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TWO WEIGHTED INEQUALITIES FOR CONVOLUTION MAXIMAL OPERATORS

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Abstract

Let $\varphi\colon \mathbb{R} \to [0,\infty)$ an integrable function such that $\varphi\chi_{(-\infty,0)} = 0$ and φ is decreasing in $(0,\infty)$. Let $\tau_h f(x) = f(x-h)$, with $h \in \mathbb{R}\setminus\{0\}$ and $f_R(x) = \frac{1}{R} f(\frac{x}{R})$, with $R > 0$. In this paper we characterize the pair of weights (u,v) such that the operators $M_{\tau_h\varphi} f(x) = \sup_{R>0} |f| * [\tau_h\varphi]_R(x)$ are of weak type (p,p) with respect to (u,v), $1 < p < \infty$.

1. Introduction

Let us consider the dilates $\varphi_R(x) = \frac{1}{R}\varphi(\frac{x}{R})$, $R > 0$, of a nonnegative integrable function φ defined on the real line. It is well known that the study of the a.e. convergence of the convolutions $f * \varphi_R$ as $R \to 0$ is related to the behavior of the maximal operator

$$M_\varphi f(x) = \sup_{R>0} |f| * \varphi_R(x).$$

If φ belongs to the set $\mathcal{F}$ of the even functions $\varphi\colon \mathbb{R} \to [0,\infty)$, decreasing in $[0,\infty)$ with $0 < \int_{\mathbb{R}} \varphi = A < \infty$, then a classical result establishes that M_φ satisfies the weighted weak type inequality

$$\int_{\{M_\varphi f > \lambda\}} u \le \frac{C}{\lambda^p} \int_{\mathbb{R}} |f|^p v, \tag{1.1}$$

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$1 \le p < \infty$, if and only if (u,v) belongs to the A_p class of Muckenhoupt, i.e., if there exists $C > 0$ such that for all $a < b$

$$\left(\int_a^b u\right)^{1/p} \left(\int_a^b v^{1-p'}\right)^{1/p'} \le C\,(b-a), \quad \text{if} \quad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and

$$Mu \le C\,v \quad \text{a.e.,} \quad \text{if} \quad p = 1,$$

where $Mf(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\,dt$ is the (two -sided) Hardy-Littlewood maximal function. The above result is a consequence of the characterization for M [**5**] and the following pointwise inequalities:

$$2\ell\varphi(\ell) Mf(x) \le M_\varphi f(x) \le AMf(x), \tag{1.2}$$

where ℓ is a positive real number such that $\varphi(\ell) > 0$ (the existence of ℓ is guaranteed since we are assuming that $\varphi \not\equiv 0$), together with the characterization of the weighted weak type inequalities for M (see [**5**] and [**7**]). The right inequality in (1.2) is a classical result (see [**7**]), the left one is an easy consequence of the inequalities

$$\frac{1}{R} \int_{\mathbb{R}} |f(y)| \varphi\left(\frac{x-y}{R}\right) dy \ge \frac{1}{R} \int_{x-\ell R}^{x+\ell R} \dots\, dy$$
$$\ge 2\ell\varphi(\ell) \left[\frac{1}{2\ell R} \int_{x-\ell R}^{x+\ell R} |f(y)|\,dy\right].$$

Sharper estimates can be obtained if the function φ is a member of the following set of functions: $\mathcal{F}^+ = \{\varphi\colon \mathbb{R} \to [0,\infty) : \varphi\chi_{(-\infty,0)} = 0$, φ decreasing in $(0,\infty)$ with $0 < \int_{\mathbb{R}} \varphi = A < \infty\}$ or $\mathcal{F}^- = \{\varphi : \varphi(-x) \in \mathcal{F}^+\}$. In fact, for almost all $x \in \mathbb{R}$ we have that, if $\varphi \in \mathcal{F}^+$ and $\ell > 0$ is such that $\varphi(\ell) > 0$, then

$$\ell\varphi(\ell) M^- f(x) \le M_\varphi f(x) \le AM^- f(x), \tag{1.3}$$

and if $\varphi \in \mathcal{F}^-$ and $\ell > 0$ is such that $\varphi(-\ell) > 0$, then

$$\ell\varphi(-\ell) M^+ f(x) \le M_\varphi f(x) \le AM^+ f(x), \tag{1.4}$$

where

$$M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(t)|\,dt \quad \text{and} \quad M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(t)|\,dt$$

are the one-sided Hardy-Littlewood maximal functions. The right inequalities were proved by M. Lorente [**2**], the left ones can be obtained as in (1.2). By (1.3) and the characterization of the weighted weak type inequalities for M^- (see [**6**] and [**3**]) we get that, if $\varphi \in \mathcal{F}^+$ and

$1 \le p < \infty$, then (1.1) holds if and only if (u,v) belongs to the Sawyer's class A_p^-, i.e., if there exists $C > 0$ such that for all $a < b < c$

$$\left(\int_b^c u\right)^{1/p} \left(\int_a^b v^{1-p'}\right)^{1/p'} \le C\,(c-a), \quad \text{if} \quad 1 < p < \infty$$

and

$$M^+ u \le C\, v \quad \text{a.e.,} \quad \text{if} \quad p = 1.$$

An analogous result holds with $\varphi \in \mathcal{F}^-$ and $(u,v) \in A_p^+$ which is the same as A_p^- but reversing the orientation of the real line.

In this paper we are interested in the behavior of the convolution maximal operator associated to a translation of a function $\varphi \in \mathcal{F}$ ($\mathcal{F}^+$ or $\mathcal{F}^-$), i.e., if $\tau_h\varphi(x) = \varphi(x-h)$ we wish to characterize (1.1) for the maximal operator

$$M_{\tau_h\varphi} f(x) = \sup_{R>0} |f| * [\tau_h\varphi]_R(x).$$

Clearly it is enough to work with functions $\varphi \in \mathcal{F}^+$ since the results for $\varphi \in \mathcal{F}^-$ are obtained similarly and the results for $\varphi \in \mathcal{F}$ follow from the corresponding ones for $\mathcal{F}^+$ and $\mathcal{F}^-$. Examples of these operators are

$$M_\alpha^- f(x) = \sup_{R>0} \frac{1}{R} \int_{x-2R}^{x-R} |f(y)| \left(\frac{x-R-y}{R}\right)^\alpha dy, \quad -1 < \alpha < 0$$

and

$$\widetilde{M}_\alpha^+ f(x) = \sup_{R>0} \frac{1}{R} \int_{x}^{x+R} |f(y)| \left(\frac{x+R-y}{R}\right)^\alpha dy, \quad -1 < \alpha < 0.$$

These operators were studied in [**1**] and [**4**] and are equal to $M_{\tau_h\varphi}$ where $\varphi(t) = t^\alpha \chi_{(0,1]}(t)$ with $h = 1$ and $h = -1$ respectively.

