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A SCHWARZ LEMMA FOR CORRESPONDENCES AND APPLICATIONS

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Abstract

A version of the Schwarz lemma for correspondences is studied. Two applications are obtained namely, the 'non-increasing' property of the Kobayashi metric under correspondences and a weak version of the Wong-Rosay theorem for convex, finite type domains admitting a 'non-compact' family of proper correspondences.

1. Introduction

Let $\mathcal{D}$ and $\mathcal{D}'$ be bounded domains in $\mathbb{C}^p$ and $\mathbb{C}^n$ respectively. A complex analytic set $A \subset \mathcal{D} \times \mathcal{D}'$ of pure dimension p that satisfies $\overline{A} \cap (\mathcal{D} \times \partial \mathcal{D}') = \emptyset$ is called a correspondence. In this situation, the natural projection $\pi \colon A \to \mathcal{D}$ is proper, surjective and a finite-to-one branched covering. The number of points in the fiber over a generic point $z \in \mathcal{D}$ is the multiplicity of A. We can also regard A as the graph of the multivalued mapping $\hat{f} := \pi' \circ \pi^{-1} \colon \mathcal{D} \to \mathcal{D}'$ where $\pi' \colon A \to \mathcal{D}'$ is the natural projection. Let $\operatorname{Cor}(\mathcal{D}, \mathcal{D}', k)$ denote the family of correspondences $\hat{f} \colon \mathcal{D} \to \mathcal{D}'$ with multiplicity at most k. If both π and π' are proper then A is called a proper correspondence. Let $\operatorname{PropCor}(\mathcal{D}, \mathcal{D}', k, k')$ denote the family of proper correspondences $\hat{f} \colon \mathcal{D} \to \mathcal{D}'$ where k and k' are the upper bounds for the multiplicities of the branched coverings $\pi \colon A \to \mathcal{D}$ and $\pi' \colon A \to \mathcal{D}'$ respectively. Here are two examples: firstly, let $f \colon \mathcal{D} \to \mathcal{D}'$ be a proper holomorphic mapping between bounded domains $\mathcal{D}$, $\mathcal{D}'$ in $\mathbb{C}^n$. Then $\operatorname{Graph}(f) \subset \mathcal{D} \times \mathcal{D}'$ defines a proper correspondence of pure dimension n. Secondly, let $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$. The graph of the multivalued mapping $\hat{f}(z, w) = (w^2, \pm\sqrt{z})$ defines a proper self correspondence of Ω.

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The purpose of this article is to prove the following version of the Schwarz lemma for correspondences and to indicate some applications. Let $\mathbb{B}^p$, $p \geq 1$, be the unit ball in $\mathbb{C}^p$ and $k > 0$ be a fixed integer. For an arbitrary domain $\mathcal{D}$ in $\mathbb{C}^n$ and a fixed point $z_0' \in \mathcal{D}$, let $\mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k, 0, z_0')$ will denote the family of correspondences $\hat{f} \in \mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k)$ with the additional property that $z_0' \in \hat{f}(0)$.

Theorem 1.1. (i) *Let $\mathcal{D} \Subset \mathbb{C}^n$ be a domain such that each point of $\partial\mathcal{D}$ is a local plurisubharmonic peak point. Fix $z_0' \in \mathcal{D}$, $k > 0$ and consider an arbitrary $\hat{f} \in \mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k, 0, z_0')$. Then for all $\eta \in (0,1)$, there exists a domain $U \Subset \mathcal{D}$ not depending on $\hat{f}$ and containing z_0' such that*

$$\hat{f}_{0,z_0'}(\eta\mathbb{B}^p) \subset U.$$

(ii) *Conversely, for each neighbourhood $U \Subset \mathcal{D}$ of z_0' there exists $\eta \in (0,1)$ not depending on $\hat{f}$ such that*

$$\hat{f}_{0,z_0'}(\eta\mathbb{B}^p) \subset U.$$

Roughly speaking, this says that the image of a smaller ball under $\hat{f}$ can be controlled. We have to be careful here though; since $\hat{f}$ is a multivalued mapping, the image $\hat{f}(\eta\mathbb{B}^p)$ will consist of at most k distinct components. The theorem is an assertion *only* about that component which contains the apriori fixed point z_0'. This will be made more precise in the next section.

