f\gamma|_{[0,t]}) + l(f\gamma|_{[t,a]}) \quad \text{by invariance} \\ &\geq d(f(w), f(z)) \\ &= d(w, z). \end{aligned}$$

It follows that we must have $l(\gamma|_{[0,t]}) = l(f\gamma|_{[0,t]})$ for all t. Our uniqueness hypothesis for γ now implies that $f(\gamma(t)) = \gamma(t)$ for all t, so f fixes each point on $\gamma([0, a])$. The identity theorem for analytic functions implies that f is the identity map. □

Remark 2. If the distance d is $\mathcal{C}^4$ then each point $w \in W$ has a neighbourhood U such that any two points of U can be joined by a unique path in U which satisfies the hypothesis of γ in the preceding corollary (such paths are usually called *length minimising geodesics*). For a taut Riemann surface W, the Kobayashi distance is $\mathcal{C}^\infty$ and hence any holomorphic map $f \in \mathrm{Hol}(W, W)$ which fixes two sufficiently close points must be the identity mapping.

Remark 3. The biholomorphism $w \mapsto \frac{1}{w}$ on the annulus $A = \{w \in \mathbb{C} : \frac{1}{2} < |w| < 2\}$ fixes the two points 1 and -1. However there is more than one length minimising geodesic joining these two points in A (with respect to the Kobayashi metric).

3. One-dimensional holomorphic retracts

In this section, following J. P. Vigué [**4**], we define analogues of the Carathéodory pseudodistance and the Kobayashi function on a complex manifold. We use these to examine the existence of holomorphic retractions of a complex manifold onto a 1-dimensional complex submanifold through two given points.

Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be ordered sequences of points in the complex manifolds X and in Y respectively. Then

$$\begin{aligned}&\operatorname{Hol}(X, x_1, \ldots, x_n, Y, y_1, \ldots, y_n)\\ &\qquad = \big\{f \in \operatorname{Hol}(X, Y) : f(x_i) = y_i \quad \forall\, i = 1, \ldots, n\big\}.\end{aligned}$$

Let W be a connected Riemann surface and d an invariant distance on W. Fix a point $w \in W$ and let X be a complex manifold with basepoint $x \in X$. Then we define a Carathéodory type function on $X \times X$ with values in $[0, \infty]$ by

$$c_{X,x}^{W,w}(x_1, x_2) = \sup\big\{d(f(x_1), f(x_2)) : f \in \operatorname{Hol}(X, x, W, w)\big\}\ \forall\, x_1, x_2 \in X.$$

The Kobayashi version $k_{X,x}^{W,w}(x_1, x_2)$ is defined as follows

(i) If $\operatorname{Hol}(W, w, w_1, w_2, X, x, x_1, x_2) = \emptyset$ for all w_1 and w_2 in W then

$$k_{X,x}^{W,w}(x_1, x_2) = \infty.$$

(ii) Otherwise $k_{X,x}^{W,w}(x_1, x_2)$ is given by

$$\inf\big\{d(w_1, w_2) : w_1, w_2 \in W,\ f \in \operatorname{Hol}(W, w, w_1, w_2, X, x, x_1, x_2)\big\}.$$

It follows from the invariance of d that

$$c_{X,x}^{W,w}(x_1, x_2) \leq k_{X,x}^{W,w}(x_1, x_2) \quad \forall\, x_1, x_2.$$

For the special case $(X, x) = (W, w)$, the invariance of d also implies

$$c_{W,w}^{W,w}(w_1, w_2) = k_{W,w}^{W,w}(w_1, w_2) = d(w_1, w_2) \quad \forall\, w_1, w_2 \in W.$$

