Publ. Mat. **46** (2002), 453–463

ℵ-PRODUCTS OF MODULES AND SPLITNESS

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Abstract

Let

$$0 \longrightarrow \prod_I^{\aleph} M_\alpha \xrightarrow{\lambda} \prod_I M_\alpha \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$$

be an exact sequence of modules, in which $\aleph$ is an infinite cardinal, λ the natural injection and γ the natural surjection. In this paper, the conditions are given mainly in the four theorems so that λ (γ respectively) is split or locally split. Consequently, some known results are generalized. In particular, Theorem 1 of [**7**] and Theorem 1.6 of [**5**] are improved.

1. Introduction

Let $\aleph$ be an infinite cardinal number, and $\{M_i \mid i \in I\}$ a family of left R-modules. As a generalization of the direct sum of modules, the $\aleph$-product of $\{M_i \mid i \in I\}$ is the submodule $\prod_I^{\aleph} M_i = \{x \in \prod_I M_i \mid |\operatorname{supp} x| < \aleph\} \leq \prod_I M_i$, in which $\operatorname{supp} x$ is the support set of $x = (x_\alpha)_{\alpha\in I}$, i.e., $\operatorname{supp} x = \{\alpha \in I \mid x_\alpha \neq 0\}$. So, given any family of left R-modules $\{M_\alpha\}_{\alpha\in I}$, we can always obtain the following exact sequence:

$$0 \longrightarrow \prod_I^{\aleph} M_\alpha \xrightarrow{\lambda} \prod_I M_\alpha \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0,$$

where λ denotes the natural injection and γ denotes the natural projection. Just like the direct sum is not a summand of the direct product in general, the same case often happens for the $\aleph$-product of modules. In other words, the natural injection λ does not split generally. Here, the questions arise naturally:

2000 *Mathematics Subject Classification.* Primary: 16A50; Secondary: 16A38.
Key words. $\aleph$-product, splitness, $\aleph$-*ACC*.
The author is supported in part by Chinese Ministry of Education and NSF of China. Also, the author would like to thank Prof. T. Y. Lam and Mathematics Department of University of California, Berkeley, for their help and hospitalities.

What conditions can make λ (γ, resp.) split, even a bit weakly, locally split? On the other hand, assume λ is split, then what information can we also obtain from that?

In the present paper, we answer the questions above, and as an application of the main results of this paper, some known results are generalized. For instance, we improve Theorem 1 of [**7**] and Theorem 1.6 of [**5**].

As usual, rings are associative with $1 \neq 0$, modules are unitary throughout this paper. A cardinal $\aleph$ is said to be regular if it is not of form of $\sum_{i \in I} \mu_i$ with $\mu_i < \aleph$ and $|I| < \aleph$.

2. Main results

Recall that a left R-module M satisfies the $\aleph$-*ACC* on annihilators, if any well-ordered ascending chain of annihilators of subsets of M has $< \aleph$ distinct elements. We first state an equivalent characterization of M has $\aleph$-*ACC* on annihilators.

Lemma 1. *Let M be a left R-module, $\aleph$ an infinite regular cardinal and I an index set with $|I| = \aleph$. Then the following are equivalent.*

(1) *M has $\aleph$-ACC on annihilators;*
(2) *The natural map* $\operatorname{Hom}_R(R/A, \prod_I M) \longrightarrow \operatorname{Hom}_R(R/A, \operatorname{Coker} \lambda)$ *is onto for every cyclic R-modules R/A, with A a left ideal generated by $\aleph$ elements.*

Proof: (1) $\Rightarrow$ (2). We only need use (1) $\Rightarrow$ (3) of Theorem 8 of [**6**].

(2) $\Rightarrow$ (1). Let $\omega_\aleph$ denote the least ordinal number with cardinality $\aleph$, we can identify I with the set of ordinals $< \omega_\aleph$. Suppose M does not have $\aleph$-*ACC* on annihilators, then by (1) $\iff$ (4) of Theorem 8 of [**6**] again, we have sets

$$S = \{m_\alpha,\, \alpha < \omega_\aleph\} \subseteq M \quad \text{and}$$
$$S = \{r_\alpha,\, \alpha < \omega_\aleph\} \subseteq R$$