It is well known that the Caratheodory and the Kobayashi metric are useful in studying the boundary behaviour of proper holomorphic mappings. Moreover some of their basic properties are consequences of the Schwarz lemma. In answering questions about the boundary behaviour of proper correspondences, S. Pinchuk [**18**] formulated and proved a version of the Schwarz lemma for the class $\mathrm{Cor}(\mathcal{D}, \Delta, k)$ where $\mathcal{D} \subset \mathbb{C}^n$ is an arbitrary domain and $\Delta \subset \mathbb{C}$ is the unit disc. This was used to obtain the 'non-increasing' property of the Caratheodory metric under correspondences and this led to the boundary continuity of proper correspondences between strongly pseudoconvex domains in $\mathbb{C}^n$. Theorem 1.1 can be considered as a 'dual' statement to Pinchuk's version of the Schwarz lemma. It particular it leads to the 'non-increasing' property of the Kobayashi metric. For an arbitrary domain $\mathcal{D}$ in $\mathbb{C}^n$ let $\mathbb{B}^k_{\mathcal{D}}(z, r)$ denote the Kobayashi ball of radius $r > 0$ centered at $z \in \mathcal{D}$. Also, let $k_{\mathcal{D}}(z, w)$ denote the Kobayashi distance in $\mathcal{D}$ between $z, w \in \mathcal{D}$ and $d(z, w)$ the usual Euclidean distance.

Theorem 1.2. (i) *Let $\mathcal{D}$ be a smoothly bounded convex domain of finite type in $\mathbb{C}^n$. Fix $k>0$ and consider an arbitrary $\hat{f}\in \mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k)$. Then for all $\eta \in (0,1)$, there exists $R = R(k, \eta, \mathcal{D}) > 0$ such that*

$$\hat{f}_{0,q}(\eta\mathbb{B}^p) \subset \mathbb{B}^k_{\mathcal{D}}(q, R)$$

where $q \in \hat{f}(0)$.

(ii) *Conversely, for all $R > 0$ there exists $\eta \in (0,1)$ not depending on $\hat{f}$ such that*

$$\hat{f}_{0,q}(\eta\mathbb{B}^p) \subset \mathbb{B}^k_{\mathcal{D}}(q, R)$$

where $q \in \hat{f}(0)$.

Two points should be noted here. Firstly, the value of $R > 0$ obtained above is independent of both $\hat{f} \in \mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k)$ and $q \in \hat{f}(0)$. Secondly, having selected a $q \in \hat{f}(0)$, the theorem is an assertion *only* about that component of $\hat{f}(\eta\mathbb{B}^p)$ that contains the chosen point q. Theorem 1.1 provides a neighbourhood U of q that contains $\hat{f}(\eta\mathbb{B}^p)$ and moreover we can also choose $R > 0$ so that $U \subset B^k_{\mathcal{D}}(q, R)$. Indeed, it is known that (see for example [**1**, Theorem 2.3.51, p. 229])

$$k_{\mathcal{D}}(q, z) \leq c - (1/2) \log d(z, \partial\mathcal{D})$$

for some positive constant c. To digress, note that this upper estimate uses neither the convexity nor the finite type assumption. It is a consequence of the smoothness of the boundary alone. Now since $U \Subset \mathcal{D}$, $d(z, \partial\mathcal{D})$ is positive for $z \in U$. The existence of R follows. Thus it may seem that Theorem 1.2 is a direct consequence of Theorem 1.1. However, the essential point is that the neighbourhood U obtained from Theorem 1.1 apriori depends on q and hence so does R. A scaling procedure for correspondences is used to remove this dependence on q. We use the one that was developed in [**11**] to scale the domain while the normality of the scaled family of correspondences follows from Pinchuk's version of the Schwarz lemma in [**18**]. The next section considers the notion of a normal family of correspondences more precisely. These ideas can also be applied to prove Theorem 1.2 in case $\mathcal{D}$ is strongly pseudoconvex. Thus we recover the boundary continuity of proper correspondences between strongly pseudoconvex domains in $\mathbb{C}^n$ as discussed in [**18**].

