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CHARACTERIZATION OF THE INESSENTIAL ENDOMORPHISMS IN THE CATEGORY OF ABELIAN GROUPS

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Abstract

An endomorphism f of an Abelian group A is said to be inessential (in the category of Abelian groups) if it can be extended to an endomorphism of any Abelian group which contains A as a subgroup. In this paper we show that f is as above if and only if $(f - v\,\mathrm{id}_A)(A)$ is contained in the maximal divisible subgroup of A for some $v \in \mathbb{Z}$.

1. Introduction

Throughout this paper, we will follow the terminology of [**2**]. Let M be an object of a category $\mathcal{C}$ and $f \in \mathrm{End}(M)$, f is called inessential (in $\mathcal{C}$) if for any monomorphism $\sigma\colon M \to N$ there exists $\tilde{f} \in \mathrm{End}(N)$ such that $\tilde{f}\sigma = \sigma f$, in other words the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\sigma} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \tilde{f}} \\ M & \xrightarrow{\sigma} & N \end{array}$$

commutes.

$\mathrm{Ines}(M)$ denotes all the inessential endomorphisms of M. M is called rigid if $\mathrm{End}(M) = \mathrm{Ines}(M)$. For a concrete category $\mathcal{C}$, the characterization of the inessential endomorphisms is one of the problems raised in [**2**]. In this paper, we take $\mathcal{C} = Ab$ the category of the Abelian groups and we show for an Abelian group A, and an endomorphism f of A, that f is inessential (in Ab) if and only if there exists $v \in \mathbb{Z}$ such that

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$(f - v\,\mathrm{id}_A)(A) \subseteq D$, where D is the maximal divisible subgroup of A. In particular if A is reduced then $\mathrm{Ines}(A) = \mathbb{Z}\,\mathrm{id}_A$. The proof of this result uses the properties of the endomorphisms of some extensions of certain direct sums of torsion cyclic groups.

From now on, the word group means Abelian group and we adopt the notations of [**3**].

2. Some constructions

Construction 1. Let $(\alpha_n)_{n\geq 0}$ be a sequence of natural numbers such that $\alpha_n < \alpha_{n+1}$ and $2\alpha_{n+1} - \alpha_n + n + 3 \leq \alpha_{n+2}$, $\forall n \in \mathbb{N}$. If we put $\theta_n = \alpha_n - \alpha_{n-1} - n$ for $n \geq 1$ then we have $\theta_n - \theta_{n-1} \geq n$, $n \geq 2$. Let $p \in \mathbb{N}^*$ and $(t_{n,m})_{n\geq m}$ be a set of nonzero natural numbers, relatively prime with p such that $t_{i,j}t_{j,k} = t_{i,k}$ if $i \geq j \geq k$.

We consider the direct product $\prod_{n\geq 1} \langle x_n \rangle$ with $o(x_n) = p^{\alpha_n}$ and denote by $\varphi_k \colon \prod_{n\geq 1} \langle x_n \rangle \to \langle x_k \rangle$ the canonical projection. For $m \geq 1$, we define the element g_m of $\prod_{n\geq 1} \langle x_n \rangle$ by

$$\varphi_n(g_m) = \begin{cases} 0 & \text{if } n < m \\ p^{\alpha_n - \alpha_m} x_n & \text{if } n \geq m. \end{cases}$$

We directly check that $o(g_m) = p^{\alpha_m}$, $x_m = g_m - p^{\alpha_{m+1}-\alpha_m} g_{m+1}$ and $\langle \{g_m/m \geq 1\} \rangle = \bigoplus_{m\geq 1} \langle g_m \rangle$.

Let $m \in \mathbb{N}^*$ and ξ a function from $\mathbb{N}$ into $\{0,1\}$, we define the element $S(m,\xi)$ of $\prod_{n\geq 1} \langle x_n \rangle$ by

$$\varphi_n(S(m,\xi)) = \begin{cases} 0 & \text{if } n < m \\ \xi(n) t_{n,m} p^{n-m+\alpha_{n-1}} x_n & \text{if } n \geq m. \end{cases}$$

We have

$$S(m,\xi) = \left(\sum_{n=m}^{r} \xi(n) t_{n,m} p^{n-m+\alpha_{n-1}} x_n \right) + t_{r+1,m} p^{r+1-m} S(r+1,\xi)$$

if $r \geq m$.

Let K_1 be the subgroup of $\prod_{n\geq 1} \langle x_n \rangle$ generated by

$\{g_m/m \geq 1\} \cup \{S(m,\xi)/m \geq 1,\ \xi \in \{0,1\}^{\mathbb{N}}\}$.

Lemma 2.1. *The direct sum* $\bigoplus_{n\geq 1}\langle x_n\rangle$ *is a subgroup of* K_1 *and for all* $\lambda \in \mathrm{End}(K_1)$ *there exist* $s, N \in \mathbb{N}$ *and* $v \in \mathbb{Z}$ *such that* $t_{s,1}p^{\alpha_n - n}\lambda(x_n) = p^{\alpha_n - n}vx_n$, $\forall\, n \geq N$.

Proof: Let $\lambda \in \mathrm{End}(K_1)$. Let us show at first that there exists $N_0 \geq 1$ such that if $n > m \geq N_0$ then $\varphi_n(p^{\alpha_m - m}\lambda(x_m)) = 0$.