Observe that in the above examples $\varphi(0+) = \lim_{t\to 0^+} \varphi(t) = +\infty$. If $\varphi \in \mathcal{F}^+$ and $\varphi(0+) < +\infty$, the weighted weak type inequalities (1.1) are equivalent to conditions A_p^-, A_p^+ or A_p as it is shown in the following theorem which we shall prove in Section 2.

Theorem 1.5. *Let $1 \le p < \infty$, $\varphi \in \mathcal{F}^+$ and $\varphi(0+) < +\infty$. Then*

(i) *If $h > 0$, (1.1) holds for $M_{\tau_h\varphi}$ if and only if $(u,v) \in A_p^-$.*
(ii) *If $h < 0$ and* $\operatorname{supp}(\varphi) \subset (0, |h|]$, (1.1) *holds for $M_{\tau_h\varphi}$ if and only if $(u,v) \in A_p^+$.*
(iii) *If $h < 0$ and* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \ne \emptyset$, (1.1) *holds for $M_{\tau_h\varphi}$ if and only if $(u,v) \in A_p$.*

When $\varphi(0+) = +\infty$, the situation is different. For example, the weighted weak type inequalities (1.1) for M_α ($\widetilde{M}_\alpha$) are equivalent to conditions which are strictly contained in A_p^- (A_p^+). Therefore, there are weights in A_p^- (A_p^+) which are not good weights for M_α ($\widetilde{M}_\alpha$).

We shall dedicate Sections 3 and 4 to characterize the good weights for $M_{\tau_h \varphi}$ assuming only some restriction on the decreasingness of φ. More precisely we shall work in the rest of the paper with functions $\varphi \in \mathcal{E}^+_{\gamma,\delta}$, with $\gamma > 0$, $\delta \in (0,1)$ and

$$\mathcal{E}^+_{\gamma,\delta} = \{\varphi \in \mathcal{F}^+ : \varphi(\gamma) > 0 \text{ and } t^\delta \varphi(t) \text{ is increasing in } (0,\gamma]\}.$$

Observe that $\varphi \in \mathcal{E}^+_{\gamma,\delta}$ implies that $t\varphi(t)$ is increasing in $(0,\gamma]$. Also notice that the functions $\varphi(t) = t^\alpha \chi_{(0,1]}(t)$, corresponding to the operators M_α and $\widetilde{M}_\alpha$, belongs to $\mathcal{E}^+_{1,-\alpha}$. Other examples belonging to $\mathcal{E}^+_{\gamma,\delta}$ for some γ and some δ are the following: $\varphi(t) = t^\alpha \left(\log \frac{1}{t}\right) \chi_{(0,1]}(t)$ with $-1 < \alpha \le 0$ and $\varphi(t) = (1 + \log \frac{1}{t})\chi_{(0,1]}(t) + t^\beta \chi_{(1,\infty)}(t)$, with $\beta < -1$.

We shall prove the following characterizations of the weighted weak type (p,p) inequalities, $1 < p < \infty$, for $M_{\tau_h \varphi}$, under the assumption $\varphi \in \mathcal{E}^+_{\gamma,\delta}$. Notice that we always may assume that $0 < \gamma \le |h|$.

Theorem 1.6. *Let $1 < p < \infty$, $h > 0$, $0 < \gamma \le h$, $\delta \in (0,1)$ and $\varphi \in \mathcal{E}^+_{\gamma,\delta}$. The following statements are equivalent.*

(i) *(1.1) holds for $M_{\tau_h \varphi}$.*

(ii) *$(u,v) \in A^-_{p,\varphi,\gamma}$, i.e., there exists $C > 0$ such that*

$$\left(\int_b^c u\right)^{1/p} \left(\int_a^b v^{1-p'}(y) \varphi^{p'}\left(\frac{b-y}{c-a}\gamma\right) dy\right)^{1/p'} \le C \frac{c-a}{\gamma},$$

for all $a < b < c$.

Theorem 1.7. *Let $1 < p < \infty$, $h < 0$, $0 < \gamma \le |h|$, $\delta \in (0,1)$, $\varphi \in \mathcal{E}^+_{\gamma,\delta}$ and assume that* $\operatorname{supp}(\varphi) \subset (0,|h|]$. *The following statements are equivalent.*

(i) *(1.1) holds for $M_{\tau_h \varphi}$.*

(ii) *$(u,v) \in \widetilde{A}^+_{p,\varphi,\gamma}$, i.e., there exists $C > 0$ such that*

$$\left(\int_a^b u\right)^{1/p} \left(\int_b^c v^{1-p'}(y) \varphi^{p'}\left(\frac{c-y}{c-a}\gamma\right) dy\right)^{1/p'} \le C \frac{c-a}{\gamma},$$

for all $a < b < c$.

Theorem 1.8. *Let* $1 < p < \infty$*,* $h < 0$*,* $0 < \gamma \leq |h|$*,* $\delta \in (0,1)$*,* $\varphi \in \mathcal{E}^+_{\gamma,\delta}$ *and assume that* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$*. The following statements are equivalent.*

(i) (1.1) *holds for* $M_{\tau_h\varphi}$*.*
(ii) $(u,v) \in \widetilde{A}^+_{p,\varphi,\gamma} \cap A_p$*.*

Taking into account the results for M_α and $\widetilde{M}_\alpha$ we see that the class of good weights for $M_{\tau_h\varphi}$ will depend on the behavior of φ close to zero. This is our starting point to analize the operator $M_{\tau_h\varphi}$. In fact, given $\varphi \in \mathcal{F}^+$, $h \in \mathbb{R}$, $h \neq 0$ and $\gamma > 0$ small enough, let us say $\gamma \leq |h|$, we write

$$\varphi = \varphi\chi_{(0,\gamma]} + \varphi\chi_{(\gamma,\infty)}.$$

Then if we denote $M_{\varphi,h,\gamma} := M_{\tau_h(\varphi\chi_{(0,\gamma]})}$ and $M_{\varphi,h,\infty} := M_{\tau_h(\varphi\chi_{(\gamma,\infty)})}$ we get the following pointwise inequalities:

$$\max\{M_{\varphi,h,\gamma}, M_{\varphi,h,\infty}\} \leq M_{\tau_h\varphi} \leq M_{\varphi,h,\gamma} + M_{\varphi,h,\infty}. \tag{1.9}$$

Therefore, $M_{\tau_h\varphi}$ satisfies (1.1) if and only if (1.1) holds for $M_{\varphi,h,\gamma}$ and $M_{\varphi,h,\infty}$. The study of $M_{\varphi,h,\infty}$ is completely similar to the study of $M_{\tau_h\varphi}$ with $\varphi(0+) < \infty$. The difficult part is concentrated in the local operator $M_{\varphi,h,\gamma}$. The operators $M_{\varphi,h,\gamma}$ have the following explicit expressions:

$$M_{\varphi,h,\gamma}f(x) = \sup_{R>0} \frac{1}{R} \int_{x-(|h|+\gamma)R}^{x-|h|R} |f(y)| \varphi\left(\frac{x-|h|R-y}{R}\right) dy \quad \text{if } h > 0$$

and

$$M_{\varphi,h,\gamma}f(x) = \sup_{R>0} \frac{1}{R} \int_{x+(|h|-\gamma)R}^{x+|h|R} |f(y)| \varphi\left(\frac{x+|h|R-y}{R}\right) dy \quad \text{if } h < 0.$$

We may observe that the operators $M_{\varphi,h,\gamma}$ are of different geometric nature depending on the sign of h. If $h > 0$, the integrals are taken over intervals $I \subset (-\infty, x)$ and φ is evaluated in a point which depends on the distance of y to the end point of I nearer to x, while if $h < 0$ the integrals are computed over intervals $I \subset (x, \infty)$ and φ is evaluated in a point which depends on the distance of y to the end point of I farer from x.