Considering normal families of correspondences also turns out to be useful in another context. To motivate this, recall the following example noted by Bedford-Bell [**2**]: Let $\Omega = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^4 < 1\}$. Consider the proper map $\eta(z, w) = (z, w^2)$ from Ω to $\mathbb{B}^2$. Choose $\{\phi_j\} \in \mathrm{Aut}(\mathbb{B}^2)$ a sequence of automorphisms converging to $(0, 1) \in \partial\mathbb{B}^2$. Such a sequence exists as $\mathrm{Aut}(\mathbb{B}^2)$ is a non-compact real Lie group. Then

$\{\eta^{-1}\circ\phi_j\circ\eta\}$ is a sequence of proper self correspondences of Ω converging to $(0,1),(0,-1)\in\partial\Omega$. Thus a sequence of correspondences can converge to strongly pseudoconvex points of $\partial\Omega$ even though Ω is not strongly pseudoconvex. In particular the Wong-Rosay theorem does not hold for correspondences. The next theorem is motivated by this example in view of the recent results that characterize bounded domains in $\mathbb{C}^n$ in terms of their automorphism groups. The reader is referred to the survey article [**13**] which gives an overview of such results along with some techniques used to prove them. Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$. Fix $z_0\in\mathcal{D}$ and a family $\mathcal{F}=\{\hat{f}^\nu:\nu\geq 1\}\subset \mathrm{PropCor}(\mathcal{D},\mathcal{D},k,k')$. The orbit of z_0 with respect to $\mathcal{F}$ is the set $\{\hat{f}^\nu(z_0):\nu\geq 1\}$. Note that for each fixed ν, $\hat{f}^\nu(z_0)$ is a finite set consisting of at most k points. A point $\zeta_0\in\partial\mathcal{D}$ is an orbit accumulation point provided the orbit of some $z\in\mathcal{D}$ with respect to some family $\{\hat{f}^\nu:\nu\geq 1\}$ contains ζ_0 in its closure. [**16**] contains a related result.

Theorem 1.3. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$. Let $\zeta_0\in\partial\mathcal{D}$ be a point near which $\partial\mathcal{D}$ is smooth, convex and of finite type, say $2m$. If ζ_0 is an orbit accumulation point with respect to a family $\mathcal{F}\subset\mathrm{PropCor}(\mathcal{D},\mathcal{D},k,k')$, then there exists $\hat{F}\in\mathrm{PropCor}(\mathcal{D},\mathcal{G},k,k')$ where,*

$$\mathcal{G}=\{(z_1,z')\in\mathbb{C}\times\mathbb{C}^{n-1}:\Im(z_1)+P(z')<0\},$$

$P(z')$ being a real non-degenerate convex polynomial of degree at most $2m$. In particular, $\mathcal{D}$ is pseudoconvex.

The non-degeneracy of $P(z')$ means that $\{P=0\}$ does not contain any non-trivial complex analytic sets. In case ζ_0 is a strongly pseudoconvex point, $\mathcal{G}$ is biholomorphic to $\mathbb{B}^n$ and hence we get a weak version of the Wong-Rosay theorem for a 'non-compact' family of proper self-correspondences. It seems very likely that Theorem 1.3 holds even in case ζ_0 is a smooth, finite type boundary point of some bounded domain in $\mathbb{C}^2$. Scaling of the domain near ζ_0 is well understood but the main difficulty is to show the normality of the family of scaled correspondences.

Finally, recall the Greene-Krantz observation (see [**12**]): let $\mathcal{D}\Subset\mathbb{C}^n$ be a domain with non-compact automorphism group. Let $\zeta_0\in\partial\mathcal{D}$ be a point where the orbit of some $z\in\mathcal{D}$ accumulates. Then $\zeta_0\notin\hat{\mathcal{D}}$, the envelope of holomorphy of $\mathcal{D}$. We show that this observation remains valid in the context of proper self correspondences as well.