If not, we can find a sequence $(m_k)_{k\geq 1}$ such that for all $k \geq 1$, there exists $n_k > m_k$ with $\varphi_{n_k}(p^{\alpha_{m_k} - m_k}\lambda(x_{m_k})) \neq 0$ and $\alpha_{n_k} \leq m_{k+1}$. Let $\zeta\colon \mathbb{N} \to \{0,1\}$ be the function defined by $\zeta(n) = 1$ if $n \in \{m_k/k \geq 1\}$ and $\zeta(n) = 0$ otherwise. We can write:

$$\lambda(S(1,\zeta)) = \sum_{i=1}^{a} c_i g_i + \sum_{j=1}^{b} d_j S(m, \xi_j).$$

If we put $t = \alpha_a$, then $p^t\lambda(S(1,\zeta)) = p^t \sum_{j=1}^{b} d_j S(m,\xi_j)$. For any k, we have

$$p^{\theta_{m_k}+1}S(1,\zeta) = p^{\theta_{m_k}+1}$$

$$\times\left[\left(\sum_{n=1}^{m_{k+1}-1}\zeta(n)t_{n,1}p^{n-1+\alpha_{n-1}}x_n\right)+t_{m_{k+1},1}p^{m_{k+1}}S(m_{k+1},\zeta)\right]\in p^{\alpha_{n_k}}K_1$$

because $\theta_{m_k} + 1 + n - 1 + \alpha_{n-1} \geq \alpha_n$ if $m_k \geq n \geq 1$, $\zeta(n) = 0$ if $m_{k+1} > n > m_k$ and $\theta_{m_k} + 1 + m_{k+1} \geq \alpha_{n_k}$. If k is large enough, then

$$\varphi_{n_k}(p^{\theta_{m_k}+1}\lambda(S(1,\zeta))) = \varphi_{n_k}\left(p^{\theta_{m_k}+1}\sum_{j=1}^{b} d_j S(m,\xi_j)\right) = 0$$

therefore $p^{\theta_{n_k}-\theta_{m_k}}$ divides $v(n_k)$, where $v(n) = \sum_{j=1}^{b} d_j\xi_j(n)$. Since the set $\{v(n)/n \in \mathbb{N}\}$ is finite and $\theta_{n_k} - \theta_{m_k} \geq n_k$, then there exists $k_1 \geq 1$ such that $v(n_k) = 0$, $\forall\, k \geq k_1$. On the other hand

$$p^{\theta_{m_k}-m_k+1}S(1,\zeta) - t_{m_k,1}p^{\alpha_{m_k}-m_k}x_{m_k} \in p^{\alpha_{n_k}}K_1,$$

therefore

$$\varphi_{n_k}\left(p^{\theta_{m_k}-m_k+1}\sum_{j=1}^{b} d_j S(m,\xi_j)\right) \neq 0$$

for k large enough. Therefore it exists $k_2 \geq 1$ such that $v(n_k) \neq 0$, $\forall\, k \geq k_2$, which is absurd. Thus there exists $N_0 \in \mathbb{N}$ such that: $p^{\alpha_n - n}\lambda(x_n) =$

$p^{\alpha_n-n}r_nx_n$, $\forall n \geq N_0$, where $r_n \in \mathbb{Z}$. Since $T(K_1) = \bigoplus_{m\geq 1}\langle g_m\rangle$ and $\alpha_k \leq \alpha_n - n$ for $k < n$, therefore $p^{\alpha_n-n}\lambda(g_n) \in p^{\alpha_n-n}(\bigoplus_{k\geq n}\langle vg_k\rangle)$. Let $m \geq N_0$ and put for $n \geq m$, $u_n = l$ if $p^{\alpha_n-m}\lambda(g_n) = p^{\alpha_n-m}\sum_{k=n}^{l} t_kg_k$ with ($p^{\alpha_n-m}t_lg_l \neq 0$ and $l > n$) and $u_n = 0$ if $p^{\alpha_n-m}\lambda(g_n) \in p^{\alpha_n-m}\langle g_n\rangle$. Since $x_n = g_n - p^{\alpha_{n+1}-\alpha_n}g_{n+1}$, it is easy to see that the sequence $(u_n)_{n\geq m}$ is decreasing. Since for $u_n \neq 0$ we have $u_n > n$, then there exists $M_m \geq m$ such that $u_n = 0$, $\forall n \geq M_m$. Therefore $p^{\alpha_n-m}\lambda(g_n) \in p^{\alpha_n-m}\langle g_n\rangle$, $\forall n \geq M_m$. Let $\xi_0(n) = 1$, $\forall n \in \mathbb{N}$. We can write: $p^{k'}\lambda(S(1,\xi_0)) = p^{k'}\sum_{j=1}^{k} m_jS(s,\xi_j)$, where $k', k, s \in \mathbb{N}$, $m_1,\ldots,m_k \in \mathbb{Z}$ and $\xi_1,\ldots,\xi_k \in \{0,1\}^{\mathbb{N}}$.

We have $p^{\theta_n-n+1}S(1,\xi_0) - t_{n,1}p^{\alpha_n-n}x_n \in p^{\alpha_n}K_1$ thus for n large enough $t_{n,1}p^{\alpha_n-n}\varphi_n(\lambda(x_n)) = p^{\theta_n-n+1}\varphi_n(\sum_{j=1}^{k} m_jS(s,\xi_j)) \implies p^{n+s-1}$ divides $t_{s,1}p^{s-1}r_n - w(n)$ where $w(n) = \sum_{j=1}^{k} m_j\xi_j(n)$. Accordingly, if $d \in \mathbb{Z}$ such that the set $\{n \in \mathbb{N}/w(n+1) - w(n) = d\}$ is infinite, then p^m divides d, $\forall m \geq N_0$, therefore $d = 0$. Since the set $\{w(n+1) - w(n)/n \in \mathbb{N}\}$ is finite, then there exist $v_0 \in \mathbb{Z}$ and $N_1 \in \mathbb{N}$ such that $w(n) = v_0$, $\forall n \geq N_1$. It is clear that p^{s-1} divides v_0. Finally if we put $v_0 = p^{s-1}v$, we can find $N \in \mathbb{N}$ such that $t_{s,1}p^{\alpha_n-n}\lambda(x_n) = p^{\alpha_n-n}vx_n$, $\forall n \geq N$. $\square$