such that $r_\alpha m_\alpha \neq 0$ for all $\alpha < \omega_\aleph$ and $r_\beta m_\alpha = 0$ for all $\alpha > \beta$. Take $x = (m_\alpha)_{\alpha < \omega_\aleph}$, then $x \in \prod_I M$. For all $\beta < \omega_\aleph$, $r_\beta x = (r_\beta m_\alpha)_{\alpha < \omega_\aleph} \in \prod_I^\aleph M$ because $r_\beta m_\alpha = 0$ for all $\beta < \alpha$. Consider the left ideal A of R generated by $\{r_\alpha,\, \alpha < \omega_\aleph\}$, and let $f\colon R \longrightarrow \prod_I M$, $r \longrightarrow rx$, then $f(A) \subseteq \prod_I^\aleph Rx$. Therefore there exists a unique homomorphism φ such that the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_I^{\aleph} M & \xrightarrow{\lambda} & \prod_I M & \xrightarrow{\gamma} & \operatorname{Coker}\lambda & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f|_A} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & A & \xrightarrow{i} & R & \longrightarrow & R/A & \longrightarrow & 0
\end{array}$$

commutes.

By hypothesis, there also exists a homomorphism $\varphi_1 \colon R/A \longrightarrow \prod_I M$ such that $\varphi = \gamma\varphi_1$, where γ represents the natural mapping. So, using Theorem 3.1 of [**2**], we can find a homomorphism $\varphi_2 \colon R \longrightarrow \prod_I^{\aleph} M$ such that $f|_A = \varphi_2 i$. Under this case, if let $\varphi_2(1) = (y_\alpha)_{\alpha<\omega_\aleph} \in \prod_I^{\aleph} M$, then $r_\beta y_\beta = r_\beta m_\beta \neq 0$ for all $\beta < \omega_\aleph$, thus $y_\alpha \neq 0$ for each $\alpha < \omega_\aleph$. This contradicts the fact that $\varphi_2(1) \in \prod_I^{\aleph} M$. □

Theorem 1. *Let $\aleph$ be a regular cardinal, I an index set with $|I| = \aleph$, and M a left R-module, together with the short exact sequence*

$$0 \longrightarrow \prod_I^{\aleph} M \xrightarrow{\lambda} \prod_I M \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0.$$

If γ is locally split, then M has $\aleph$-ACC on annihilators. Moreover, if M is faithful, then R has $\aleph$-ACC on annihilators.

Proof: Let A be a left ideal generated by $\aleph$ elements. Take any $f \in \operatorname{Hom}_R(R/A, \operatorname{Coker}\lambda)$. Note that R/A is cyclic, and generated by $\bar{1}$. So let $f(\bar{1}) = x \in \operatorname{Coker}\lambda$, then by hypothesis, there exists a $\psi \colon \operatorname{Coker}\lambda \longrightarrow \prod_I M$ such that $\gamma\psi(x) = x$. Thus, let $\psi_1 = \psi f$, then $\psi_1 \in \operatorname{Hom}_R(R/A, \prod_I M)$ and $\gamma\psi_1 = \gamma\psi f = f$. Using Lemma 1, it follows that M has $\aleph$-*ACC* on annihilators. Furthermore, suppose M is faithful and γ is locally split, consider $\{sm : s \in S, m \in M\}$, then $x\{sm; s \in S, m \in M\} = 0 \iff xS = 0$. So the left annihilator of S is the annihilator of a subset of M. This completes the whole proof. □

Observation. Take $\aleph = \aleph_0$ in the theorem above, obviously $\prod_I^{\aleph_0} M = \oplus_{i=1}^{\infty} M$. Now if $\oplus_{i=1}^{\infty} M$ is a direct summand of $\prod_{i=1}^{\infty} M$, then the exact sequence

$$0 \longrightarrow \bigoplus_{i=1}^{\infty} M \xrightarrow{\lambda} \prod_{i=1}^{\infty} M \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$$

splits. Of course, γ is locally split. So in this case, Theorem 1 above implies that M has *ACC* on annihilators. Furthermore, if we let M be a faithful left module, then Theorem 1 above shows also that R must have ACC on annihilators, which is exactly the Corollary 2 of [**4**]. In particular, when $\oplus_{i=1}^{\infty} R$ is a direct summand of $\prod_{i=1}^{\infty} R$, we get that R has ACC on annihilators, a well-known result.

It is well known that a ring R is left coherent $\iff \prod_I M_i$ is flat for any family of right flat R-modules and many attempts have been made to generalize it, mainly by means of direct or large subdirect products of various special modules. For example, n-coherent rings, $\aleph$-coherent rings and so on.