Proposition 1.1. *Let $\mathcal{D}\Subset\mathbb{C}^n$ and fix $\mathcal{F}=\{\hat{f}^\nu:\nu\geq 1\}\subset\mathrm{PropCor}(\mathcal{D},\mathcal{D},k,k')$. Suppose that there exists $z\in\mathcal{D}$ such that its orbit with respect to $\mathcal{F}$ accumulates at $\zeta_0\in\partial\mathcal{D}$. Then $\zeta_0\notin\hat{\mathcal{D}}$.*

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2. Some preliminary notions

Let $\mathcal{D}$, $\mathcal{D}'$ be bounded domains in $\mathbb{C}^n$ and $\hat{f} \in \mathrm{Cor}(\mathcal{D}, \mathcal{D}', k)$. Pick $a \in K \subset \mathcal{D}$ and $b \in \hat{f}$. Define a correspondence $\hat{f}_{a,b} \in \mathrm{Cor}(K^\circ, \mathcal{D}', k)$, where K° refers to the interior of K, as follows. Consider the germ of $\hat{f}$ at (a, b) and decompose it into finitely many irreducible germs of branches of $\hat{f}$ at (a, b). Analytic continuation of each of these irreducible germs along all possible paths in K defines $\hat{f}_{a,b} \in \mathrm{Cor}(K^\circ, \mathcal{D}', k)$. Equivalently, the graph of this new correspondence is the union of all those irreducible components of $\mathrm{Graph}(\hat{f}) \cap (K \times \mathcal{D}')$ which contain (a, b). Furthermore, if $S \subset K$, $\hat{f}_{a,b}(S)$ will denote the image of S under only those branches of $\hat{f}$ which are obtained by continuation of all irreducible germs of branches of $\hat{f}$ at (a, b) along all paths in S.

Let $\mathcal{F} = \{\hat{f}^\nu : \nu \geq 1\} \subset \mathrm{Cor}(\mathcal{D}, \mathcal{D}', k)$. Following [**14**], we say that $\mathcal{F}$ is compactly divergent if for all $K \Subset \mathcal{D}$ and $K' \Subset \mathcal{D}'$, there exists ν_0 so that $\hat{f}^\nu(K) \cap K' = \emptyset$ for all $\nu \geq \nu_0$. On the other hand, $\{\hat{f}^\nu : \nu \geq 1\}$ converges to $\hat{f} \in \mathrm{Cor}(\mathcal{D}, \mathcal{D}', k)$ if there exists a sequence $(z_0, z'_\nu) \in \mathrm{Graph}(\hat{f}^\nu)$ with $z'_\nu \to z'_0 \in \mathcal{D}'$ and for all $K \Subset \mathcal{D}$ with $z_0 \in K$,

(i) $\hat{f}^\nu_{z_0, z'_\nu} \in \mathrm{Cor}(K^\circ, \mathcal{D}', k)$ converges to $\hat{f}_{K^\circ}$ for some $\hat{f}_{K^\circ} \in \mathrm{Cor}(K^\circ, \mathcal{D}', k)$ in the sense that the graph of $\hat{f}^\nu_{z_0, z'_\nu}$ over K converges in the Hausdorff metric to the graph of $\hat{f}_{K^\circ}$ over K,

and

(ii) $\bigcup_{K \Subset \mathcal{D}} \mathrm{Graph}(\hat{f}_{K^\circ}) = \hat{f}$.

A family $\mathcal{F}$ is said to be normal if every sequence in it has either a compactly divergent or a convergent subsequence.

Finally, $B(z, r)$ will denote the euclidean ball centered at z with radius $r > 0$. For a set $K \in \mathbb{C}^n$ and $\epsilon > 0$, $K(\epsilon)$ will denote the ϵ neighbourhood of K. The class of plurisubharmonic functions on an arbitrary domain Ω will be denoted by $\mathrm{PSH}(\Omega)$.

3. Proofs of Theorems

The statement of Theorem 1.1 involves a domain having a local plurisubharmonic peak function at each of its boundary points. The existence of such functions has been established for a wide class of domains. For example, the class of so called $\mathcal{B}$ regular domains introduced in [**20**] satisfies this property. We refer the reader to [**7**], [**9**], [**10**] and [**20**] which deal with such constructions.