The paper is organized as follows: Section 2 and 3 are devoted to the proof of Theorems 1.5 and 1.6 respectively, while we give the proofs of Theorems 1.7 and 1.8 in Section 4.

Throughout the paper h, γ and δ are real numbers, $h \neq 0$, $\gamma > 0$ with $\gamma \leq |h|$, $0 < \delta < 1$ and the classes $\mathcal{E}^+_{\gamma,\delta}$ are the ones defined above. The functions u and v will be weights, i.e., positive measurable functions. Finally, p' stands for the conjugate exponent of p, $1 < p < \infty$, and the letter C means a positive constant that may change from one line to another.

2. Proof of Theorem 1.5

Let $\varphi \in \mathcal{F}^+$ and $\varphi(0+) < +\infty$. Without loss generality we can assume that $\varphi(0) = \varphi(0+)$. The proof of Theorem 1.5 is based on the following lemma.

Lemma 2.1. *Let $\ell > 0$ be such that $\varphi(\ell) > 0$. There exist positive constants C_1 and C_2 such that*

(i) *If $h > 0$,*

$$C_1\varphi(\ell)hM^-f(x) \leq M_{\tau_h\varphi}f(x) \leq \left(\varphi(0)h + \int_0^\infty \varphi\right) M^-f(x).$$

(ii) *If $h < 0$ and* $\operatorname{supp}(\varphi) \subset (0, |h|]$,

$$C_2\varphi(\ell)|h|M^+f(x) \leq M_{\tau_h\varphi}f(x) \leq \varphi(0)|h|M^+f(x).$$

(iii) *If $h < 0$, $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$ and $\ell > |h|$,*

$$2\varphi(\ell)\min\{|h|, \ell+h\}Mf(x) \leq M_{\tau_h\varphi}f(x) \leq 2\left(\varphi(0)|h| + \int_{|h|}^\infty \varphi\right) Mf(x).$$

Before proving the above lemma we define the following maximal operators:

$$N_\mu^- f(x) = \sup_{T>0} \frac{1}{T} \int_{x-\mu T}^{x-T} |f(y)|\,dy \quad \text{for} \quad \mu > 1$$

and

$$N_\eta^+ f(x) = \sup_{T>0} \frac{1}{T} \int_{x+\eta T}^{x+T} |f(y)|\,dy \quad \text{for} \quad 0 < \eta < 1.$$

The above operators are pointwise equivalent to M^- and M^+ respectively. In fact, we have the following proposition.

Proposition 2.2. *There exist positive constants C_1 and C_2 such that*

(i) $C_1M^-f(x) \leq N_\mu^-f(x) \leq \mu M^-f(x)$ *and*
(ii) $C_2M^+f(x) \leq N_\eta^+f(x) \leq M^+f(x)$.

Proof: The right inequalities in (i) and (ii) are obvious. In order to prove the left inequality in (i), we may assume that $M^-f(x)<\infty$. Let s be such that $1/\mu<s<1$. Then, there exists $T>0$ such that

$$\begin{aligned} s\,M^-f(x) &\le \frac{1}{\mu T}\int_{x-\mu T}^{x}|f(y)|\,dy \\ &= \frac{1}{\mu T}\int_{x-\mu T}^{x-T}|f(y)|\,dy \\ &\quad + \frac{1}{\mu T}\int_{x-T}^{x}|f(y)|\,dy \\ &\le \frac{1}{\mu}N_\mu^- f(x) + \frac{1}{\mu}M^-f(x). \end{aligned}$$

Then, since $s>1/\mu$ we obtain (i) with $C_1=\mu s-1$. The left inequality in (ii) is proved similarly. In fact, assume that $M^+f(x)<\infty$ and let s be such that $\eta<s<1$. Then, there exists $T>0$ such that

$$\begin{aligned} s\,M^+f(x) &\le \frac{1}{T}\int_{x}^{x+T}|f(y)|\,dy \\ &= \frac{1}{T}\int_{x}^{x+\eta T}|f(y)|\,dy \\ &\quad + \frac{1}{T}\int_{x+\eta T}^{x+T}|f(y)|\,dy \\ &\le \eta M^+f(x) + N_\eta^+ f(x). \end{aligned}$$

Then, since $\eta<s$ we obtain (ii) with $C_2=s-\eta$. □

Proof of Lemma 2.1: (i) First, notice that $\tau_h(\varphi)$ is dominated by $\varphi(0)\chi_{(0,h]}+\tau_h(\varphi)\in\mathcal{F}^+$. Therefore, by (1.3) we get the right inequality of (i). On the other hand, we fix $\mu=\frac{h+\ell}{h}>1$ and since φ is decreasing we have that

$$\begin{aligned} \frac{1}{R}\int_{\mathbb{R}}|f(y)|\varphi\left(\frac{x-y-hR}{R}\right)dy &\ge \frac{1}{R}\int_{x-(\ell+h)R}^{x-hR}\dots\,dy \\ &\ge h\varphi(\ell)\left[\frac{1}{hR}\int_{x-\mu hR}^{x-hR}|f(y)|\,dy\right]. \end{aligned}$$

Taking supremum over $R > 0$ we have that $M_{\tau_h \varphi} f(x) \geq h\varphi(\ell) N_\mu^- f(x)$ and using Proposition 2.2(i) we obtain statement (i).

(ii) By the hypothesis on h and on the support of φ we can easily see that $\tau_h(\varphi)$ is dominated by $\varphi(0)\chi_{[h,0]} \in \mathcal{F}^-$ and by (1.4) we get that $M_{\tau_h\varphi} f(x) \leq \varphi(0)|h| M^+ f(x)$. Let us fix $\eta = \frac{|h|-\ell}{|h|}$. Then (ii) follows by the inequalities

$$\begin{aligned}\frac{1}{R}\int_{\mathbb{R}} |f(y)|\varphi\left(\frac{x-y-hR}{R}\right) dy &\geq \frac{1}{R}\int_{x+(|h|-\ell)R}^{x+|h|R} \dots\, dy \\ &\geq |h|\varphi(\ell)\left[\frac{1}{|h|R}\int_{x+\eta|h|R}^{x+|h|R} |f(y)|\, dy\right],\end{aligned}$$

taking supremum over $R > 0$ and applying Proposition 2.2(ii).