Construction 2. Let $(\alpha_n)_{n\geq 0}$ be as in Construction 1, and let p and q be two natural numbers different from zero and relatively prime, we consider the two direct products

$$\prod_{n\geq 1}\langle x_n\rangle \text{ and } \prod_{n\geq 1}\langle y_n\rangle \text{ with } o(x_n) = p^{\alpha_n} \text{ and } o(y_n) = q^{\alpha_n}, \quad \forall n \geq 1.$$

The elements h_m of $\prod_{n\geq 1}\langle y_n\rangle$ are defined in the same way as the g_m of $\prod_{n\geq 1}\langle x_n\rangle$ (see Construction 1). The elements $S_1(m,\xi)$ (respectively $S_2(m,\xi)$) of $\prod_{n\geq 1}\langle x_n\rangle$ (respectively $\prod_{n\geq 1}\langle y_n\rangle$) are defined like $S(m,\xi)$ of Construction 1 with $t_{n,m} = q^{n-m}$ (respectively $t_{n,m} = p^{n-m}$).

We put $R(m,\xi) = S_1(m,\xi) + S_2(m,\xi) \in (\prod_{n\geq 1} \langle x_n\rangle) \oplus (\prod_{n\geq 1} \langle y_n\rangle)$, then we have,

$$R(m,\xi)=\left(\sum_{n=m}^{r}\xi(n)(pq)^{n-m}(p^{\alpha_{n-1}}x_n+q^{\alpha_{n-1}}y_n)\right)+(pq)^{r+1-m}R(r+1,\xi)$$

if $r \geq m$.

Let K_2 be the subgroup of $(\prod_{n\geq 1} \langle x_n\rangle) \oplus (\prod_{n\geq 1} \langle y_n\rangle)$ generated by $\{g_m/m \geq 1\} \cup \{h_m/m \geq 1\} \cup \{R(m,\xi)/m \geq 1,\ \xi \in \{0,1\}^{\mathbb{N}}\}$.

Lemma 2.2. *The direct sum* $(\bigoplus_{n\geq 1} \langle x_n\rangle) \oplus (\bigoplus_{n\geq 1} \langle y_n\rangle)$ *is a subgroup of* K_2 *and for all* $\lambda \in \mathrm{End}(K_2)$, *there exist* $v \in \mathbb{Z}$, $N \in \mathbb{N}$ *such that* $p^{\alpha_n - n}\lambda(x_n) = p^{\alpha_n-n}vx_n$ *and* $q^{\alpha_n-n}\lambda(y_n) = q^{\alpha_n-n}vy_n$, $\forall\, n \geq N$.

Proof: Let $\mu\colon (\prod_{n\geq 1} \langle x_n\rangle) \oplus (\prod_{n\geq 1} \langle y_n\rangle) \to \prod_{n\geq 1} \langle x_n\rangle$ be the canonical projection. Then $\mu(K_2)$ is the group K_1 of Construction 1 (with $t_{n,m} = q^{n-m}$). Let $\lambda \in \mathrm{End}(K_2)$. There exists $\lambda_1 \in \mathrm{End}(\mu(K_2))$ such that $\lambda_1(\mu(X)) = \mu(\lambda(X))$, $\forall\, X \in K_2$. By Lemma 2.1 there exist $s_1, N_1 \in \mathbb{N}$ and $v_1 \in \mathbb{Z}$ such that $q^{s_1}p^{\alpha_n-n}\lambda_1(x_n) = p^{\alpha_n-n}v_1x_n$, $\forall\, n \geq N_1$, therefore $q^{s_1}p^{\alpha_n-n}\lambda(x_n) = p^{\alpha_n-n}v_1x_n$, $\forall\, n \geq N_1$. In the same way there are $s_2, N_2 \in \mathbb{N}$ and $v_2 \in \mathbb{Z}$ such that $p^{s_2}q^{\alpha_n-n}\lambda(y_n) = q^{\alpha_n-n}v_2y_n$, $\forall\, n \geq N_2$. We can take $s_1 = s_2 = s$ and $N_1 = N_2 = N$. Let $\xi_0(n) = 1$, $\forall\, n \in \mathbb{N}$, we can write: $(pq)^l\lambda(R(1,\xi_0)) = (pq)^l \sum_{j=1}^{k} m_jR(m,\xi_j)$ where $l, k, m \in \mathbb{N}^*$, $m_1, \dots, m_k \in \mathbb{Z}$ and $\xi_1, \dots, \xi_k \in \{0,1\}^{\mathbb{N}}$. We can take $m \geq 1 + s$. By applying μ to this equality, we obtain:

$$p^l\lambda(S(1,\xi_0)) = p^l\sum_{j=1}^{k} m_jS(m,\xi_j).$$

Then for n large enough p^{n+m-1} divides $q^{m-1-s}p^{m-1}v_1 - v(n)$ where $v(n) = \sum_{j=1}^{k} m_j\xi_j(n)$ (see the proof of Lemma 2.1). Let $d \in \mathbb{Z}$ such that the set $\{n \in \mathbb{N}/v(n) = d\}$ is infinite, then $d = q^{m-1-s}p^{m-1}v_1$ in the same way $d = p^{m-1-s}q^{m-1}v_2$. If we put $v_1 = q^sv$ and $v_2 = p^sv$, then we can find $N \in \mathbb{N}$ such that

$$p^{\alpha_n-n}\lambda(x_n)=p^{\alpha_n-n}vx_n \text{ and } q^{\alpha_n-n}\lambda(y_n)=q^{\alpha_n-n}vy_n, \quad \forall\, n \geq N. \qquad \square$$