Proof of Theorem 1.1: (i) Let z, z' denote coordinates in $\mathbb{B}^p$, $\mathcal{D}$ respectively. The graph of $\hat{f} \in \mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k, 0, z_0')$ is an analytic set which can be described by using canonical defining equations (cf. [**6**, §4]) as follows:

$$\mathrm{Graph}(\hat{f}) = \left\{ \Phi_I(z, z') := \sum_{|J| \le k} \psi_{IJ}(z)(z')^J = 0 : |I| = k \right\} \tag{3.1}$$

where $\psi_{IJ} \in \mathcal{O}(\mathbb{B}^p)$ for all multi-indices I, J. Since the projection $\pi\colon \mathrm{Graph}(\hat{f}) \to \mathbb{B}^p$ is proper, we can regard $\mathrm{Graph}(\hat{f})$ as an analytic set in $\mathbb{B}^p \times \mathbb{C}^n$. Fix a family of increasing, relatively compact domains $\mathcal{D}_\nu$ which exhaust $\mathcal{D}$. Suppose the theorem is false. Then there exists $\eta \in (0,1)$, and a sequence of correspondences $\hat{f}^\nu \in \mathrm{Cor}(\mathbb{B}^p, \mathcal{D}, k, 0, z_0')$ such that

$$\hat{f}^\nu_{0,z_0'}(\eta\mathbb{B}^p) \not\subset \mathcal{D}_\nu.$$

Each $\hat{f}^\nu$ can be described as above by writing,

$$\mathrm{Graph}(\hat{f}^\nu) = \left\{ \Phi_I^\nu(z, z') := \sum_{|J| \le k_\nu} \psi^\nu_{IJ}(z)(z')^J = 0 : |I| = k_\nu,\ k_\nu \le k \right\} \tag{3.2}$$

where $\psi^\nu_{IJ}(z) \in \mathcal{O}(\mathbb{B}^p)$ for all ν, I, J. After taking a subsequence, assume that the multiplicity $k_\nu = k'$ for all ν and we write $k' = k$ for brevity. For each ν, let $\gamma_\nu \subset \eta\mathbb{B}^p$ be a path parametrized by $[0,1]$ with the following properties:

(i) $\gamma_\nu(0) = 0$, $\gamma_\nu(1) = z_\nu$,

and

(ii) analytic continuation of some irreducible germ of the branch of $\hat{f}^\nu$ at $(0, z_0')$ along γ_ν results in a branch, say $\hat{f}^\nu_{j(\nu)}$, $j(\nu) \le k$, such that $\hat{f}^\nu_{j(\nu)}(z_\nu) := z_\nu' \in \partial\mathcal{D}_\nu$.

After passing to a further subsequence, we may assume that $z_\nu \to z_0 \in \eta\overline{\mathbb{B}}^p$ and that $z_\nu' \to \tilde{z}_0' \in \partial\mathcal{D}$. The family $\{\psi^\nu_{IJ} : \nu \ge 1\}$ is normal since $\mathcal{D}$ is bounded and hence a further subsequence, still denoted by ψ^ν_{IJ}, converges uniformly on compact subsets of $\mathbb{B}^p$ to $\psi_{IJ} \in \mathcal{O}(\mathbb{B}^p)$ for all I, J. Consider the analytic set $V \subset \mathbb{B}^p \times \mathbb{C}^n$ defined by

$$V = \left\{ \Phi_I(z, z') := \sum_{|J| \le k} \psi_{IJ}(z)(z')^J = 0 : |I| = k \right\}. \tag{3.3}$$

Note that $(z_0, \tilde{z}_0') \in V$. Then V is a pure p dimensional analytic set. Indeed, fix $(z_1, z_1') \in V$ and consider

$$\left\{ z' : \Phi_I^\nu(z_1, z') = \sum_{|J| \le k} \psi_{IJ}^\nu(z_1)(z')^J = 0 : |I| = k \right\} \tag{3.4}$$

for a fixed ν. These are the canonical defining equations for the system of points $\hat{f}^\nu(z_1)$, which we will denote by