As in the cases of the usual Carathéodory and Kobayashi functions it is straightforward to show that for all $f \in \operatorname{Hol}(X, x, Y, y)$ and $x_1, x_2 \in X$

$$c_{X,x}^{W,w}(x_1, x_2) \geq c_{Y,y}^{W,w}(f(x_1), f(x_2))$$

and

$$k_{X,x}^{W,w}(x_1, x_2) \geq k_{Y,y}^{W,w}(f(x_1), f(x_2)).$$

If we take the usual Kobayashi distance on W as our invariant distance d, then it is easy to see that the resulting function $k_{X,x}^{W,w}$ satisfies

$$k_{X,x}^{W,w} \leq k$$

where k denotes the usual Kobayashi function given by

$$k(x,x_1)=\inf\left\{\tanh^{-1}\left|\frac{z-w}{1-\overline{w}z}\right| : \exists\, w,z\in\mathbb{D}\right.$$
$$\left.\text{with } \operatorname{Hol}(\mathbb{D},w,z,X,x,x_1)\neq\emptyset\right\}$$

where $\mathbb{D}$ denotes the unit disc in the complex plane.

We now use the functions $c^{W,w}_{X,x}$ and $k^{W,w}_{X,x}$ to give a criterion for deciding when there exists a holomorphic retraction of a complex manifold X onto a submanifold biholomorphic to W which passes through two given points of X. First we recall the definition of a holomorphic retract.

Definition 3.1. A *holomorphic retraction* of X is a holomorphic mapping $\rho\colon X\to X$ such that

$$\rho|_{\rho(X)} \text{ is the identity map on } \rho(X).$$

The set $\rho(X)$ is called a *holomorphic retract of* X. It is closed and analytic.

Proposition 3.2. *Let x and x_1 be distinct points in a complex manifold X and let d be an invariant distance on a Riemann surface W. Assume that there exists a holomorphic retraction $\rho\colon X\to X$ such that $x,x_1\in\rho(X)$ and $\rho(X)$ is biholomorphic to W. Then there exists some point $w\in W$ for which*

$$0<c^{W,w}_{X,x}(x,x_1)=k^{W,w}_{X,x}(x,x_1)<\infty.$$

Proof: Let $i\colon W\to X$ be a biholomorphism of W onto $\rho(X)$. Put $w=i^{-1}(x)$ and $w_1=i^{-1}(x_1)$. The three inequalities

(i) $d(w,w_1)\geq k^{W,w}_{X,x}(i(w),i(w_1))=k^{W,w}_{X,x}(x,x_1)$

(ii) $c^{W,w}_{X,x}(x,x_1)\geq d(i^{-1}\rho(x),i^{-1}\rho(x_1))=d(w,w_1)$

(iii) $c^{W,w}_{X,x}(x,x_1)\leq k^{W,w}_{X,x}(x,x_1)$

combine to give

$$0<d(w,w_1)\leq c^{W,w}_{X,x}(x,x_1)\leq k^{W,w}_{X,x}(x,x_1)\leq d(w,w_1)<\infty$$

since $w\neq w_1$. The result follows. □

By strengthening our hypotheses, we can prove the following converse to this proposition.

Theorem 3.3. *Let x and x_1 be distinct points in a connected, taut complex manifold X. Let d be a $\mathcal{C}^1$ invariant distance on a taut Riemann surface W. If there is some point $w \in W$ for which*

(a) $c^{W,w}_{X,x}(x, x_1) = k^{W,w}_{X,x}(x, x_1) < \infty$ *and*

(b) the open ball $B(w,r)$ has compact closure in W (where $r = k^{W,w}_{X,x}(x,x_1)$),

then there exists a holomorphic retraction $\rho\colon X \to X$ such that $x, x_1 \in \rho(X)$ and $\rho(X)$ is biholomorphic to W.

Proof: Assume that there is a point $w \in W$ which satisfies the hypotheses *(a)* and *(b)*. By tautness of X we can find maps $f, f_1, f_2, \ldots$ in $\mathrm{Hol}(W, w, X, x)$ and a sequence of points $z_n \in W$ such that

(i) $f_n \to f$ uniformly on compact sets,

(ii) $f_n(z_n) = x_1$ for each n,

(iii) $\lim_{n\to\infty} d(w, z_n) = k^{W,w}_{X,x}(x, x_1)$.