Construction 3. Let $(\alpha_n)_{n\geq 0}$ be as in Construction 1 and $(\beta_n)_{n\geq 1}$ be a sequence of nonzero natural numbers. Let $p, q_1, \ldots, q_n, \ldots$ be nonzero relatively prime natural numbers. Let us consider the group $(\prod\limits_{n\geq 1}\langle x_n\rangle)\oplus(\prod\limits_{n\geq 1}\langle z_n\rangle)$ with $o(x_n)=p^{\alpha_n}$ and $o(z_n)=q_n^{\beta_n}$, $\forall\, n\geq 1$, the elements g_m and $S(m,\xi)$ of $\prod\limits_{n\geq 1}\langle x_n\rangle$ are defined as in Construction 1 with

$$t_{n,m}=\begin{cases}1 & \text{if } n=m\\ q_1\cdots q_m & \text{if } n=m+1\\ (q_1\cdots q_m)^{n-m}\left(\prod\limits_{j=1}^{n-m-1} q_{m+j}^{n-m-j}\right) & \text{if } n\geq m+2,\end{cases}$$

the element $R(m,\xi)$ of $\prod\limits_{n\geq 1}\langle z_n\rangle$ is defined as follows

$$\varphi_n(R(m,\xi))=\begin{cases}0 & \text{if } n<m\\ \xi(n)p^{n-m}t_{n,m}z_n & \text{if } n\geq m\end{cases}$$

where $\varphi_k\colon \prod\limits_{n\geq 1}\langle z_n\rangle\to\langle z_k\rangle$ is the canonical projection. If we put

$$T(m,\xi)=S(m,\xi)+R(m,\xi)\in\left(\prod_{n\geq 1}\langle x_n\rangle)\oplus(\prod_{n\geq 1}\langle z_n\rangle\right),$$

we have

$$T(m,\xi)=\left(\sum_{n=m}^{r}\xi(n)t_{n,m}p^{n-m}(p^{\alpha_{n-1}}x_n+z_n)\right)+t_{r+1,m}p^{r+1-m}T(r+1,\xi),$$

if $r\geq m$.

Let K_3 be the subgroup of $(\prod\limits_{n\geq 1}\langle x_n\rangle)\oplus(\prod\limits_{n\geq 1}\langle z_n\rangle)$ generated by $\{g_n/n\geq 1\}\cup\{z_n/n\geq 1\}\cup\{T(m,,\xi)/m\geq 1,\ \xi\in\{0,1\}^{\mathbb{N}}\}$.

Lemma 2.3. *The direct sum* $(\bigoplus\limits_{n\geq 1}\langle x_n\rangle)\oplus(\bigoplus\limits_{n\geq 1}\langle z_n\rangle)$ *is a subgroup of* K_3 *and for all* $\lambda\in\mathrm{End}(K_3)$, *there exist* $v\in\mathbb{Z}$ *and* $N,s\in\mathbb{N}$ *such that* $t_{s,1}p^{\alpha_n-n}\lambda(x_n)=p^{\alpha_n-n}vx_n$ *and* $t_{s,1}\lambda(z_n)=vz_n$, $\forall\, n\geq N$.

Proof: Let $\mu\colon(\prod\limits_{n\geq 1}\langle x_n\rangle)\oplus(\prod\limits_{n\geq 1}\langle z_n\rangle)\to\prod\limits_{n\geq 1}\langle x_n\rangle$ be the canonical projection. Then $\mu(K_3)=K_1$ is the group of Construction 1. Let $\lambda\in\mathrm{End}(K_3)$, the endomorphism λ_1 of K_1 defined by $\lambda_1(\mu(X))=\mu(\lambda(X))$,

$\forall X \in K_3$, is well defined. According to Lemma 2.1 there exist $s, N_0 \in \mathbb{N}$ and $v \in \mathbb{Z}$ such that $t_{s,1}p^{\alpha_n-n}\lambda_1(x_n) = p^{\alpha_n-n}vx_n$, $\forall n \geq N_0$. It is clear that $\lambda(z_n) \in \langle z_n\rangle$, $\forall n \geq 1$. Putting $\lambda(z_n) = k_n z_n$, $\forall n \geq 1$, we consider $\xi_0 \colon \mathbb{N} \to \{0,1\}$ with $\xi_0(n) = 1$, $\forall n \in \mathbb{N}$ we can write: $p^l r\lambda(T(1,\xi_0)) = p^l r \sum_{j=1}^{k} d_j T(m,\xi_j)$ where r and p are relatively prime and $m \geq s$. By applying μ to this equality, we obtain:

$$p^l \lambda_1(S(1,\xi_0)) = p^l \sum_{j=1}^{k} d_j S(m,\xi_j).$$

Following the same steps as in Lemmas 2.1 and 2.2, we can find $N_1 \in \mathbb{N}$ such that p^{n+m-1} divides $t_{m,s}p^{m-1}v - v(n)$, $\forall n \geq N_1$, with $v(n) = \sum_{j=1}^{k} d_j\xi_j(n)$. Then there exists N_2 such that $v(n) = t_{m,s}p^{m-1}v$, $\forall n \geq N_2$. If $n \geq m$, then $q_n^{\beta_n}$ divides $t_{m,1}p^{m-1}k_n - v(n)$. Finally there exists $N \in \mathbb{N}$ such that $t_{s,1}p^{\alpha_n-n}\lambda(x_n) = vp^{\alpha_n-n}x_n$ and $t_{s,1}\lambda(z_n) = vz_n$, $\forall n \geq N$. $\square$

3. Characterization of the inessential endomorphisms in the category of the Abelian groups

In the following, we suppose that A is a group, and f an endomorphism of A satisfying the following property.