Let $\aleph$ be an infinite cardinal. According to [**5**], a left module M is said to be $\aleph$-finitely generated if for any subset $S \subseteq M$ such that $|S| < \aleph$, there exists a f.g. submodule N of M such that $S \subseteq N$; A ring R is said to be left $\aleph$-coherent if any f.g. left ideal I of R is $\aleph$-finitely presented (in the sense that, I has the following resolution: $0 \longrightarrow K \longrightarrow F \longrightarrow I \longrightarrow 0$, in which F is f.g. free and K is $\aleph$-finitely generated). It is also shown in [**5**] that R is left $\aleph$-coherent if and only if $\prod_I^{\aleph} R$ is right flat for any index set I (see, [**5**, Theorem 1.6]). Now, we point out that this result can be improved as follows.

Theorem 2. *Let $\aleph$ be an infinite cardinal, I a set with $|I| = \aleph$. Then the following are equivalent.*

1. *R is a left $\aleph$-coherent ring;*
2. *For any resolution of $\prod_I^{\aleph}$: $0 \longrightarrow K \xrightarrow{i} P \longrightarrow \prod_I^{\aleph} R \longrightarrow 0$ in which P is projective and i represents the natural injection, i is locally split. In other words, $\prod_I^{\aleph} R$ is right flat.*

Proof: (1) $\Rightarrow$ (2). By [**5**, Theorem 1.6], $\prod_I^{\aleph} R$ is right flat. Directly, by [**3**, p. 163, Exercise 38; p. 154, Corollary 4.86; p. 129, Theorem 4.23], $\prod_I^{\aleph} R$ is right flat $\iff$ For any resolution: $0 \longrightarrow K \xrightarrow{i} P \longrightarrow \prod_I^{\aleph} R \longrightarrow 0$ with projective P, i is locally split.

(2) $\Rightarrow$ (1). Let I be a finitely generated left ideal of R, say $I = Rr_1 + \cdots + Rr_n$. Then there is the exact sequence of left R-modules,

$$0 \longrightarrow K \xrightarrow{i} R^n \xrightarrow{p} I = Rr_1 + \cdots + Rr_n \longrightarrow 0$$

where $p\colon R^n \longrightarrow I$ is defined via $e_i \longrightarrow r_i$ $(i = 1, \ldots, n)$. We need show that K is $\aleph$-finitely generated. Consider the right R-module homomorphism $q\colon R \longrightarrow R^n$, $x \longrightarrow (r_1x, \ldots, r_nx)$, and let $E_R = R^n/\operatorname{Imq}$. Then

E_R is f.p., and

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(E_R, R) & \longrightarrow & \mathrm{Hom}(R^n, R) & \longrightarrow & \mathrm{Hom}(\mathrm{Imq}, R) \\
 & & \Big\downarrow \theta_2 & & \Big\downarrow \theta & & \Big\downarrow \theta_1 \\
0 & \longrightarrow & K & \longrightarrow & R^n & \xrightarrow{p} & R
\end{array}$$

commutes, where $\theta\colon \mathrm{Hom}(R^n, R) \longrightarrow R^n$ defined by $\phi \longrightarrow (\phi(e_1), \ldots, \phi(e_n))$ and $\theta_1\colon \mathrm{Hom}(\mathrm{Imq}, R) \longrightarrow R$ via $\varphi \longrightarrow \varphi((r_1, \ldots, r_n))$, is a monomorphism. Therefore $E^* = \mathrm{Hom}(E_R, R) \simeq K$. Now, we identify I with the set of ordinals $< \omega_\aleph$ again. For $\beta < \omega_\aleph$, let $\{u_\alpha,\, \alpha < \beta\}$ be a subset of E^*, consider the map $u\colon E \longrightarrow \prod_I^\aleph R$, $e \longrightarrow (x_\alpha)_{\alpha<\omega_\aleph}$, in which

$$x_\alpha = \begin{cases} u_\alpha(e), & \text{if } \alpha < \beta; \\ 0, & \text{if } \alpha \geq \beta. \end{cases}$$

Note that E is f.p. and $\prod_I^\aleph R$ is right flat, and hence by [**3**, p. 133, Theorem 4.32] again, there exists a free R-module R^m and homomorphisms $v\colon E \longrightarrow R^m$ and $w\colon R^m \longrightarrow \prod_I^\aleph R$ such that $wv = u$. That is, we have the following commutative diagram:

$$\begin{array}{ccc}
E & \xrightarrow{\quad u \quad} & \prod_I^\aleph R \\
 & {}_{v}\searrow \quad \nearrow_{w} & \\
 & R^m &
\end{array}$$