$$\hat{f}^\nu(z_1) = \{f_1^\nu(z_1), \dots, f_k^\nu(z_1)\}. \tag{3.5}$$

But $\psi_{IJ}^\nu(z_1) \to \psi_{IJ}(z_1)$ as $\nu \to \infty$ and so $\Phi_I^\nu(z_1, z')$ converges uniformly on compact subsets of $\mathbb{C}^n$ to $\Phi_I(z_1, z')$. As $\mathcal{D}$ is bounded, two conclusions may be drawn from [**6**]. Firstly, the system of points $\hat{f}^\nu(z_1) \to \hat{f}^\infty(z_1) := \{f_1^\infty(z_1), \dots, f_k^\infty(z_1)\}$ and secondly that $\Phi_I(z_1, z')$ are the canonical defining equations for the system $\hat{f}^\infty(z_1)$. Thus the fibre above z_1 in V is discrete and hence the complex subspace $L := \{z_1\} \times \mathbb{C}^n$ is such that (z_1, z') is isolated in $V \cap L$. We conclude that $\operatorname{codim}_{(z_1,z')} V \ge n$. On the other hand, the analytic sets $\operatorname{Graph}(\hat{f}^\nu)$ are all pure p dimensional and hence it follows that $\dim_{(z_1,z')} V \ge p$. Therefore V is pure p dimensional.

Let ϕ be a local plurisubharmonic peak function at $\tilde{z}_0' \in \partial\mathcal{D}$. Let V_1 be an irreducible component of V containing $(z_0, \tilde{z}_0')$ which by the above argument is isolated in the fibre over z_0 in V. Choose neighbourhoods $z_0 \in U$ and $\tilde{z}_0' \in U'$ in $\mathbb{B}^p$ and $\mathbb{C}^n$ respectively so that the projection $\pi \colon V_1 \cap (U \times U') \to U$ is proper. Let $g_1, \dots, g_k$ be the branches of π^{-1} which are locally well defined and holomorphic on $U \setminus \sigma$, where $\sigma \subset U$ is an analytic set of dimension at most $p-1$. The function $\rho(z) = \max(\phi \circ g_1, \dots, \phi \circ g_k) \in \operatorname{PSH}(U \setminus \sigma)$ is bounded above as $\phi \le 1$. It therefore extends as a plurisubharmonic function on U. Moreover, $\rho \le 1$ and $\rho(z_0) = 1$. It follows that $\rho \equiv 1$ on U by the maximum principle. In particular, one of the branches, say $g_1(z) \equiv \tilde{z}_0'$ for all $z \in U$. It follows that $U \times \{\tilde{z}_0'\} \subset V_1 \cap (U \times U')$. The irreducibility of the analytic set $\mathbb{B}^p \times \{\tilde{z}_0'\}$ implies that $V_1 = \mathbb{B}^p \times \{\tilde{z}_0'\}$. Since V_1 was an arbitrary component, it follows that there exists a unique component namely $\mathbb{B}^p \times \{\tilde{z}_0'\}$ that contains $(z_0, \tilde{z}_0')$. A similar argument shows that if $(z, \tilde{z}_0') \in V$ where z is different from z_0, there is again a unique component of V, namely $\mathbb{B}^p \times \{\tilde{z}_0'\}$ that contains it. In particular, if V' is another component of V distinct from $\mathbb{B}^p \times \{\tilde{z}_0'\}$, then $V' \cap (\mathbb{B}^p \times \{\tilde{z}_0'\}) = \emptyset$. If there exists $(\tilde{z}, \tilde{z}') \in V$ with $\tilde{z}' \in \partial\mathcal{D}$, the same process as described above can be used to identify the 'constant component' $\mathbb{B}^p \times \{\tilde{z}'\}$ of V. However $(0, z_0') \in V$ shows that not all irreducible components of V are 'constant' components. Let $\{\tilde{z}'_1, \dots, \tilde{z}'_l\} \in \partial\mathcal{D}$ be all the distinct points which correspond to these 'constant' components. Let $\pi\colon V \to \mathbb{B}^p$, $\pi'\colon V \to \mathbb{C}^n$ be the projections and $\hat{V} := \pi' \circ \pi^{-1}$ denote the branches of V over $\mathbb{B}^p$. It follows that all the 'non-constant' branches of V are contained in $\mathcal{D}$ for all $z \in \mathbb{B}^p$. Moreover, since $\mathcal{D}$ is bounded, continuity of roots (cf. [**6**, §4]) of the canonical defining equations forces all the 'non-constant' branches to be contained in some $K \Subset \mathcal{D}$ for all $z \in \eta\mathbb{B}^p$.