(iii) The function $\tau_h(\varphi)$ is dominated by a sum of two functions: $\phi_1 = \varphi(0)\chi_{[h,0]} \in \mathcal{F}^-$ and $\phi_2 = \tau_h(\varphi)\chi_{(0,\infty)} \in \mathcal{F}^+$. Therefore, using (1.3) and (1.4) we get that $M_{\tau_h\varphi} f(x) \leq \varphi(0)|h| M^+ f(x) + \left(\int_{|h|}^\infty \varphi\right) M^- f(x) \leq 2\left(\varphi(0)|h| + \int_{|h|}^\infty \varphi\right) Mf(x)$. On the other hand, if $\nu = \min\{|h|, \ell + h\}$, then

$$\begin{aligned}\frac{1}{R}\int_{\mathbb{R}} |f(y)|\varphi\left(\frac{x-y-hR}{R}\right) dy &\geq \frac{1}{R}\int_{x-(\ell+h)R}^{x+|h|R} \dots\, dy \\ &\geq 2\nu\varphi(\ell)\left[\frac{1}{2\nu R}\int_{x-\nu R}^{x+\nu R} |f(y)|\, dy\right].\end{aligned}$$

Therefore, taking supremum over $R > 0$ we complete the proof of the lemma. □

Now, Theorem 1.5 follows from Lemma 2.1 together with the characterizations of the weighted weak type (p,p) inequalities for M^-, M^+ and M.

3. Proof of Theorem 1.6

We shall start studying the local part $M_{\varphi,h,\gamma}$. More precisely, we shall prove the following theorem.

Theorem 3.1. *Let $1 < p < \infty$, $h > 0$, $0 < \gamma \leq h$, $\delta \in (0,1)$ and $\varphi \in \mathcal{E}^+_{\gamma,\delta}$. The following statements are equivalent.*

(i) (1.1) *holds for $M_{\varphi,h,\gamma}$.*
(ii) $(u,v) \in A^-_{p,\varphi,\gamma}$.

First, we notice that if φ, h and γ are as in Theorem 3.1 and $\beta = \frac{h+\gamma}{h} > 1$ then we have

$$M_{\varphi,h,\gamma} f(x) = \sup_{R>0} \frac{1}{R} \int_{x-\beta hR}^{x-hR} |f(y)| \varphi\left(\frac{x-hR-y}{R}\right) dy.$$

In order to prove Theorem 3.1, we define the following noncentered version of this operator

$$N_{\varphi,h,\gamma} f(x) = \sup_{(a,b)\in\mathcal{A}_x} \frac{\gamma}{b-a} \int_a^b |f(y)| \varphi\left(\frac{b-y}{b-a}\gamma\right) dy,$$

where $\mathcal{A}_x = \{(a,b) : b < x \text{ and } b-a \geq \frac{\gamma}{h}(x-b)\}$. The operators $M_{\varphi,h,\gamma}$ and $N_{\varphi,h,\gamma}$ are pointwise equivalent for $\varphi \in \mathcal{E}^+_{\gamma,\delta}$.

Proposition 3.2. *If $h > 0$, $0 < \gamma \leq h$, $\beta = \frac{h+\gamma}{h}$, $\delta \in (0,1)$ and $\varphi \in \mathcal{E}^+_{\gamma,\delta}$, then*

$$M_{\varphi,h,\gamma} f(x) \leq N_{\varphi,h,\gamma} f(x) \leq \left(\frac{\beta}{\gamma\varphi(\gamma)} \int_0^\gamma \varphi(y)\, dy + 2\right) M_{\varphi,h,\gamma} f(x).$$

Proof: The first inequality is obvious. To prove the second one let us consider $x \in \mathbb{R}$ and $(a,b) \in \mathcal{A}_x$. Let R be the positive number such that $a = x - \beta hR$. Observe that $x - b \leq hR$. Let m be the nonnegative integer number such that $x - \frac{hR}{\beta^m} \leq b < x - \frac{hR}{\beta^{m+1}}$. Then

$$\begin{aligned}\int_a^b |f(y)| \varphi\left(\frac{b-y}{b-a}\gamma\right) dy &= \left(\sum_{k=0}^{m-1} \int_{x-\frac{hR}{\beta^{k-1}}}^{x-\frac{hR}{\beta^k}} + \int_{x-\frac{hR}{\beta^{m-1}}}^{x-\frac{hR}{\beta^m}} + \int_{x-\frac{hR}{\beta^m}}^{b}\right)(\dots\, dy)\\ &= I + II + III,\end{aligned}$$

where I is understod to be zero if $m = 0$. For fixed k, $0 \le k \le m-1$, let $T = R\beta^{-k}$. Since φ is decreasing we have

$$\begin{aligned}
&\int_{x-\frac{hR}{\beta^{k-1}}}^{x-\frac{hR}{\beta^k}} |f(y)|\varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy \\
&\qquad = \int_{x-\beta hT}^{x-hT} |f(y)|\varphi\left(\frac{x-hT-y}{T}\right)\frac{\varphi\left(\frac{b-y}{b-a}\gamma\right)}{\varphi\left(\frac{x-hT-y}{T}\right)}\,dy \\
&\qquad \le \frac{1}{\varphi(\gamma)}\varphi\left(\frac{b-(x-\frac{hR}{\beta^k})}{b-a}\gamma\right)\frac{R}{\beta^k}\,M_{\varphi,h,\gamma}f(x) \\
&\qquad \le \frac{\beta}{\gamma\varphi(\gamma)}\left[\int_{x-\frac{hR}{\beta^k}}^{x-\frac{hR}{\beta^{k+1}}}\varphi\left(\frac{b-t}{b-a}\gamma\right)\,dt\right]M_{\varphi,h,\gamma}f(x).
\end{aligned}$$

Summing up in k, $0 \le k \le m-1$, we get that

$$\begin{aligned}
I &\le \frac{\beta}{\gamma\varphi(\gamma)}\left[\int_a^b \varphi\left(\frac{b-t}{b-a}\gamma\right)\,dt\right]M_{\varphi,h,\gamma}f(x) \\
&= \frac{b-a}{\gamma}\left[\frac{\beta}{\gamma\varphi(\gamma)}\left(\int_0^\gamma \varphi\right)M_{\varphi,h,\gamma}f(x)\right].
\end{aligned}$$

In order to estimate II, let $T = \frac{R}{\beta^m}$. Using that $a = x-\beta hR$, $b \ge x-\frac{hR}{\beta^m}$ and the fact that $\beta^{m+1} \ge \beta$ we can easily prove that $\frac{\gamma}{b-a} \le \frac{1}{T}$. Then, since φ is decreasing and $t\varphi(t)$ is increasing in $(0,\gamma]$ (which follows from $t^\delta\varphi(t)$ is increasing in $(0,\gamma]$) we get that

$$\begin{aligned}
II &\le \int_{x-\beta hT}^{x-hT} |f(y)|\varphi\left(\frac{x-hT-y}{b-a}\gamma\right)\,dy \\
&\le \frac{b-a}{\gamma}\left[\frac{1}{T}\int_{x-\beta hT}^{x-hT}|f(y)|\varphi\left(\frac{x-hT-y}{T}\right)\,dy\right] \le \frac{b-a}{\gamma}M_{\varphi,h,\gamma}f(x).
\end{aligned}$$