Suppose $\{e_1, \ldots, e_m\}$ is the basis of R^m, and let p_i be the ith coordinal projection from R^m to R resp. Let $v_1 = p_1 v, \ldots, v_m = p_m v$, then for any $e \in E$, $u(e) = wv(e) = w(e_1 v_1(e) + \cdots + e_m v_m(e)) = w(e_1)v_1(e) + \cdots + w(e_m)v_m(e)$. More explicitly, for $\alpha < \beta$, $u_\alpha(e) = x_\alpha = (w(e_1))_\alpha v_1(e) + \cdots + (w(e_m))_\alpha v_m(e)$. Thus $u_\alpha = (w(e_1))_\alpha v_1 + \cdots + (w(e_m))_\alpha v_m$. Furthermore, $\{u_\alpha,\, \alpha < \beta\} \subseteq Rv_1 + \cdots + Rv_m$, a f.g. submodule of E^*, this shows E^* is ℵ-finitely generated, and so K is ℵ-finitely generated. The proof is completed. □

Corollary 1. *Let ℵ be a regular cardinal, I a set with $|I| = \aleph$. Suppose $\prod_I^\aleph R$ is a direct summand of $\prod_I R$ and $\prod_I R$ is right flat, then R is ℵ-coherent and has ℵ-ACC on annihilators.*

Proof: From Theorem 1 and Theorem 2, we deduce this corollary immediately. □

From now on, let's focus on those properties of single elements of $\prod_I M$ such that

$$0 \longrightarrow \prod_I^{\aleph} M \xrightarrow{\lambda} \prod_I M \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$$

splits, where $\aleph$ is a regular cardinal and I is any infinite index set with $|I| \geq \aleph$. Take $\underline{x} \in \prod_I M$, then $\underline{x} = (x_\alpha)_{\alpha\in I}$. Construct a family of ideals of R, $\Gamma(\underline{x})$, as follows:

$$\Gamma(\underline{x}) = \{P_K(\underline{x}) = \operatorname{ann}_R\{x_\alpha\}_{\alpha\in K} : K \subseteq I \text{ and } |I\backslash K| < \aleph\}.$$

Motivated by the concept of $\aleph$-*ACC* on annihilators of modules, we say $\Gamma(\underline{x})$ has $\aleph$-*ACC* if any well-ordered ascending chain of $\Gamma(\underline{x})$ has $< \aleph$ distinct elements. With this in hand, we now state the following theorem:

Theorem 3. *Let $\aleph$ be a regular cardinal, $\{M_\alpha\}_{\alpha\in I}$ a family of injective left modules with $|I| \geq \aleph$. If $\Gamma(\underline{x})$ has $\aleph$-ACC for any $\underline{x} \in \prod_{\alpha\in I} M_\alpha$, then*

$$0 \longrightarrow \prod_I^{\aleph} M_\alpha \xrightarrow{\lambda} \prod_I M_\alpha \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$$

splits. In other words, $\prod_I^{\aleph} M_\alpha$ is injective.

For the proof of Theorem 3, we first need the following lemma.

Lemma 2. *Let $\{M_\alpha\}_{\alpha\in I}$ be a family of modules, $\aleph$ a regular cardinal and $|I| \geq \aleph$. For $\underline{x} \in \prod_{\alpha\in I} M_\alpha$, let $I_{\underline{x}} = \{r \in R \mid r\underline{x} \in \prod_I^{\aleph} M_\alpha\}$. If $\Gamma(\underline{x})$ has $\aleph$-ACC, then there exists $\underline{y} \in \prod_I^{\aleph} M_\alpha$ such that $a\underline{x} = a\underline{y}$ for any $a \in I_{\underline{x}}$.*

Proof: We assert that if $\Gamma(\underline{x})$ has $\aleph$-*ACC*, then $\Gamma(\underline{x})$ has a maximum element. Otherwise, take $P_{K_1}(\underline{x}) \in \Gamma(\underline{x})$, since $P_{K_1}(\underline{x})$ is not the maximum element, there exists $P_{K_2}(\underline{x}) \in \Gamma(\underline{x})$ such that $P_{K_1}(\underline{x}) \subsetneq P_{K_2}(\underline{x})$. In general, for an ordinal $\beta < \omega_\aleph$, assume we have found $P_{K_\alpha}(\underline{x})$ for all $\alpha < \beta$ such that

$$P_{K_1}(\underline{x}) \subsetneq P_{K_2}(\underline{x}) \subsetneq \cdots \subsetneq P_{K_\alpha}(\underline{x})$$

and

$$P_{K_\alpha}(\underline{x}) \subsetneq P_{K_{\alpha+1}}(\underline{x}) \text{ when } \alpha + 1 < \beta.$$

