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ON THE GEOMETRIC STRUCTURE OF THE LIMIT SET OF CONFORMAL ITERATED FUNCTION SYSTEMS

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Abstract

We consider infinite conformal function systems on $\mathbb{R}^d$. We study the geometric structure of the limit set of such systems. Suppose this limit set intersects some l-dimensional C^1-submanifold with positive Hausdorff t-dimensional measure, where $0 < l < d$ and t is the Hausdorff dimension of the limit set. We then show that the closure of the limit set belongs to some l-dimensional affine subspace or geometric sphere whenever d exceeds 2 and analytic curve if d equals 2.

1. Introduction

We work on the setting introduced by Mauldin and Urbański in [**6**]. There they consider uniformly contractive countable collections of conformal injections defined on some open, bounded and connected set $\Omega \subset \mathbb{R}^d$. It is needed that there exists some compact set $X \subset \Omega$ with nonempty interior such that each contraction maps this set to some subset of itself. For this kind of setting, even without the conformality assumption, there is always so called limit set associated. We denote it with E and we are particularly interested in the properties of this set. The conformality assumption is basically needed for good behavior of the derivatives. As usual, open set condition (OSC), introduced by Moran in [**8**], is used for getting decent separation for those previously mentioned subsets of X. We also need bounded distortion property (BDP), which says that the value of the norm of derivatives cannot vary too much. This is actually

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just a consequence of our previous assumptions. And finally the boundary of X is assumed to be "smooth" enough. For example all convex sets are like that. From this property one may verify that the limit set is a Borel set. In the finite case it is always compact. Together with OSC it also guarantees some properties for the natural Borel regular probability measure, so called conformal measure, associated to this set.

This is one way to generalize similar kind of situation for finite collection of similitudes; the setting introduced by Hutchinson in [**2**]. Mattila proved in [**3**] for the limit set E of this kind of setting the following result: Either E lies on an l-dimensional affine subspace or $\mathcal{H}^t(E \cap M) = 0$ for every l-dimensional C^1-submanifold $M \subset \mathbb{R}^d$. Here $0 < l < d$ and $\mathcal{H}^t$ denotes the t-dimensional Hausdorff measure, where $t = \dim_H(E)$, the Hausdorff dimension of the set E. The main result in this note is a generalization for this. Our approach is based on an extensive use of tangents. Before going into more detailed preliminaries we should mention that Springer has proved in [**10**] similar result in the plane and Mauldin, Mayer and Urbański have studied similar behavior for connected limit sets in [**5**] and [**7**].

As usual, let I be a countable set with at least two elements. Put $I^* = \bigcup_{n=1}^{\infty} I^n$ and $I^\infty = I^{\mathbb{N}} = \{(i_1, i_2, \dots) : i_j \in I \text{ for } j \in \mathbb{N}\}$. Thus, if $\mathtt{i} \in I^*$ there is $k \in \mathbb{N}$ such that $\mathtt{i} = (i_1, \dots, i_k)$, where $i_j \in I$ for all $j = 1, \dots, k$. We call this k as the *length* of $\mathtt{i}$ and we denote $|\mathtt{i}| = k$. If $\mathtt{i} \in I^\infty$ we denote $|\mathtt{i}| = \infty$. For $\mathtt{i} \in I^* \cup I^\infty$ we put $\mathtt{i}|_k = (i_1, \dots, i_k)$ whenever $1 \le k < |\mathtt{i}|$.

Choose Ω to be open, bounded and connected set on $\mathbb{R}^d$. Now for each $i \in I$ we define an injective mapping $\varphi_i \colon \Omega \to \Omega$ such that it is *contractive*, that is, there exists $0 < s_i < 1$ such that

$$|\varphi_i(x) - \varphi_i(y)| \le s_i |x - y| \tag{1.1}$$

whenever $x, y \in \Omega$. A mapping with equality in (1.1) is called *similitude*. We assume also that φ_i is *conformal*, that is, $|\varphi_i'|^d = |J\varphi_i|$, where J stands for normal Jacobian and the norm in the left side is just the normal "sup-norm" for linear mappings. Here the derivative exists $\mathcal{H}^d$-almost all points using Rademacher's theorem. This definition for conformality is usually better known as 1-quasiconformality; see [**12**]. Note that a conformal mapping is always C^∞ by [**9**, Theorem 4.1] of Reshetnyak. We assume also that there exists a compact set X with $\operatorname{int}(X) \ne \varnothing$ such that $\varphi_i(X) \subset X$ for every $i \in I$. The use of the bounded set Ω here is essential, since conformal mappings contractive in whole $\mathbb{R}^d$ are similitudes, as d exceeds 2. We call a collection $\{\varphi_i : i \in I\}$ as *conformal iterated function system* (CIFS) if the following conditions (1)–(4) are satisfied:

(1) Mappings φ_i are *uniformly contractive*, that is, $s := \sup_{i\in I} s_i < 1$.