(E) : For any exact sequence $0 \to A \xrightarrow{\sigma} B$ there exists $\tilde{f} \in \mathrm{End}(B)$ such that the following diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\sigma} & B \\ & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}} \\ 0 & \longrightarrow & A & \xrightarrow{\sigma} & B \end{array}$$

is commutative.

Let $(\alpha_n)_{n\geq 0}$ be a sequence as in Construction 1.

Lemma 3.1. *For all $a \in A$ and any $q \in \mathbb{N}^*$, there exists $v \in \mathbb{Z}$ such that $(f(a) - va) \in \bigcap_{n\geq 0} q^n A$.*

Proof: Let us consider the free group $L = \bigoplus_{n\geq 1} \langle e_n\rangle$. We put $G = A \oplus L$, $G_0 = \langle\{a - q^{\alpha_n}e_n/n \geq 1\}\rangle$ and $\overline{G} = G/G_0$. The homomorphism $\sigma \colon A \to \overline{G}$ defined by $\sigma(b) = b + G_0$ is a monomorphism, and if $x_n = \overline{e_n} + \sigma(A)$ ($\overline{e_n} = e_n + G_0$) then $\overline{G}/\sigma(A) = \bigoplus_{n\geq 1} \langle x_n\rangle$ and $o(x_n) = q^{\alpha_n}$, $\forall n \geq 1$. Let

K_1 be a subgroup of $\prod\limits_{n\geq 1}\langle x_n\rangle$ defined in Construction 1 (with $t_{n,m}=1$, $\forall\, n\geq m$). There exists a commutative diagram, whose rows are exact, and which has the following form:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \overline{G} & \xrightarrow{\pi} & \overline{G}/\sigma(A) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\sigma} & B & \xrightarrow{\mu} & K_1 & \longrightarrow & 0
\end{array}$$

(see [**3**, 24.6]). We can find $\tilde{f}\in \mathrm{End}(B)$ and $\lambda\in \mathrm{End}(K_1)$ such that $\tilde{f}\sigma=\sigma f$ and $\lambda\mu=\mu\tilde{f}$. By Lemma 2.1, there are $v\in\mathbb{Z}$ and $N\in\mathbb{N}$ such that $q^{\alpha_n-n}\lambda(x_n)=vq^{\alpha_n-n}x_n$, $\forall\, n\geq N$. For $n\geq N$, $\mu[q^{\alpha_n-n}(\tilde{f}(\overline{e_n})-ve_n)]=0$, therefore $(f(a)-va)\in q^nA$, so $(f(a)-va)\in\bigcap\limits_{n\geq 0}q^nA$. $\square$

Corollary 3.2. *If $A^1=0$, then for all $a\in T(A)$ there exists $v_a\in\mathbb{Z}$ such that $f(a)=v_a a$ where $T(A)$ is the torsion part of A.*

Proof: Let us put $q=o(a)$ and let $v\in\mathbb{Z}$ such that $(f(a)-va)\in\bigcap\limits_{n\geq 0}q^nA$. Let p be a prime number, if p divides q then $(f(a)-va)\in\bigcap\limits_{n\geq 0}p^nA$ and if p and q are relatively prime, we also have $(f(a)-va)\in\bigcap\limits_{n\geq 0}p^nA$, thus $f(a)=va$. $\square$

Lemma 3.3. *If $A^1=0$, then there exists $v\in\mathbb{Z}$ such that $f(a)=va$, $\forall\, a\in T(A)$.*

Proof: We suppose that $T(A)$ is bounded, then there exists $x_0\in T(A)$ such that $\langle x_0\rangle$ is a direct summand of $T(A)$ and $o(x_0).T(A)=0$. If $f(x_0)=vx_0$, then $\forall\, a\in T(A)$, $f(a)=va$. We now suppose that $T(A)$ is not bounded. If p is prime number, we denote by T_p the p-component of $T(A)$.

1st case: There exists a prime number p such that T_p is not bounded. Let S be a basic subgroup of T_p, we can write

$$S=\left(\bigoplus_{n\geq 1}\langle a_n\rangle\right)\oplus S_0 \text{ with } o(a_n)=p^{r_n} \text{ and } 1\leq r_n<r_{n+1},\quad \forall\, n\geq 1.$$

For each $n \geq 1$, we consider a_n as an element of the group $\langle X_n \rangle$ with $p^{\alpha_n} X_n = a_n$. There exists a group G such that:

$$A \leq G,$$

$$\left(\bigoplus_{n \geq 1} \langle X_n \rangle \right) \leq G,$$

$$A + \left(\bigoplus_{n \geq 1} \langle X_n \rangle \right) = G$$

and

$$A \cap \left(\bigoplus_{n \geq 1} \langle X_n \rangle \right) = \bigoplus_{n \geq 1} \langle a_n \rangle.$$

We put $x_n = X_n + A$, then $G/A = \bigoplus_{n \geq 1} \langle x_n \rangle$ and $o(x_n) = p^{\alpha_n}$, $\forall\, n \geq 1$.