To estimate III, let $T = \frac{x-b}{h}$. Enlarging the interval $(x-\frac{hR}{\beta^m}, b)$ and using that $b-a \ge \frac{\gamma}{h}(x-b)$ and that the function $t\varphi(t)$ is increasing in

$(0,\gamma]$ we get

$$III=\int_{x-\frac{hR}{\beta^m}}^{x+(b-x)}|f(y)|\varphi\left(\frac{b-y}{b-a}\gamma\right)dy$$
$$\le\int_{x-\beta hT}^{x-hT}|f(y)|\varphi\left(\frac{x-hT-y}{b-a}\gamma\right)dy\le\frac{b-a}{\gamma}M_{\varphi,h,\gamma}f(x).$$

Putting together the estimates for I, II and III we are done. □

Proof of Theorem 3.1: We observe first that, by Proposition 3.2, statement (i) is equivalent to the same weighted weak type (p,p) inequality for $N_{\varphi,h,\gamma}$.

(i) ⇒ (ii). Let $a<b<c$. Assume that $b-a\ge\frac{\gamma}{h}(c-b)$. For every natural number n, let us consider the function $f(y)=v_n^{1-p'}(y)\varphi_n^{p'-1}\left(\frac{b-y}{c-a}\gamma\right)\chi_{(a,b)}(y)$, where $v_n=v+1/n$ and $\varphi_n=\min\{\varphi,n\}$. Since $t\varphi(t)$ is increasing and $\varphi\ge\varphi_n$ we have for all $x\in(b,c)$,

$$N_{\varphi,h,\gamma}f(x)\ge\frac{\gamma}{b-a}\int_a^b v_n^{1-p'}(y)\varphi_n^{p'-1}\left(\frac{b-y}{c-a}\gamma\right)\varphi\left(\frac{b-y}{b-a}\gamma\right)dy$$
$$\ge\frac{\gamma}{c-a}\int_a^b v_n^{1-p'}(y)\varphi_n^{p'}\left(\frac{b-y}{c-a}\gamma\right)dy\equiv\lambda.$$

This means that $(b,c)\subset\{N_{\varphi,h,\gamma}f\ge\lambda\}$. Then by (i) (with $N_{\varphi,h,\gamma}$) we get the inequality

$$\left(\int_b^c u\right)^{1/p}\left(\int_a^b v_n^{1-p'}(y)\varphi_n^{p'}\left(\frac{b-y}{c-a}\gamma\right)dy\right)^{1/p'}\le C\frac{c-a}{\gamma}.$$

Letting n tend to ∞, we obtain $A^-_{p,\varphi,\gamma}$ with $b-a\ge\frac{\gamma}{h}(c-b)$.

Assume now that $b-a<\frac{\gamma}{h}(c-b)$. Let $\overline{a}<a$ such that $b-\overline{a}=\frac{\gamma}{h}(c-a)$. If $f(y)=v_n^{1-p'}(y)\varphi_n^{p'-1}\left(\frac{b-y}{b-\overline{a}}\gamma\right)\chi_{(a,b)}(y)$ then for all $x\in(b,c)$ we obtain

$$N_{\varphi,h,\gamma}f(x)\ge\frac{\gamma}{b-\overline{a}}\int_a^b v_n^{1-p'}(y)\varphi_n^{p'}\left(\frac{b-y}{b-\overline{a}}\gamma\right)dy$$
$$=\frac{h}{c-a}\int_a^b v_n^{1-p'}(y)\varphi_n^{p'}\left(\frac{b-y}{c-a}h\right)dy\equiv\lambda.$$

Applying (i) with $N_{\varphi,h,\gamma}$ and letting n tend to ∞ we have

$$\left(\int_b^c u\right)^{1/p}\left(\int_a^b v^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{c-a}h\right)dy\right)^{1/p'}\le C\frac{c-a}{h}.$$

Now, $A^-_{p,\varphi,\gamma}$ follows since $\gamma\le h$ and $t\varphi(t)$ is increasing in $(0,\gamma]$.

(ii) $\Rightarrow$ (i). This implication follows from the following proposition and the fact that the maximal operator $M_u^- g(x)=\sup_{h<x}\frac{\int_h^x |g|u}{\int_h^x u}$ is of weak type $(1,1)$ with respect to the measure $u(x)\,dx$. □

Proposition 3.3. *Let $1<p<\infty$, $h>0$, $0<\gamma\le h$, $\delta\in(0,1)$ and $\varphi\in\mathcal{E}^+_{\gamma,\delta}$. Assume that $(u,v)\in A^-_{p,\varphi,\gamma}$. Then, there exists $C>0$ such that for every measurable function f*

$$N_{\varphi,h,\gamma}f(x)\le C\left[M_u^-\left(|f|^p vu^{-1}\right)(x)\right]^{1/p}.$$

Proof: Let $x\in\mathbb{R}$ and $(a,b)\in\mathcal{A}_x=\{(a,b): b<x \text{ and } b-a\ge\frac{\gamma}{h}(x-b)\}$. First, let us assume that $4\int_b^x u>\int_a^x u$. Since $(u,v)\in A^-_{p,\varphi,\gamma}$, by Hölder inequality, we have

$$\begin{aligned}
&\int_a^b |f(y)|\varphi\left(\frac{b-y}{b-a}\gamma\right)dy\\
&\qquad\le\left(\int_a^b |f|^p v\right)^{1/p}\left(\int_a^b v^{1-p'}(y)\varphi^{p'}\left(\frac{b-y}{b-a}\gamma\right)dy\right)^{1/p'}\\
&\qquad\le C\left(\int_a^x |f|^p v\right)^{1/p}\left(\int_b^x u\right)^{-1/p}\frac{x-a}{\gamma}\\
&\qquad\le C\frac{b-a}{\gamma}\left(\frac{\gamma+h}{\gamma}\left[M_u^-\left(|f|^p vu^{-1}\right)(x)\right]^{1/p}\right).
\end{aligned}$$

Now, assume that $4\int_b^x u\le\int_a^x u$. Let $\{x_i\}$ be the increasing sequence in $[a,x]$ defined by $x_0=a$ and

$$\int_{x_{i+1}}^x u=\int_{x_i}^{x_{i+1}} u=\frac{1}{2}\int_{x_i}^x u.$$

Let N be such that $x_N\le b<x_{N+1}$ (observe that $N\ge 2$). Then we have

$$\int_a^b |f(y)|\varphi\left(\frac{b-y}{b-a}\gamma\right)dy=\sum_{i=0}^{N-2}\int_{x_i}^{x_{i+1}}\dots dy+\int_{x_{N-1}}^b\dots dy=I+II.$$