Case 1: If β is an isolated ordinal, then $\beta - 1 < \beta$. Because $P_{K_{\beta-1}}(\underline{x})$ is not a maximum element, there exists $P_{K'}(\underline{x}) \in \Gamma(\underline{x})$ such that

$$P_{K_{\beta-1}}(\underline{x}) \subsetneq P_{K'}(\underline{x}).$$

Let $K_\beta = K'$, then $\forall\, \alpha < \beta$, $P_{K_\alpha}(\underline{x}) \subsetneq P_{K_\beta}(\underline{x})$.

Case 2: If β is a limiting ordinal, take $K^* = \cap_{\alpha<\beta} K_\alpha$, then $|I \backslash K^*| = \sum_{\alpha<\beta} |I \backslash K_\alpha| < \aleph$ since $\aleph$ is regular. So $P_{K^*}(\underline{x}) \in \Gamma(\underline{x})$, and once more, since $P_{K^*}(\underline{x})$ is not a maximum one, there is $P_T(\underline{x}) \in \Gamma(\underline{x})$ such that $P_{K^*}(\underline{x}) \subsetneq P_T(\underline{x})$. Let $K_\beta = T$, then for any $\alpha < \beta$, we have γ such that $\alpha < \gamma < \beta$ because β is a limiting ordinal, as a result,

$$P_{K_\beta}(\underline{x}) \supsetneq P_{K^*}(\underline{x}) \supsetneq P_{K_\gamma}(\underline{x}) \supsetneq P_{K_\alpha}(\underline{x}).$$

So, in a word, we have inductively defined a sequence $\{P_{K_\alpha}(\underline{x}) : \alpha < \omega_\aleph\}$, which has obviously $\aleph$ distinct elements. It contradicts the fact that $\Gamma(\underline{x})$ has $\aleph$-*ACC*.

Now suppose $P_{K_*}(\underline{x})$ and $P_{K_{**}}(\underline{x})$ are two maximum elements of $(\Gamma(\underline{x}), \subseteq)$. Note that $K_* \cap K_{**} \subseteq K_*$ and $K_* \cap K_{**} \subseteq K_{**}$, then $P_{K_* \cap K_{**}}(\underline{x}) \in \Gamma(\underline{x})$ and $P_{K_*}(\underline{x}) \subseteq P_{K_* \cap K_{**}}(\underline{x})$. Thus, $P_{K_*}(\underline{x}) = P_{K_* \cap K_{**}}(\underline{x})$. Similarly, $P_{K_{**}}(\underline{x}) = P_{K_* \cap K_{**}}(\underline{x})$. This implies, $\Gamma(\underline{x})$ has only one maximum element. Recalling the proof of the foregoing assertion, we also find that each element of $\Gamma(\underline{x})$ must be contained in a maximum element. Therefore, up to now, we can say there exists $P_{K_0}(\underline{x}) \in \Gamma(\underline{x})$ such that $P_K(\underline{x}) \subseteq P_{K_0}(\underline{x})$ for all $P_K(\underline{x}) \in \Gamma(\underline{x})$.

Let's consider $\underline{y} = (y_\alpha)_{\alpha \in I}$, in which

$$y_\alpha = \begin{cases} x_\alpha, & \alpha \in I \backslash K_0 \\ 0, & \alpha \in K_0. \end{cases}$$

Obviously, $\underline{y} \in \prod_I^\aleph M_\alpha$. For any $a \in I_{\underline{x}}$, since $a\underline{x} \in \prod_I^\aleph M_\alpha$, $\operatorname{supp} a\underline{x} = S \subseteq I$ satisfies $|\operatorname{supp} a\underline{x}| < \aleph$. So $P_{I \backslash S}(\underline{x}) \subseteq P_{K_0}(\underline{x})$ and $a \in P_{I \backslash s}(\underline{x})$. Consequently, $a\underline{x} = (ax_\alpha)_{\alpha \in I} = a\underline{y}$. This completes the proof of Lemma 2. $\square$