By [**3**, Proposition 24.6], there exists a commutative diagram, whose rows are exact, and has the following form:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & G \arrow[r, "\pi"] \arrow[d] & G/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\sigma"] & B \arrow[r, "\mu"] & K_1 \arrow[r] & 0
\end{tikzcd}.$$

K_1 is the group of Construction 1 (with $t_{n,m} = 1$, $\forall\, n \geq m$). There are $\tilde{f} \in \mathrm{End}(B)$ and $\lambda \in \mathrm{End}(K_1)$ such that $\tilde{f}\sigma = \sigma f$ and $\lambda\mu = \mu\tilde{f}$. There exist $v \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $p^{\alpha_n - n}\lambda(x_n) = vp^{\alpha_n - n}x_n$, $\forall\, n \geq N$ (Lemma 2.1). We have for each $n \geq N$, $\mu[p^{\alpha_n - n}(\tilde{f}(X_n) - vX_n)] = 0$, so that $(f(a_n) - va_n) \in p^n A$.

Let us put $f(a_n) = k_n a_n$ (Corollary 3.2), then we have p^n divides $k_n - v$, $\forall\, n \geq N$. By using again Corollary 3.2, we can establish easily that $f(a_n) = va_n$, $\forall\, n \geq 1$. Let $b \in T_q$ with $q \neq p$, and put $o(b) = q^s$. Let us consider the free group $L = \bigoplus_{n \geq 0} \langle e_n \rangle$. Let L_0 be the subgroup of L generated by $\{q^s e_0\} \cup \{q^{\alpha_n} e_n - e_0 / n \geq 1\}$.

We consider b as an element of $\overline{L} = L/L_0$ by identifying b with $\overline{e_0} = e_0 + L_0$. There exists a group G_1 such that $A \leq G_1$,

$$\left(\bigoplus_{n\geq 1}\langle X_n\rangle\right) \oplus \overline{L} \leq G_1,$$

$$A + \left(\left(\bigoplus_{n\geq 1}\langle X_n\rangle\right) \oplus \overline{L}\right) = G_1$$

and

$$A \cap \left(\left(\bigoplus_{n\geq 1}\langle X_n\rangle\right) \oplus \overline{L}\right) = \left(\bigoplus_{n\geq 1}\langle a_n\rangle\right) \oplus \langle b\rangle.$$

We put $x_n^{'} = X_n + A$ and $y_n = \overline{e_n} + A$, then $o(x_n^{'}) = p^{\alpha_n}$, $o(y_n) = q^{\alpha_n}$. $\forall\, n \geq 1$, and $G_1/A = (\bigoplus_{n\geq 1}\langle x_n\rangle) \oplus (\bigoplus_{n\geq 1}\langle y_n\rangle)$.

Let K_2 be the group of Construction 2, there exists a commutative diagram, whose rows are exact, and has the following form:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & G_1/A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{\sigma_1} & B_1 & \xrightarrow{\mu_1} & K_2 & \longrightarrow & 0 \end{array}.$$

There exist $\tilde{f}_1 \in \mathrm{End}(B_1)$ and $\lambda_1 \in \mathrm{End}(K_2)$ such that $\tilde{f}_1\sigma_1 = \sigma_1 f$ and $\lambda_1\mu_1 = \mu_1\tilde{f}_1$. By Lemma 2.2, there exist $k \in \mathbb{Z}$ and $M \in \mathbb{N}$ such that $p^{\alpha_n - n}\lambda_1(x_n^{'}) = kp^{\alpha_n-n}x_n^{'}$ and $q^{\alpha_n-n}\lambda_1(y_n) = kq^{\alpha_n-n}y_n$, $\forall\, n \geq M$. For $n \geq M$, $p^{\alpha_n-n}\mu_1(\tilde{f}(X_n) - kX_n) = 0$ and $q^{\alpha_n-n}\mu_1(\tilde{f}(\overline{e_n}) - k\overline{e_n}) = 0$ and so $(f(a_n) - ka_n) \in p^nA$ and $(f(b) - kb) \in q^nA$. Then $k = v$ and $f(b) = vb$. Therefore it is easy to deduce that $f(a) = va$, $\forall\, a \in T(A)$.

2nd case: T_p is bounded for any prime number p. We can write $T(A) = \bigoplus_{n\geq 1} T_{p_n}$ and let for each $n \geq 1$, $b_n \in T_{p_n}$ such that $\langle b_n\rangle$ is a direct summand of T_{p_n} and $o(b_n)T_{p_n} = 0$.

We put $o(b_n) = p_n^{\beta_n}$ and we consider b_n as an element of the group $\langle Z_n\rangle$ with $p_n^{\beta_n}Z_n = b_n$. We take $m \geq 1$, there exists a group H such that $A \leq H$, $(\bigoplus_{n\geq m}\langle Z_n\rangle) \leq H$, $H = A + (\bigoplus_{n\geq m}\langle Z_n\rangle)$ and $A \cap (\bigoplus_{n\geq m}\langle Z_n\rangle) = \bigoplus_{n\geq m}\langle b_n\rangle$. If $z_n = Z_n + A$, then $H/A = \bigoplus_{n\geq m}\langle z_n\rangle$. By using Lemma 2.3 and [**3**, Proposition 24.6], as before, we can find $r_1 \in \mathbb{N}^*$ whose only prime factors are $p_m, \dots, p_{m'}$, $m^{'} \geq m$ and $(v_1, N_1) \in \mathbb{Z} \times \mathbb{N}$ such that $r_1 f(b_n) = v_1 b_n$, $\forall\, n \geq N_1$. in the same way there exists $r_2 \in \mathbb{N}$ whose only prime factors are $p_{m'+1}, \dots, p_{m''}$ (in particular $r_2 \wedge r_1 = 1$) and $(v_2, N_2) \in \mathbb{Z} \times \mathbb{N}$ such that $r_2 f(b_n) = v_2 b_n$, $\forall\, n \geq N_2$. If $v = \gamma_1 v_1 + \gamma_2 v_2$ (where $\gamma_1 r_1 + \gamma_2 r_2 = 1$) then for $N = \sup(N_1, N_2)$ we have $f(b_n) = v b_n$, $\forall\, n \geq N$.