First we estimate II. By the condition $A^-_{p,\varphi,\gamma}$, the monotonicities of φ and $t\varphi(t)$ in $(0,\gamma]$ and the inequality $\int_{x_{N-1}}^x u \le 4\int_b^x u$, we get

$$
\begin{aligned}
II \le & \left(\int_{x_{N-1}}^b |f|^p v\right)^{1/p}\left(\int_{x_{N-1}}^b v^{1-p'}(y)\varphi^{p'}\!\left(\frac{b-y}{x-a}\gamma\right)\,dy\right)^{1/p'} \\
\le & \left(\int_{x_{N-1}}^x |f|^p v\right)^{1/p}\left(\int_{x_{N-1}}^b v^{1-p'}(y)\varphi^{p'}\!\left(\frac{b-y}{x-x_{N-1}}\gamma\right)\,dy\right)^{1/p'}\frac{x-a}{x-x_{N-1}} \\
\le & C\left(\int_{x_{N-1}}^x |f|^p v\right)^{1/p}\left(\int_b^x u\right)^{-1/p}\frac{x-a}{\gamma} \\
\le & C\frac{b-a}{\gamma}\left(\frac{\gamma+h}{\gamma}\left[M_u^-\left(|f|^p v u^{-1}\right)(x)\right]^{1/p}\right).
\end{aligned}
$$

Now we shall estimate I. Notice that for each i, $0 \le i \le N-2$, there exists $q_i = \frac{Ux_{i+1}-b}{U-1}$ where $U = \frac{b-a}{x_{i+1}-x_i} > 1$ such that $q_i \in [x_i, x_{i+1}]$ and

$$
\frac{b-y}{b-a} \ge \frac{x_{i+1}-y}{x_{i+1}-x_i} \quad \text{if and only if} \quad y \ge q_i.
$$

Then we can write

$$
\int_{x_i}^{x_{i+1}} |f(y)|\varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy = \int_{x_i}^{q_i}\dots\,dy + \int_{q_i}^{x_{i+1}}\dots\,dy = III + IV.
$$

Since φ is decreasing, the Hölder inequality, the hypothesis $(u,v) \in A^-_{p,\varphi,\gamma}$ and the definition of the sequence $\{x_i\}$ give

$$
\begin{aligned}
IV \le & \int_{q_i}^{x_{i+1}} |f(y)|\varphi\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right)\,dy \le \int_{x_i}^{x_{i+1}}\dots\,dy \\
\le & \left(\int_{x_i}^{x_{i+1}} |f|^p v\right)^{1/p}\left(\int_{x_i}^{x_{i+1}} v^{1-p'}(y)\varphi^{p'}\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right)\,dy\right)^{1/p'} \\
\le & C\left(\int_{x_i}^{x} |f|^p v\right)^{1/p}\left(\int_{x_{i+1}}^{x_{i+2}} u\right)^{-1/p}\frac{x_{i+2}-x_i}{\gamma} \\
\le & C\,\frac{x_{i+2}-x_i}{\gamma}\left[M_u^-\left(|f|^p v u^{-1}\right)(x)\right]^{1/p}.
\end{aligned}
$$

To estimate III we shall use that $\frac{b-y}{b-a} < \frac{x_{i+1}-y}{x_{i+1}-x_i}$ if and only if $y < q_i$ and the fact that $t^\delta\varphi(t)$ is increasing in $(0,\gamma]$. Then,

$$III = \int_{x_i}^{q_i} |f(y)|\varphi\left(\frac{b-y}{b-a}\gamma\right)\,dy \le \int_{x_i}^{q_i} |f(y)|\varphi\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right) g(y)\,dy,$$

where $g(y) = \left(\frac{b-y}{b-a}\right)^{-\delta}\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\right)^{\delta}$. Since g is decreasing in (x_i, q_i), we have

$$III \le \left(\frac{b-x_i}{b-a}\right)^{-\delta} \int_{x_i}^{x_{i+1}} |f(y)|\varphi\left(\frac{x_{i+1}-y}{x_{i+1}-x_i}\gamma\right)\,dy.$$

Using the same argument as in the boundedness of IV and the increasingness of $(b-y)^{-\delta}$ we get that

$$\begin{aligned} III &\le C\left(\frac{b-x_i}{b-a}\right)^{-\delta} \frac{x_{i+2}-x_i}{\gamma}\left[M_u^-\left(|f|^p vu^{-1}\right)(x)\right]^{1/p} \\ &\le \frac{C}{\gamma}\left(\int_{x_i}^{x_{i+2}}\left(\frac{b-y}{b-a}\right)^{-\delta} dy\right)\left[M_u^-\left(|f|^p vu^{-1}\right)(x)\right]^{1/p}. \end{aligned}$$

Now, adding up in i, we get that

$$I \le C\frac{b-a}{\gamma}\left(\frac{2-\delta}{1-\delta}\left[M_u^-\left(|f|^p vu^{-1}\right)(x)\right]^{1/p}\right).$$

Finally, putting together the estimates of I and II, we are done. □

As a consequence of Theorem 1.5 we get the following characterization of the weak type inequalities for $M_{\varphi,h,\infty}$.

Theorem 3.4. *Let $\varphi \in \mathcal{F}^+$, $h > 0$ and $0 < \gamma \le h$. Then* (1.1) *holds for $M_{\varphi,h,\infty}$ if and only if $(u,v) \in A_p^-$.*

Proof: Let $\psi = \tau_{-\gamma}(\varphi\chi_{(\gamma,\infty)})$. It is clear that $\psi \in \mathcal{F}^+$ and $\tau_h(\varphi\chi_{(\gamma,\infty)}) = \tau_{h+\gamma}(\psi)$. Then $M_{\varphi,h,\infty}$ is equal to the operator $M_{\tau_{h+\gamma}\psi}$. Therefore, since $h+\gamma > 0$ and $\psi(0+) = \varphi(\gamma) < +\infty$, applying Theorem 1.5(i) we are done. □

Now, we can prove Theorem 1.6.

Proof of Theorem 1.6: (i) ⇒ (ii). This is an easy consequence of Theorem 3.1 and the fact that (i) implies statement (i) in Theorem 3.1.

(ii) ⇒ (i). By (1.9) we only have to see that $M_{\varphi,h,\gamma}$ and $M_{\varphi,h,\infty}$ satisfy (1.1). On one hand, by Theorem 3.1, $(u,v) \in A_{p,\varphi,\gamma}^-$ implies that $M_{\varphi,h,\gamma}$ verifies (1.1). On the other hand, since φ is decreasing it is easy to prove that $A^-_{p,\varphi,\gamma} \subset A^-_p$. Therefore, Theorem 3.4 gives that $M_{\varphi,h,\infty}$ is of weak type (p,p) with respect to the pair (u,v). □

4. Proof of Theorems 1.7 and 1.8

As in the proof of Theorem 1.6, the hard work in the proof of Theorems 1.7 and 1.8 is in the study of the local part $M_{\varphi,h,\gamma}$. For $h<0$ we shall prove the following theorem.