Denoting $\varphi_{\mathtt{i}} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_{|\mathtt{i}|}}$ for each $\mathtt{i} \in I^*$, we get from this property that for every $n \in \mathbb{N}$

$$d\big(\varphi_{\mathtt{i}|_n}(X)\big) \le s^n\, d(X) \tag{1.2}$$

whenever $\mathtt{i} \in I^\infty$. Here d means the diameter of a given set. Now we may define a mapping $\pi\colon I^\infty \to X$ such that

$$\big\{\pi(\mathtt{i})\big\} = \bigcap_{n=1}^{\infty} \varphi_{\mathtt{i}|_n}(X). \tag{1.3}$$

We set

$$E = \pi(I^\infty) = \bigcup_{\mathtt{i}\in I^\infty} \bigcap_{n=1}^{\infty} \varphi_{\mathtt{i}|_n}(X) \tag{1.4}$$

and we call this set as the *limit set* of the corresponding CIFS. Our aim is to study this set. Observe that E satisfies the natural invariance equality, $E = \bigcup_{i\in I} \varphi_i(E)$.

(2) *Bounded distortion property* (BDP), that is, there exists $K \ge 1$ such that $|\varphi_{\mathtt{i}}'(x)| \le K|\varphi_{\mathtt{i}}'(y)|$ for every $\mathtt{i} \in I^*$ and $x, y \in \Omega$.

For a finite collection of mappings, this is just a consequence of smoothness (see [**6**, Lemma 2.2]) and in the infinite case it follows from Koebe's distortion theorem as d equals 2 and [**11**, Theorem 1.1] whenever d exceeds 2. Using these assumptions we may prove that each mapping φ_i is a diffeomorphism and that there exists $D \ge 1$ such that

$$D^{-1} \le \frac{d\big(\varphi_{\mathtt{i}}(E)\big)}{||\varphi_{\mathtt{i}}'||} \le D \tag{1.5}$$

for every $\mathtt{i} \in I^*$. Here $||\varphi_{\mathtt{i}}'|| = \sup_{x\in\Omega} |\varphi_{\mathtt{i}}'(x)|$.

(3) *Open set condition* (OSC) holds for $\operatorname{int}(X)$, that is, $\varphi_i\big(\operatorname{int}(X)\big) \cap \varphi_j\big(\operatorname{int}(X)\big) = \varnothing$ for every $i \ne j$.

(4) There exists $r_0 > 0$ such that

$$\inf_{x\in\partial X}\, \inf_{0<r<r_0} \frac{\mathcal{H}^d\big(B(x,r)\cap \operatorname{int}(X)\big)}{\mathcal{H}^d\big(B(x,r)\big)} > 0, \tag{1.6}$$

where ∂X is the boundary of the set X.

We should mention that in [**6**], instead of assumption (4), it was used so called cone condition, which says that for each boundary point x there exists some "cone" in the interior of X with vertex x. However, assumption (4) is sufficient for our setting, as was remarked also in [**6**]. Using these assumptions it follows that E is a Borel set. Suppose there exists a Borel regular probability measure m on E such that

$$m\big(\varphi_{\mathtt{i}}(A)\big) = \int_A |\varphi_{\mathtt{i}}'(x)|^t \, dm(x), \tag{1.7}$$

where $t = \dim_H(E)$, $A \subset X$ is a Borel set and $\mathtt{i} \in I^*$. Then OSC and assumption (4) are crucial to derive that $m\big(\varphi_i(X) \cap \varphi_j(X)\big) = 0$ for every $i \neq j$ (see [**6**, Section 3] for details). If this measure exists, we call it a *t-conformal measure* and the corresponding CIFS *regular*. Observe that finite CIFS's are always regular (see [**6**, Section 3] for details). If we consider measure theoretical Jacobian J_m for function $\varphi_{\mathtt{i}}$ defined as

$$J_m\varphi_{\mathtt{i}}(x) = \lim_{r \searrow 0} \frac{m\big(\varphi_{\mathtt{i}}\big(B(x,r)\big)\big)}{m\big(B(x,r)\big)} \tag{1.8}$$