As $\overline{B(w,r)}$ is compact and W is locally compact, there exists $\epsilon > 0$ such that $\overline{B(w, r+\epsilon)}$ is compact. By *(iii)*, there exists N such that $z_n \in B(w, r+\epsilon)$ for all $n \geq N$. Compactness of $\overline{B(w, r+\epsilon)}$ implies that z_n has a convergent subsequence. Passing to this subsequence if necessary, we may assume that z_n converges to z (say). Since d is continuous, we obtain

$$d(w, z) = \lim_{n\to\infty} d(w, z_n) = k^{W,w}_{X,x}(x, x_1). \tag{1}$$

As the set $\{z, z_1, z_2, \ldots\}$ is compact, conditions *(i)* and *(ii)* imply that

$$f(z) = \lim_{n\to\infty} f_n(z_n) = x_1.$$

Note that z and w are distinct. Otherwise we would would have $x = f(w) = f(z) = x_1$ which contradicts the hypothesis that x and x_1 are distinct.

Next we use the tautness of W to construct a sequence $g_n \in \mathrm{Hol}(X, x, W, w)$ which converges uniformly on compact sets (to g say) such that

$$\lim_{n\to\infty} d(g_n(x), g_n(x_1)) = c^{W,w}_{X,x}(x, x_1).$$

Let $w_1 = g(x_1)$. As $(g_n(x), g_n(x_1))$ converges to $(g(x), g(x_1)) = (w, w_1)$ and d is continuous, we obtain

$$d(w, w_1) = c^{W,w}_{X,x}(x, x_1). \tag{2}$$

Since $g(f(w)) = w$ and $g(f(z)) = w_1$, equations (1) and (2) yield $d(w,z) = d(w, g(f(z)))$. As $w \neq z$, Theorem 2.3 implies that $gf \in \mathrm{Aut}(W)$. Now set $\theta = (gf)^{-1}$ and define $\rho\colon X \to X$ by

$$\rho = f\theta g.$$

It is easy to verify that ρ is a holomorphic retraction and that g maps $\rho(X)$ biholomorphically onto W. □

Corollary 3.4. *Let x and x_1 be distinct points in a connected, taut complex manifold X. Let d be a complete, invariant, $\mathcal{C}^1$ distance on a taut Riemann surface W. If there is some point $w \in W$ for which*

$$c_{X,x}^{W,w}(x, x_1) = k_{X,x}^{W,w}(x, x_1) < \infty$$

then there exists a holomorphic retraction $\rho\colon X \to X$ such that $x, x_1 \in \rho(X)$ and $\rho(X)$ is biholomorphic to W.

Proof: As W is locally compact and d is complete and inner, the Hopf-Rinow theorem (see [**3**]) implies that each open ball $B(w,s)$ $(s > 0)$ has compact closure. Thus all of the hypotheses of the previous theorem are satisfied and the corollary follows. □

Remark 4. The Kobayashi distance for a taut Riemann surface W is both $\mathcal{C}^\infty$ and complete and thus may validly be used in applying the preceding corollary. However, for explicit calculation it may be simpler to use an invariant distance on W other than Kobayashi's.

Remark 5. The proof of Theorem 3.3 implies that $k_{X,x}^{W,w}(x, x_1) > 0$. In fact, it is not difficult to show that if x and x_1 are distinct points in a taut complex manifold X, then $k_{X,x}^{W,w}(x, x_1) > 0$ for *any* invariant distance d on *any* Riemann surface W.

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References

[1] H. Cartan, Sur les fonctions de plusieurs variables complexes: l'itération des transformations intérieures d'un domaine borné, *Math. Z.* **35** (1932), 760–773.

[2] J. M. Lee, *"Riemannian manifolds. An introduction to curvature"*, Graduate Texts in Mathematics **176**, Springer-Verlag, New York, 1997.

[3] W. Rinow, *"Die innere Geometrie der metrischen Räume"*, Die Grundlehren der mathematischen Wissenschaften **105**, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.

[4] J. P. Vigué, Géodésiques complexes et rétractes holomorphes de dimension 1, *Ann. Mat. Pura Appl. (4)* **176** (1999), 95–112.

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