Theorem 4.1. *Let* $1<p<\infty$, $h<0$, $0<\gamma\le|h|$, $\delta\in(0,1)$ *and* $\varphi\in\mathcal{E}^+_{\gamma,\delta}$. *The following statements are equivalent.*

(i) (1.1) *holds for* $M_{\varphi,h,\gamma}$.

(ii) $(u,v)\in\widetilde{A}^+_{p,\varphi,\gamma}$.

Before proving the theorem we shall show that it suffices to prove it for the case $|h|=\gamma$, i.e. $h=-\gamma$. First, if $|h|>\gamma$ and

$$\phi(x)=\varphi\chi_{(0,\gamma]}+\varphi(\gamma)\chi_{(\gamma,|h|]}, \tag{4.2}$$

we have that $\phi\in\mathcal{E}^+_{|h|,\delta}$. Furthermore, the following lemma shows that the operators $M_{\varphi,h,\gamma}$ and $M_{\phi,h,|h|}$ are pointwise equivalent.

Lemma 4.3. *Let* $\gamma>0$, $\delta\in(0,1)$ *and* $\varphi\in\mathcal{E}^+_{\gamma,\delta}$. *Assume* $h<0$ *such that* $|h|>\gamma$ *and let* ϕ *be as in* (4.2). *Then there exists* $C>0$ *such that*

$$C\,M_{\phi,h,|h|}f(x)\le M_{\varphi,h,\gamma}f(x)\le M_{\phi,h,|h|}f(x).$$

Proof: The second inequality is obvious since $\varphi\le\phi$. To prove the first one, we fix $\eta=\frac{|h|-\gamma}{|h|}$. Since φ is decreasing we have that

$$\frac{1}{R}\int_{x+(|h|-\gamma)R}^{x+|h|R}|f(y)|\varphi\left(\frac{x+|h|R-y}{R}\right)dy \ge \varphi(\gamma)|h|\left[\frac{1}{|h|R}\int_{x+\eta|h|R}^{x+|h|R}|f(y)|\,dy\right].$$

Taking supremum over $R>0$ we have that $M_{\varphi,h,\gamma}f(x)\ge\varphi(\gamma)|h|N^+_\eta f(x)$, where N^+_η is the operator defined in Section 2. Then, by Proposition 2.2(ii) we get

$$M_{\varphi,h,\gamma}f(x)\ge C\varphi(\gamma)|h|M^+f(x). \tag{4.4}$$

Now, by the definition of ϕ we obtain

$$\frac{1}{R}\int_x^{x+|h|R}|f(y)|\phi\left(\frac{x+|h|R-y}{R}\right)dy=\frac{\varphi(\gamma)}{R}\int_x^{x+(|h|-\gamma)R}|f(y)|\,dy$$
$$+\frac{1}{R}\int_{x+(|h|-\gamma)R}^{x+|h|R}|f(y)|\varphi\left(\frac{x+|h|R-y}{R}\right)dy.$$

Taking supremum over $R>0$ and using (4.4) we have

$$\begin{aligned}M_{\phi,h,|h|}f(x)&\leq\varphi(\gamma)(|h|-\gamma)M^+f(x)+M_{\varphi,h,\gamma}f(x)\\&\leq\varphi(\gamma)|h|M^+f(x)+M_{\varphi,h,\gamma}f(x)\\&\leq C\;M_{\varphi,h,\gamma}f(x),\end{aligned}$$

as we wished to prove. □

Once Lemma 4.3 has been proved we are able to show that Theorem 4.1 for $|h|>\gamma$ follows from Theorem 4.1 from $h=-\gamma$. In fact, let us assume that Theorem 4.1 is proved for $h=-\gamma$. By Lemma 4.3, we can easily see that (i) is equivalent to $(u,v)\in\widetilde{A}^+_{p,\phi,|h|}$, i.e., there exists $C>0$ such that

$$\left(\int_a^b u\right)^{1/p}\left(\int_b^c v^{1-p'}(y)\phi^{p'}\left(\frac{c-y}{c-a}|h|\right)dy\right)^{1/p'}\leq C\frac{c-a}{|h|},$$

for all $a<b<c$. It only remains to prove that $\widetilde{A}^+_{p,\phi,|h|}$ and $\widetilde{A}^+_{p,\varphi,\gamma}$ are equivalent. The implication $(u,v)\in\widetilde{A}^+_{p,\phi,|h|}\Rightarrow(u,v)\in\widetilde{A}^+_{p,\varphi,\gamma}$ is a consequence of the increasingness of $t\phi(t)$ in $(0,|h|]$ while the converse follows from the fact that ϕ is decreasing.

Proof of Theorem 4.1 for $h=-\gamma$: Notice that in this case

$$\begin{aligned}M_{\varphi,h,\gamma}f(x)&=\sup_{R>0}\frac{1}{R}\int_x^{x+|h|R}|f(y)|\varphi\left(\frac{x+|h|R-y}{R}\right)dy\\&=\sup_{c>x}\frac{\gamma}{c-x}\int_x^c|f(y)|\varphi\left(\frac{c-y}{c-x}\gamma\right)dy.\end{aligned}$$

(i) $\Rightarrow$ (ii). Let $a<b<c$. Let v_n and φ_n be as in the proof of Theorem 3.1 and let us consider $f(y)=v_n^{1-p'}(y)\varphi_n^{p'-1}\left(\frac{c-y}{c-a}\gamma\right)\chi_{(b,c)}(y)$.

Using that $t\varphi(t)$ is increasing in $(0,\gamma]$ and $\varphi \geq \varphi_n$, we have for all $x \in (a,b)$,

$$M_{\varphi,h,\gamma}f(x) \geq \frac{\gamma}{c-x}\int_b^c v_n^{1-p'}(y)\varphi_n^{p'-1}\left(\frac{c-y}{c-a}\gamma\right)\varphi\left(\frac{c-y}{c-x}\gamma\right)dy$$
$$\geq \frac{\gamma}{c-a}\int_b^c v_n^{1-p'}(y)\varphi_n^{p'}\left(\frac{c-y}{c-a}\gamma\right)dy \equiv \lambda.$$

Then (ii) follows applying (i) and letting n tend to ∞.

The implication (ii) $\Rightarrow$ (i) follows, as in the proof of Theorem 3.1, from the following proposition.