We now suppose $n_1 < N$ and put $p = p_{n_1}$, $\beta = \beta_{n_1}$. Let $L = \bigoplus\limits_{n \geq 0} \langle e_n \rangle$ be the free group and $\overline{L} = L/L_1$ where $L_1 = \langle \{p^\beta e_0\} \cup \{p^{\alpha_n} e_n - e_0 / n \geq 1\} \rangle$ there exists a group H_1 such that $A \leq H_1$, $\overline{L} \oplus (\bigoplus\limits_{n \geq N} \langle Z_n \rangle) \leq H_1$, $A + (\overline{L} \oplus (\bigoplus\limits_{n \geq N} \langle Z_n \rangle)) = H_1$ and $A \cap (\overline{L} \oplus (\bigoplus\limits_{n \geq N} \langle Z_n \rangle)) = \langle b_{n_1} \rangle \oplus (\bigoplus\limits_{n \geq N} \langle b_n \rangle)$.

Now put $x_n = \overline{e_n} + A$ ($\overline{e_n} = e_n + L_1$) and $z_n = Z_n + A$, then $o(x_n) = p^{\alpha_n}$, $o(z_n) = p_n^{\beta_n}$ and

$$H_1/A = \left(\bigoplus_{n \geq 1} \langle x_n \rangle \right) \oplus \left(\bigoplus_{n \geq 1} \langle y_n \rangle \right).$$

By applying again Lemma 2.3 and [**3**, Proposition 24.6] we show that $f(a_{n_1}) = v a_{n_1}$. Thus $f(a_n) = v a_n$, $\forall\, n \geq 1$ and thereafter $f(a) = va$, $\forall\, a \in T(A)$. □

Lemma 3.4. *If $A^1 = 0$ and $T(A) = 0$, then there exists $v \in \mathbb{Z}$ such that $f = v\, \mathrm{id}_A$.*

Proof: Let $a \in A$ with $a \neq 0$. There exists a prime number p such that $a \notin \bigcap\limits_{n \geq 0} p^n A$. According to Lemma 3.1 there exists $v \in \mathbb{Z}$ such that $(f(a) - va) \in \bigcap\limits_{n \geq 0} p^n A$. Let $q \in \mathbb{N}^*$, there exists $v_q \in \mathbb{Z}$ such that $(f(a) - v_q a) \in \bigcap\limits_{n \geq 0} (pq)^n A$.

We have $(v - v_q)a \in \bigcap\limits_{n \geq 0} p^n A$, this implies $v_q = v$, and thereafter $f(a) = va$. Since A is torsion-free, it is easy to establish that $f(b) = vb$, $\forall\, b \in A$. □

Lemma 3.5. *If $A^1 = 0$, then there exists $v \in \mathbb{Z}$ such that $f = v\, \mathrm{id}_A$.*

Proof: By Lemma 3.3, there exists $v \in \mathbb{Z}$ such that $f(x) = vx$, $\forall\, x \in T(A)$.

Let $a \in A$, we will show that $f(a) \in \langle a \rangle$. We Suppose that $(f(a) - va) \neq 0$, then there exists a prime number p such that $(f(a) - va) \notin$

$\bigcap\limits_{n\geq 0} p^n A$. By Lemma 3.1, there exists $r\in\mathbb{Z}$ such that $(f(a)-ra)\in \bigcap\limits_{n\geq 0} p^n A$, there also exists for all $q\in\mathbb{N}^*$ an $r_q\in\mathbb{Z}$ such that $(f(a)-r_q a)\in \bigcap\limits_{n\geq 0} (pq)^n A$. Assume $r_q\neq r$ for some number q.

Since $(r-r_q)a\in \bigcap\limits_{n\geq 0} p^n A$, then there exists $s\in\mathbb{N}$ such that $p^s a\in \bigcap\limits_{n\geq 0} p^n A$. Therefore $\forall\, n\in\mathbb{N}$, there exists $a_n\in A$ such that $p^s(a-p^n a_n)=0$, it follows that $f(a-p^n a_n)=v(a-p^n a_n)$ and hence $(f(a)-va)\in p^n A$, which is absurd. Thus $r_q=r$, $\forall\, q\in\mathbb{N}^*$ and thereafter $f(a)=ra$.

Now, we will distinguish two cases:

1st case: $T(A)$ is not bounded. Let $a\in A$ with $o(a)=\infty$ and put $f(a)=ra$. $\forall\, x\in T(A)$, $f(a+x)=r^{'}(a+x)=ra+vx$ which implies that $r=r^{'}$ and $(v-r^{'})x=0$, since $T(A)$ is not bounded so $r=v$.

2nd case: $T(A)$ is bounded, let $m\in\mathbb{N}^*$ such that $mT(A)=0$.