Proof of Theorem 3: Let f be a homomorphism from A to $\prod_I^{\aleph} M_\alpha$, where A is any left ideal of R. Note $\prod_I M_\alpha$ is injective, there is a homomorphism $\phi\colon R \longrightarrow \prod_{\alpha\in I} M_\alpha$ such that the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \prod_I^{\aleph} M_\alpha & \longrightarrow & \prod_I M_\alpha \\ & & \uparrow f & & \uparrow \phi \\ 0 & \longrightarrow & A & \longrightarrow & R \end{array}$$

commutes. Suppose $\underline{x} = (x_\alpha)_{\alpha\in I} = \phi(1)$, then $\Gamma(\underline{x})$ has $\aleph$-*ACC* by assumption. So, using Lemma 2, we can find $\underline{y} \in \prod_I^{\aleph} M_\alpha$, such that $a\underline{x} = a\underline{y}$ for all $a \in I_{\underline{x}}$. Obviously, $A \subseteq I_{\underline{x}}$, so if we define a homomorphism $\phi_1\colon R \longrightarrow \prod_I^{\aleph} M_\alpha$ via $1 \longrightarrow \underline{y}$, then we have $\phi_1|_A = f$ immediately. This shows that $\prod_I^{\aleph} M_\alpha$ is injective. The proof of Theorem 3 is completed. □

As an application of our Theorem 1 and Theorem 3, we claim that the Theorem III of [**1**] can be obtained easily as our next corollary, i.e.,

Corollary 2. *Let $\aleph$ be a regular cardinal. For an injective left R-module M, the following statements ar equivalent.*

(1) *$\prod_I^{\aleph} M$ is injective, for any index set I;*
(2) *$\prod_I^{\aleph} M$ is injective, for some index set I with $|I| = \aleph$;*
(3) *M has $\aleph$-ACC on annihilators.*

Proof: (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Since $\prod_I^{\aleph} M$ is injective, the exact sequence

$$0 \longrightarrow \prod_I^{\aleph} M \xrightarrow{\ \lambda\ } \prod_I M \xrightarrow{\ \gamma\ } \operatorname{Coker}\lambda \longrightarrow 0$$

splits. So, by Theorem 1, M has $\aleph$-*ACC* on annihilators.

(3) $\Rightarrow$ (1). We only need consider the case $|I| \geq \aleph$. In fact, if $|I| < \aleph$, then $\prod_I^{\aleph} M = \prod_I M$, of course, is injective. Now assume M has $\aleph$-*ACC* on annihilators, naturally $\Gamma(\underline{x})$ has $\aleph$-*ACC* for all $\underline{x} \in \prod_I M$, so (1) is obtained by Theorem 3 immediately. □

Proposition 1. *Let $\aleph_1$, $\aleph_2$ be two infinite cardinals with $\aleph_1 \leq \aleph_2$, $\{M_\alpha\}_{\alpha\in I}$ a family of left R-modules over an infinite set I with $|I| \geq \aleph_1$. Then we have*

(1) *The exact sequence:* $0 \longrightarrow \prod_{\alpha\in I}^{\aleph_1} M_\alpha \xrightarrow{\lambda} \prod_{\alpha\in I}^{\aleph_2} M_\alpha \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$ *is always a pure exact sequence. So, if* $\prod_I^{\aleph_2} M_\alpha$ *is projective, then* λ *is locally split.*

(2) *If* $0 \longrightarrow \prod_{\alpha\in I}^{\aleph_1} M_\alpha \xrightarrow{\lambda} \prod_{\alpha\in I}^{\aleph_2} M_\alpha \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$ *splits, then* $0 \longrightarrow \prod_{\alpha\in J}^{\aleph_1} M_\alpha \xrightarrow{\lambda} \prod_{\alpha\in J}^{\aleph_2} M_\alpha \xrightarrow{\gamma} \operatorname{Coker}\lambda \longrightarrow 0$ *is also split, for all* $J \subseteq I$. *In particular, if* $\prod_I^{\aleph_1} M_\alpha$ *is a direct summand of* $\prod_I M_\alpha$, *then* $\prod_J^{\aleph_1} M_\alpha$ *is also a direct summand of* $\prod_J M_\alpha$ *for all* $J \subseteq I$.

Proof: (1) Given any f.g. left ideal of $R : \langle r_1, \ldots, r_n\rangle$, we have the exact sequence: $0 \longrightarrow \langle r_1, \ldots, r_n\rangle \longrightarrow R \longrightarrow R/\langle r_1, \ldots, r_n\rangle \longrightarrow 0$.