for each point $x \in E$, we notice using conformality of $\varphi_{\mathtt{i}}$ and (1.7) that $J_m\varphi_{\mathtt{i}}^{-1}\big(\varphi_{\mathtt{i}}(x)\big) = \big(J_m\varphi_{\mathtt{i}}(x)\big)^{-1} = |\varphi_{\mathtt{i}}'(x)|^{-t} = \big|\big(\varphi_{\mathtt{i}}^{-1}\big)'\big(\varphi_{\mathtt{i}}(x)\big)\big|^t$ for m-almost all $x \in E$ and for all $\mathtt{i} \in I^*$ (recall also [**4**, Theorem 2.12]). Thus for example, using BDP

$$\begin{aligned} m\big(\varphi_{\mathtt{i}}^{-1}(A)\big) &= \int_A |(\varphi_{\mathtt{i}}^{-1})'(x)|^t \, dm(x) \\ &\le ||(\varphi_{\mathtt{i}}^{-1})'||^t m(A) \le K^t ||\varphi_{\mathtt{i}}'||^{-t} m(A) \end{aligned} \tag{1.9}$$

whenever $A \subset \varphi_{\mathtt{i}}(X)$ is Borel. Furthermore by setting $\Phi|_{\varphi_i(X)}(x) = \varphi_i^{-1}(x)$ for all $i \in I$ we get at m-almost every point defined mapping $\Phi \colon \bigcup_{i \in I} \varphi_i(X) \to X$ for which

$$m\big(\Phi(A)\big) = \int_A |\Phi'(x)|^t \, dm(x). \tag{1.10}$$

In fact, m is ergodic and equivalent to some invariant (with respect to the function Φ) measure (see [**6**, Section 3] for details). In the finite case, the t-conformal measure is *Ahlfors regular*, that is, there exists $C \ge 1$ such that

$$C^{-1} \le \frac{m\big(B(x,r)\big)}{r^t} \le C \tag{1.11}$$

for all $x \in E$ and $r > 0$ small enough (see [**6**, Lemma 3.14]).

Let $0 < l < d$ be an integer and $G(d,l)$ be the collection of all l-dimensional subspaces of $\mathbb{R}^d$. We denote by P_V the orthogonal projection onto $V \in G(d,l)$ and we put $Q_V = P_{V^\perp}$, where $V^\perp$ is the orthogonal complement of V. For $x \in \mathbb{R}^d$ we denote $V + x = \{v + x : v \in V\}$. If $a \in \mathbb{R}^d$, $V \in G(d,l)$ and $0 < \delta < 1$ we set

$$\begin{aligned} X(a,V,\delta) &= \{x \in \mathbb{R}^d : |Q_V(x-a)| < \sqrt{\delta}|x-a|\} \\ &= \{x \in \mathbb{R}^d : d(x, V+a) < \sqrt{\delta}|x-a|\}. \end{aligned} \tag{1.12}$$

Note that the closure of $X(a,V,\delta)$ equals to the complement of $X(a,V^\perp,1-\delta)$. We say that V is a (t,l)-*tangent plane* for E at a if

$$\lim_{r \searrow 0} \frac{m\big(B(a,r) \setminus X(a,V,\delta)\big)}{r^t} = 0 \tag{1.13}$$

for all $0 < \delta < 1$. Furthermore we say that V is a *strong l-tangent plane* for given set $A \subset \mathbb{R}^d$ at a if for every $0 < \delta < 1$ there exists $r > 0$ such that

$$A \cap B(a,r) \subset X(a,V,\delta). \tag{1.14}$$

For example, an l-dimensional C^1-submanifold has a strong l-tangent plane at all of it's points. Observe that these two tangents are exactly the same for E in the case of Ahlfors regular m. However, we shall not need this fact here. Recall also that t-dimensional *upper density* of a measure μ at a is defined as

$$\Theta^{*t}(\mu,a) = \limsup_{r \searrow 0} \frac{\mu\big(B(a,r)\big)}{r^t}. \tag{1.15}$$

2. Main result

The main result of this note is the following theorem.

Theorem 2.1. *Suppose CIFS is given, $t = \dim_H(E)$ and $0 < l < d$. Then either $\mathcal{H}^t(E \cap M) = 0$ for every l-dimensional C^1-submanifold $M \subset \mathbb{R}^d$ or the closure of E is contained in some l-dimensional affine subspace or l-dimensional geometric sphere whenever d exceeds 2 and analytic curve if d equals 2.*

Using this theorem we are able to find minimal amount of essential directions for where the set E is spread out. It also follows that if t is an integer, then the limit set is always either t-rectifiable or purely t-unrectifiable. See [**4**] for definitions and properties for these concepts. Provided that d exceeds 2, it is also easy to see that if at least one of our conformal mappings is similitude, the latter case of the theorem concerns only affine subspaces.