Proposition 4.5. *Let $1 < p < \infty$, $\gamma > 0$, $\delta \in (0,1)$, $\varphi \in \mathcal{E}^+_{\gamma,\delta}$, $h = -\gamma$ and $(u,v) \in \widetilde{A}^+_{p,\varphi,\gamma}$. Then, there exists $C > 0$ such that for every measurable function f*

$$M_{\varphi,h,\gamma}f(x) \leq C\left[M_u^+\left(|f|^p vu^{-1}\right)(x)\right]^{1/p}.$$

Proof: Let $x \in \mathbb{R}$. Let $\{x_i\}$ be the decreasing sequence in $[x,c]$ defined by $x_0 = c$ and

$$\int_x^{x_{i+1}} u = \int_{x_{i+1}}^{x_i} u = \frac{1}{2}\int_x^{x_i} u.$$

Then,

$$\int_x^c |f(y)|\varphi\left(\frac{c-y}{c-x}\gamma\right)dy = \sum_{i=0}^{\infty}\int_{x_{i+1}}^{x_i}|f(y)|\varphi\left(\frac{c-y}{c-x}\gamma\right)dy.$$

The rest of the proof follows in a similar way as in the proof of Proposition 3.3. In fact, by taking $q_i = \frac{Ux_i-c}{U-1}$ with $U = \frac{c-x}{x_i-x_{i+1}} > 1$ we can prove that $\frac{c-y}{c-x} \geq \frac{x_i-y}{x_i-x_{i+1}}$ if and only if $y \in [q_i, x_i]$. Then

$$\int_{x_{i+1}}^{x_i}|f(y)|\varphi\left(\frac{c-y}{c-x}\gamma\right)dy = \int_{x_{i+1}}^{q_i}\dots dy + \int_{q_i}^{x_i}\dots dy = I + II.$$

Since φ is decreasing, the Hölder inequality, the hypothesis $(u,v) \in \widetilde{A}^{+}_{p,\varphi,\gamma}$ and the definition of the sequence $\{x_i\}$ give

$$\begin{aligned} II &\leq \int_{q_i}^{x_i} |f(y)|\varphi\left(\frac{x_i-y}{x_i-x_{i+2}}\gamma\right)\,dy \leq \int_{x_{i+1}}^{x_i} \dots\,dy \\ &\leq \left(\int_{x_{i+1}}^{x_i} |f|^p v\right)^{1/p} \left(\int_{x_{i+1}}^{x_i} v^{1-p'}(y)\varphi^{p'}\left(\frac{x_i-y}{x_i-x_{i+2}}\gamma\right)\,dy\right)^{1/p'} \\ &\leq C\left(\int_{x}^{x_i} |f|^p v\right)^{1/p}\left(\int_{x_{i+2}}^{x_{i+1}} u\right)^{-1/p} \frac{x_i-x_{i+2}}{\gamma} \\ &\leq C\,\frac{x_i-x_{i+2}}{\gamma}\left[M_u^+\left(|f|^p v u^{-1}\right)(x)\right]^{1/p}. \end{aligned}$$

To estimate I we shall use that $\frac{c-y}{c-x} < \frac{x_i-y}{x_i-x_{i+1}}$ if and only if $y < q_i$ and the fact that $t^{\delta}\varphi(t)$ is increasing in $(0,\gamma]$. Then,

$$I = \int_{x_{i+1}}^{q_i} |f(y)|\varphi\left(\frac{c-y}{c-x}\gamma\right)\,dy \leq \int_{x_{i+1}}^{q_i} |f(y)|\varphi\left(\frac{x_i-y}{x_i-x_{i+2}}\gamma\right)g(y)\,dy,$$

where $g(y) = \left(\frac{c-y}{c-x}\right)^{-\delta}\left(\frac{x_i-y}{x_i-x_{i+2}}\right)^{\delta}$. Since g is decreasing in (x_{i+2},q_i), we have

$$I \leq \left(\frac{c-x_{i+2}}{c-x}\right)^{-\delta}\int_{x_{i+1}}^{x_i} |f(y)|\varphi\left(\frac{x_i-y}{x_i-x_{i+2}}\gamma\right)\,dy.$$

With the same argument as in the boundedness of IV in the proof of Theorem 3.1, using that $(c-y)^{-\delta}$ is increasing, we get that

$$\begin{aligned} I &\leq C\left(\frac{c-x_{i+2}}{c-x}\right)^{-\delta}\frac{x_i-x_{i+2}}{\gamma}\left[M_u^+\left(|f|^p v u^{-1}\right)(x)\right]^{1/p} \\ &\leq \frac{C}{\gamma}\left(\int_{x_{i+2}}^{x_i}\left(\frac{c-y}{c-x}\right)^{-\delta}dy\right)\left[M_u^+\left(|f|^p v u^{-1}\right)(x)\right]^{1/p}. \end{aligned}$$

Now, adding up in i, we obtain

$$I + II \leq C\frac{c-x}{\gamma}\left(\frac{2-\delta}{1-\delta}\left[M_u^+\left(|f|^p v u^{-1}\right)(x)\right]^{1/p}\right),$$

and we are done. □

As in the case $h > 0$, we obtain the characterizations for $M_{\varphi,h,\infty}$ from Theorem 1.5.

Theorem 4.6. *Let* $1 \leq p < \infty$, $\varphi \in \mathcal{F}^+$, $h < 0$ *and* $0 < \gamma \leq |h|$. *Then*

(i) *If* $\operatorname{supp}(\varphi) \subset (0, |h|]$ *and* $\gamma = |h|$, *then* $M_{\varphi,h,\infty} \equiv 0$.
(ii) *If* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$ *and* $\gamma = |h|$, (1.1) *holds for* $M_{\varphi,h,\infty}$ *if and only if* $(u, v) \in A_p^-$.
(iii) *If* $\operatorname{supp}(\varphi) \subset (0, |h|]$ *and* $\gamma < |h|$, (1.1) *holds for* $M_{\varphi,h,\infty}$ *if and only if* $(u, v) \in A_p^+$.
(iv) *If* $\operatorname{supp}(\varphi) \cap (|h|, \infty) \neq \emptyset$ *and* $\gamma < |h|$, (1.1) *holds for* $M_{\varphi,h,\infty}$ *if and only if* $(u, v) \in A_p$.

Proof: (i) is obvious. As in the proof of Theorem 3.4, taking $\psi = \tau_{-\gamma}(\varphi \chi_{(\gamma,\infty)}) \in \mathcal{F}^+$ the operator $M_{\varphi,h,\infty}$ is equal to $M_{\tau_{h+\gamma}\psi}$. In the case (ii), $M_{\tau_{h+\gamma}\psi} = M_\psi$ and therefore, (ii) follows from one of the results cited in the introduction. In the cases (iii) and (iv) we have that $h + \gamma < 0$ and applying Theorem 1.5(ii) and (iii) we are done. □

Now we shall prove Theorems 1.7 and 1.8.

Proof of Theorem 1.7: The proof follows as the proof of Theorem 1.6 using Theorem 4.1, Theorem 4.6(i) and (iii), the inequalities (1.9) and the fact that $\widetilde{A}^+_{p,\varphi,\gamma} \subset A_p^+$ which is a consequence of the decreasingness of φ. □

Proof of Theorem 1.8: It follows from Theorem 4.1, Theorem 4.6(ii) and (iv) and inequalities (1.9). □

Remark 4.7. We have not studied in this paper the case $p = 1$. The study of the weighted weak type inequality $(1, 1)$ for $M_{\tau_h\varphi}$ will appear in a forthcoming paper on weighted restricted weak type inequalities for this operator and $1 \leq p < \infty$ (notice that the restricted weak type $(1, 1)$ inequality for $M_{\tau_h\varphi}$ is equivalent to the weak type $(1, 1)$ inequality [**8**]).

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