For any homomorphism $\varphi\colon R/\langle r_1, \ldots, r_n\rangle \longrightarrow \operatorname{Coker}\lambda$, it induces $f\colon R \longrightarrow \prod_I^{\aleph_2} M_\alpha$ and $f_1\colon \langle r_1, \ldots, r_n\rangle \longrightarrow \prod_I^{\aleph_1} M_\alpha$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle r_1, \ldots, r_n\rangle & \longrightarrow & R & \longrightarrow & R/\langle r_1, \ldots, r_n\rangle & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow \varphi & & \\ 0 & \longrightarrow & \prod_I^{\aleph_1} M_\alpha & \longrightarrow & \prod_I^{\aleph_2} M_\alpha & \longrightarrow & \operatorname{Coker}\lambda & \longrightarrow & 0 \end{array}$$

commutes. Let $f(1) = (x_\alpha)_{\alpha\in I}$, $f_1(r_1) = (x_\alpha^{(1)})_{\alpha\in I}, \ldots,$ and $f_1(r_n) = (x_\alpha^{(n)})_{\alpha\in I}$, then $|\operatorname{supp} f_1(r_1)| < \aleph_1, \ldots, |\operatorname{supp} f_1(r_n)| < \aleph_1$. So

$$r_i(x_\alpha)_{\alpha\in I} = (r_i x_\alpha)_{\alpha\in I} = (x_\alpha^{(i)})_{\alpha\in I}$$

for each $i = 1, \ldots, n$. Consequently,

$$r_i x_\alpha = \begin{cases} 0, & \alpha \in I \backslash \operatorname{supp} f_1(r_i); \\ r_i x_\alpha \neq 0, & \alpha \in \operatorname{supp} f_1(r_i). \end{cases}$$

Construct $y = (y_\alpha)_{\alpha\in I}$, via

$$y_\alpha = \begin{cases} x_\alpha, & \alpha \in \cup_{i=1}^n \operatorname{supp} f_1(r_i); \\ 0, & \alpha \notin \cup_{i=1}^n \operatorname{supp} f_1(r_i). \end{cases}$$

Since $|\cup_{i=1}^n \operatorname{supp} f_1(r_i)| = \sum_{i=1}^n |\operatorname{supp} f_1(r_i)| < \aleph_1$, $y \in \prod_I^{\aleph_1} M_\alpha$. Define $\phi\colon R \longrightarrow \prod_I^{\aleph_1} M_\alpha$ via $\phi(1) = y$, then $f_1 = \phi|_{\langle r_1,\ldots,r_n\rangle}$. By [**2**, Theorem 3.1], the natural map $\operatorname{Hom}(R/\langle r_1, \ldots, r_n\rangle, \prod_I^{\aleph_2} M_\alpha) \longrightarrow$

$\mathrm{Hom}(R/\langle r_1,\dots,r_n\rangle, \mathrm{Coker}\,\lambda)$ is onto. Moreover, if $\prod_{\alpha\in I}^{\aleph_2} M_\alpha$ is projective, then by [**3**], $\mathrm{Coker}\,\lambda$ is flat and so λ is locally split.

(2) We only need note that $\prod_J^{\aleph_1} M_\alpha \oplus \prod_{I\setminus J}^{\aleph_1} M_\alpha = \prod_I^{\aleph_1} M_\alpha$ and $\prod_J^{\aleph_1} M_\alpha \subseteq \prod_J^{\aleph_2} M_\alpha$, the proof is completed directly. □

Any infinite set I carries a well-ordering $<$, such that $(I,<)$ is order isomorphism to $[0,\theta)$, we always can identify it with the set of ordinals $<\theta$. Let $\underline{x} \in \prod_I M_\alpha$, then we can write $\underline{x}$ as $(x_\alpha)_{\alpha<\theta}$. Now let $P_\alpha(\underline{x}) = \{r \in R \mid rx_\beta = 0 \text{ for all } \beta \geq \alpha\}$, obviously, $P_0(\underline{x}) = \mathrm{ann}(\underline{x}) \subseteq P_\alpha(\underline{x}) \subseteq P_\beta(\underline{x})$ for all $\alpha \leq \beta < \theta$.