The proof is divided into three parts. We start with an easy lemma which provides a useful dichotomy for our purposes.

Lemma 2.2. *Suppose CIFS is given,* $t = \dim_H(E)$ *and* $0 < l < d$*. Then either* $\mathcal{H}^t(E \cap M) = 0$ *for every* l*-dimensional* C^1*-submanifold* $M \subset \mathbb{R}^d$ *or the system is regular and at least one point of* E *has a* (t, l)*-tangent plane.*

Proof: Assume $\mathcal{H}^t(E \cap M) > 0$ for some M. Since $\mathcal{H}^t(E) < \infty$, the regularity of the system is guaranteed by [**6**, Theorem 4.17]. From [**1**, 2.10.19(4)] of Federer we get that $\Theta^{*t}(m|_{E \setminus M}, x) = 0$ for $\mathcal{H}^t$-almost every $x \in E \cap M$. Let $a \in E \cap M$ be such point and $V \in G(d, l)$ be a strong l-tangent plane for M at a. This means that for any given $0 < \delta < 1$ there exists $r_\delta > 0$ such that

$$M \cap B(a, r) \subset X(a, V, \delta) \tag{2.1}$$

whenever $r < r_\delta$. Thus for every $0 < \delta < 1$

$$\begin{aligned} \limsup_{r \searrow 0} \frac{m\big(B(a,r) \setminus X(a, V, \delta)\big)}{r^t} &\leq \limsup_{r \searrow 0} \frac{m\big(B(a,r) \setminus \big(M \cap B(a,r)\big)\big)}{r^t} \\ &= \Theta^{*t}(m|_{E \setminus M}, a) = 0 \end{aligned} \tag{2.2}$$

and V is a (t, l)-tangent plane for E at a. □

With this dichotomy in mind it is enough to study what happens if one point of E has a (t, l)-tangent plane.

Theorem 2.3. *Suppose CIFS is regular,* $t = \dim_H(E)$ *and* $0 < l < d$*. If one point of* E *has a* (t, l)*-tangent plane then* m*-almost all of* E *is contained in the set* $f(V)$*, where* $V \in G(d, l)$ *and* f *is some conformal mapping.*

Proof: Suppose $a \in E$ has a (t, l)-tangent plane $V \in G(d, l)$ and let $\varepsilon > 0$. Now for each $j \in \mathbb{N}$ there exists $r_{j,0} > 0$ such that

$$m\big(B(a, r) \setminus X(a, V, \tfrac{1}{j})\big) \leq \varepsilon r^t \tag{2.3}$$

whenever $r < r_{j,0}$. Let $\mathtt{i} \in I^\infty$ be such that $\pi(\mathtt{i}) = a$. For each $j \in \mathbb{N}$ choose some fixed radius $r_j < r_{j,0}$ and $n_j \in \mathbb{N}$ such that

$$\varphi_{\mathtt{i}|_{n_j}}(E) \subset B(a, r_j) \qquad \text{and}$$

$$\varphi_{\mathtt{i}|_{n_j}}(E) \setminus B(a, \tfrac{r_j}{2}) \neq \emptyset. \tag{2.4}$$

Since trivially $a \in \varphi_{\mathtt{i}|_n}(E)$ for all n, we have

$$\tfrac{r_j}{2} \leq d\big(\varphi_{\mathtt{i}|_{n_j}}(E)\big) \leq 2r_j. \tag{2.5}$$

For each j define mapping $\psi_j : \mathbb{R}^d \to \mathbb{R}^d$ such that $\psi_j(x) = ||\varphi'_{\mathtt{i}|_{n_j}}||^{-1}(x-a) + a$. Then

$$\psi_j\big(X(a, V, \tfrac{1}{j})\big) = X(a, V, \tfrac{1}{j}) \tag{2.6}$$

and

$$|\psi_j(x) - \psi_j(y)| = ||\varphi'_{\mathtt{i}|_{n_j}}||^{-1}|x-y| \tag{2.7}$$

for every $x, y \in \mathbb{R}^d$. Now $F_j := \psi_j \circ \varphi_{\mathtt{i}|_{n_j}}$ is clearly a conformal mapping from Ω to $\mathbb{R}^d$. Since for arbitrary $x, y \in \Omega$