We consider the exact sequence $0\to T(A)\to A\to mA\to 0$. By [**3**, Proposition 24.6] it is easy to see that the endomorphism g of mA defined by $g(ma)=mf(a)$ satisfies the property (E), since $T(mA)=0$ and $(mA)^1=0$ then according to Lemma 3.4 there exists $r\in\mathbb{Z}$ such that $mf(a)=rma$, $\forall\, a\in A$.

We suppose $T(A)\neq A$. Let $a\in A$ with $o(a)=\infty$ and $f(a)=r_a a$, therefore $m(r_a-a)a=0$ and hence $r_a=r$.

Let $x\in T(A)$, then $f(a+x)=r(a+x)=ra+vx$ which implies that $(r-v)x=0$, thus $f(x)=rx$.

Finally $\forall\, b\in A$, $f(b)=rb$. $\square$

Theorem 3.6. *If A is reduced, then there exists $v\in\mathbb{Z}$ such that $f=v\,\mathrm{id}_A$.*

Proof: Let $x\in A$ such that $\langle x\rangle$ is a direct summand of A.

We can write $A=\langle x\rangle\oplus A_0$. Let S be a divisible group such that $x\in S$.

Let $\sigma\colon A\to S\oplus A_0$, $\sigma(nx+a_0)=nx+a_0$. Then there exists $\tilde{f}\in \mathrm{End}(S\oplus A_0)$ such that $\tilde{f}\sigma=\sigma f$. If we put $f(x)=mx+a_0$ with $m\in\mathbb{Z}$ and $a_0\in A_0$, we get $a_0=\tilde{f}(x)-mx\in S\cap A_0=0$ which implies that $a_0=0$ and thereafter $f(x)=mx$. Therefore if $\langle x\rangle$ is a direct summand of A, then $f(x)\in\langle x\rangle$.

By Lemma 3.5 there exists $v\in\mathbb{Z}$ such that $(f-v\,\mathrm{id})(A)\subseteq A^1$. We put $\rho=f-v\,\mathrm{id}_A$.

Show first that $\rho(T(A)) = 0$. Let B be p-basic subgroup of $T(A)$ (p is a prime number), $B = \bigoplus_{i\in I}\langle x_i\rangle$ and $\forall\, i \in I$, $\langle x_i\rangle$ is a direct summand of A. We put for $i \in I$, $f(x_i) = m_i x_i$. We have $(m_i - v)x_i \in A^1$ which implies that $m_i x_i = v x_i = f(x_i)$.

Then $\rho(B) = 0$ and thereafter $\rho(T(A))$ is p divisible. Therefore, $\rho(T(A))$ is divisible and so $\rho(T(A)) = 0$. Let us put $A/T(A) = (D/T(A)) \oplus (R/T(A))$ with $D/T(A)$ divisible, $R/T(A)$ reduced, $T(A) \leq D$ and $A^1 \leq D$.

The homomorphism $\overline{\rho}\colon A/T(A) \to A$ where $\overline{\rho}(a + T(A)) = \rho(a)$ is well defined and $\overline{\rho}(D/T(A)) = \rho(D) = 0$ because $D/T(A)$ is divisible and A is reduced. There exists a torsion-free divisible group C such that $A/D \leq C$. By [**3**, Proposition 24.6], there exists a commutative diagram, whose rows are exact, and has the following form:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A/D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] & A_1 \arrow[r, "\mu"] & C \arrow[r] & 0
\end{tikzcd}.$$

Let D_1 be the maximal divisible subgroup of A_1, $D \cap D_1$ is then divisible.

In fact if $x \in D \cap D_1$ and $n \in \mathbb{N}^*$, we can write $x = ny$ with $y \in D_1$ and $\mu(x) = n\mu(y) = 0$ so $\mu(y) = 0$ because C is torsion-free, therefore $y \in D$. Since A is reduced, then $D \cap D_1 = 0$. there exists $f_1 \in \mathrm{End}(A_1)$ such that $f_1(a) = f(a)$, $\forall\, a \in A$.

If we put $\rho_1 = f_1 - v\,\mathrm{id}_{A_1}$, we have $\rho_1(A) = \rho(A) \subseteq A^1 \subseteq D$, from an other side the homomorphism $\overline{\rho_1}\colon A_1/D \to A_1$ such that $\overline{\rho_1}(a_1 + D) = \rho_1(a_1)$, for $a_1 \in A_1$, is well defined. Thus $\rho_1(A_1)$ is divisible and thereafter $\rho_1(A_1) \subseteq D_1$. We then conclude that $\rho(A) \subseteq D \cap D_1 = 0$ which implies that $\rho = 0$. $\square$

Corollary 3.7. *Let A be a group and f be an endomorphism of A, f satisfies (E) if and only if there exists $v \in \mathbb{Z}$ such that $(f - v\,\mathrm{id}_A)(A) \subseteq D$, where D is maximal divisible subgroup of A.*

Proof: According to [**3**, Proposition 24.6], the endomorphism $\overline{f}$ of $\overline{A} = A/D$, $(\overline{f}(\overline{a}) = \overline{f(a)})$ satisfies (E). By Theorem 3.6, there exists $v \in \mathbb{Z}$ such that $(f - v\,\mathrm{id}_A)(A) \subseteq D$.

The second assertion is easy to establish. $\square$

We end this paper by the following remarks:

1. Let C be a reduced group. C is rigid (according to the terminology of [**2**]) if and only if C is torsion cyclic or C is torsion-free and $\mathrm{End}(C) \cong \mathbb{Z}$.
2. A group A is rigid if and only if $A = D \oplus C$ with D divisible and C reduced rigid.
3. For any cardinal m there exists a rigid group of cardinality m ([**1**], [**4**] and [**5**]).

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