Theorem 4. *Let $\aleph$ be an infinite cardinal, I a set with $|I| \geq \aleph$. We identify I with the set of ordinals $<\theta$. If $\prod_{\alpha<\theta}^{\aleph} M_\alpha$ is a direct summand of $\prod_{\alpha<\theta} M_\alpha$, then $|\{P_\alpha(\underline{x}) \mid \alpha<\theta\}| < \aleph$ for any $\underline{x} \in \prod_{\alpha<\theta} M_\alpha$.*

Proof: It is easy to see $\theta \geq \omega_\aleph$. Assume to the contrary that there exists a $\underline{x} = (x_\alpha)_{\alpha<\theta}$ with $|\{P_\alpha(\underline{x}) \mid \alpha<\theta\}| \geq \aleph$. Then we can find a strictly infinite ascending chain:

$$\alpha_1 < \alpha_2 < \cdots < \alpha_\mu < \cdots < \theta$$

and

$$P_{\alpha_1}(\underline{x}) \subsetneq P_{\alpha_2}(\underline{x}) \subsetneq \cdots \subsetneq P_{\alpha_\mu}(\underline{x}) \subsetneq \cdots \quad \mu < \omega_\aleph.$$

Since $P_{\alpha_\mu}(\underline{x}) \subsetneq P_{\alpha_{\mu+1}}(\underline{x})$, there exists $a_\mu \in P_{\alpha_{\mu+1}}(\underline{x})$ but $a_\mu \notin P_{\alpha_\mu}(\underline{x})$. Thus, $\exists\, x_{\gamma_\mu} \in M_{\gamma_\mu}$ such that $a_\mu x_{\gamma_\mu} \neq 0$ and $\alpha_{\mu+1} > \gamma_\mu \geq \alpha_\mu$. In connection with $\{M_{\gamma_\mu}\}_{\mu<\omega_\aleph}$, we construct the p-functors sequence: $\{U_\mu : \mu < \omega_\aleph\}$, as follows: $U_\mu \colon R\text{-Mod} \longrightarrow Z\text{-Mod},\ M \longrightarrow \mathrm{ann}_M P_{\gamma_\mu}(\underline{x})$. In this case, $x_{\gamma_\mu} \in U_\mu(M_{\gamma_\mu}) - U_{\mu+1}(M_{\gamma_\mu})$, so, using Lemma 1 of [**6**], $\prod_{\mu<\omega_\aleph}^{\aleph} M_{\gamma_\mu}$ is not a direct summand of $\prod_{\mu<\omega_\aleph} M_{\gamma_\mu}$. By Proposition 1, $\prod_{\alpha<\theta}^{\aleph} M_\alpha$ is not a direct summand of $\prod_{\alpha<\theta} M_\alpha$, a contradiction. □

Remark. Take $\aleph = \aleph_0$ in Theorem 4 above, we get: if $\oplus_{\alpha<\theta} M_\alpha$ is a direct summand of $\prod_{\alpha<\theta} M_\alpha$, then $|\{P_\alpha(\underline{x}) \mid \alpha<\theta\}|$ is finite for all $\underline{x} \in \prod_{\alpha<\theta} M_\alpha$. So, we can say we have generalized Theorem 1 of [**7**]. In addition, combining this theorem with Theorem 3 before, we also obtain the following result, which is exactly Theorem II of [**1**].

Corollary 3. *Let $\aleph$ be a regular cardinal. The following are equivalent:*

(1) *R is a left $\aleph$-ACC ring;*

(2) *The exact sequence $0 \longrightarrow \prod_I^{\aleph} M_\alpha \xrightarrow{\lambda} \prod_I M_\alpha \xrightarrow{\gamma} \mathrm{Coker}\,\lambda \longrightarrow 0$ splits for any family of injective modules $\{M_\alpha\}_{\alpha\in I}$.*

Proof: (1) ⇒ (2). By Theorem 3.

(2) ⇒ (1). Otherwise, there exists an ascending chain:

$$I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_\alpha \subsetneq \cdots \quad \alpha < \omega_\aleph.$$

Let $E(R/I_\alpha)$ be the injective hull of R/I_α, and let $\overline{1}_\alpha$ be the image of $\eta_\alpha i_\alpha$, in which η_α is the natural map $R \longrightarrow R/I_\alpha$, i_α is the injection from R/I_α to $E(R/I_\alpha)$. Obviously, $\underline{x} = (\overline{1}_\alpha)_{\alpha<\omega_\aleph} \in \prod_{\alpha<\omega_\aleph} E(R/I_\alpha)$, and $I_\alpha = P_\alpha(\underline{x})$. Consequently,

$$|\{P_\alpha(\underline{x}) \mid \alpha < \omega_\aleph\}| = \aleph.$$

By Theorem 4, $\prod^{\aleph}_{\alpha<\omega_\aleph} E(R/I_\alpha)$ is not a direct summand of $\prod_{\alpha<\omega_\aleph} E(R/I_\alpha)$, a contradiction. □

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Primera versió rebuda el 29 de novembre de 2001,
darrera versió rebuda el 6 de juny de 2002.