$$\begin{aligned} K^{-1}|x-y| &\le ||\varphi'_{\mathtt{i}|_{n_j}}||^{-1}||(\varphi^{-1}_{\mathtt{i}|_{n_j}})'||^{-1}|x-y| \\ &\le ||\varphi'_{\mathtt{i}|_{n_j}}||^{-1}|\varphi_{\mathtt{i}|_{n_j}}(x) - \varphi_{\mathtt{i}|_{n_j}}(y)| \\ &= |F_j(x) - F_j(y)| \le |x-y| \end{aligned} \tag{2.8}$$

using BDP and mean value theorem, we notice that F_j is bi-Lipschitz with constants K^{-1} and 1 for every $j \in \mathbb{N}$. Using now Ascoli-Arzela's theorem we shall find an uniformly converging subsequence, say, $F_{j_k} \to F$, as $k \to \infty$. According to [**12**, Corollaries 37.3 and 13.3] of Väisälä we notice that F^{-1} is conformal. Since

$$\begin{aligned} m\big(E\setminus F_j^{-1}\big(X(a,V,\tfrac{1}{j})\big)\big) &= m\big(\varphi^{-1}_{\mathtt{i}|_{n_j}}\big(\varphi_{\mathtt{i}|_{n_j}}(E)\setminus X(a,V,\tfrac{1}{j})\big)\big) \\ &\le K^t||\varphi'_{\mathtt{i}|_{n_j}}||^{-t} m\big(\varphi_{\mathtt{i}|_{n_j}}(E)\setminus X(a,V,\tfrac{1}{j})\big) \\ &\le (DK)^t d\big(\varphi_{\mathtt{i}|_{n_j}}(E)\big)^{-t} m\big(B(a,r_j)\setminus X(a,V,\tfrac{1}{j})\big) \\ &\le (2DK)^t r_j^{-t}\varepsilon r_j^t \end{aligned} \tag{2.9}$$

for every $j \in \mathbb{N}$ using (1.9), (1.5), (2.4), (2.3) and (2.5), we conclude

$$m\big(E\setminus F^{-1}(V+a)\big) \le (2DK)^t\varepsilon. \tag{2.10}$$

We finish the proof by letting $\varepsilon \searrow 0$. □

Notice that the inclusion in the previous theorem holds for the closure of E, since any set of full m-measure is dense in E and any l-dimensional C^1-submanifold is closed in $\mathbb{R}^d$. Now the main theorem follows as a corollary just by recalling that conformal mappings are complex analytic in the plane and by Liouville's theorem Möbius transformations elsewhere (see [**9**, Theorem 4.1] of Reshetnyak).

Remark. The proof of the main theorem was found in January '01 and it was supposed to be part of the author's thesis. Since recently there has been some interest for similar kind of questions (particularly [**7**]), it was decided to be published independently.

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References

[1] H. Federer, *"Geometric measure theory"*, Die Grundlehren der mathematischen Wissenschaften **153**, Springer-Verlag New York Inc., New York, 1969.

[2] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30(5)** (1981), 713–747.

[3] P. Mattila, On the structure of self-similar fractals, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **7(2)** (1982), 189–195.

[4] P. Mattila, *"Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability"*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.

[5] R. D. Mauldin, V. Mayer and M. Urbański, Rigidity of connected limit sets of conformal IFSs, *Michigan Math. J.* **49(3)** (2001), 451–458.

[6] R. D. Mauldin and M. Urbański, Dimensions and measures in infinite iterated function systems, *Proc. London Math. Soc. (3)* **73(1)** (1996), 105–154.

[7] V. Mayer and M. Urbański, Finer geometric rigidity of connected limit sets of conformal IFS, Preprint (2001), *Proc. Amer. Math. Soc.* (to appear).

[8] P. A. P. Moran, Additive functions of intervals and Hausdorff measure, *Proc. Cambridge Philos. Soc.* **42** (1946), 15–23.

[9] Yu. G. Reshetnyak, *"Stability theorems in geometry and analysis"*, Mathematics and its Applications **304**, Kluwer Academic Publishers Group, Dordrecht, 1994.

[10] O. B. Springer, Order two density and self-conformal sets, Ph. D. thesis, University of Bristol (1993).

[11] M. Urbański, Rigidity of multi-dimensional conformal iterated function systems, *Nonlinearity* **14(6)** (2001), 1593–1610.

[12] J. Väisälä, *"Lectures on n-dimensional quasiconformal mappings"*, Lecture Notes in Mathematics **229**, Springer-Verlag, Berlin-New York, 1971.

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