Choose $\epsilon > 0$ small enough so that $\partial\mathcal{D}(\epsilon) \cap K(\epsilon) = \emptyset$ and $z'_0 \notin \partial\mathcal{D}(\epsilon)$. The fact that $\psi^\nu_{IJ}(z)$ converge uniformly on $\psi_{IJ}(z)$ on $\eta\mathbb{B}^p$ combined with the boundedness of $\mathcal{D}$ and the continuity of roots now shows that all the branches of $\hat{f}^\nu$ and $\hat{V}$, for all large ν, are contained in the disjoint union $\partial\mathcal{D}(\epsilon) \cup K(\epsilon)$ for $z \in \eta\mathbb{B}^p$. Since $z'_0 \notin \partial\mathcal{D}(\epsilon)$, we must have $\hat{f}^\nu_{0,z'_0}(\eta\mathbb{B}^p) \subset K(\epsilon) \Subset \mathcal{D}$ for ν large enough. This contradicts the assumption that $\hat{f}^\nu_{0,z'_0}(\eta\mathbb{B}^p) \not\subset \mathcal{D}_\nu$ for all ν.

(ii) Given a neighbourhood $U \Subset \mathcal{D}$ of z'_0, choose $R > 0$ so that $B(z_0, R) \Subset U$. Corollary 5 of [**14**] now ensures the existence of $\eta \in (0,1)$, independent of $\hat{f}$ such that

$$\hat{f}_{0,z'_0}(\eta\mathbb{B}^p) \subset B(z_0, R) \subset U$$

holds. $\square$

Let us briefly recall the scaling of convex domains developed in [**11**] (see [**8**] also). Let $\mathcal{D}$ be as in the hypotheses of Theorem 1.2. Without loss of generality $0 \in \partial\mathcal{D}$ and we may write the defining function of $\partial\mathcal{D}$ in a neighbourhood, say U of the origin as

$$\mathcal{D} \cap U = \{z = (z_1, z') \in U : \rho(z) = \Im(z_1) + \phi(\Re(z_1), z') < 0\} \tag{3.6}$$

where ϕ is a smooth convex function. Let $(q_\nu) \in \mathcal{D}$ be a sequence converging to $0 \in \partial\mathcal{D}$. Let $q^1_\nu \in \partial\mathcal{D}$ be closest to q_ν. Denote the complex line containing q^1_ν and q_ν by l^ν_1 and set $\tau^\nu_1 = d(q_\nu, q^1_\nu)$. Consider the orthogonal complement $(l^\nu_1)^\perp$ in $\mathbb{C}^n$. Since $\mathcal{D}$ is of finite type, the distances from q_ν to $\partial\mathcal{D}$ in $(l^\nu_1)^\perp$ is uniformly bounded and there exists $q^2_\nu \in \partial\mathcal{D}$ where the largest distance is reached. Denote the complex line containing q_ν and q^2_ν by l^ν_2 and set $\tau^\nu_2 = d(q_\nu, q^2_\nu)$. Consider the orthogonal complement of the complex subspace generated by l^ν_1, l^ν_2 and find the largest distance from q_ν to $\partial\mathcal{D}$ therein. Repeating this process we get orthogonal lines $l^\nu_1, l^\nu_2, \dots, l^\nu_n$. Let U_ν be a unitary mapping of $\mathbb{C}^n$ sending l^ν_j to the z_j axis and q^j_ν to the positive imaginary axis $\Im(z_j)$. Let T_ν, T'_ν be translations sending q_ν to the origin, and the origin to $(-\tau^\nu_1 i, 0')$ respectively. The composition $h^\nu = T'_\nu \circ U_\nu \circ T_\nu$ gives a coordinate system centered at q^1_ν.

Define the dilations

$$\Lambda^\nu(z) = (z_1/\tau_1^\nu, z_2/\tau_2^\nu, \ldots, z_n/\tau_n^\nu) \tag{3.7}$$

and the dilated domains

$$\mathcal{D}_\nu = \{z : \rho \circ (h^\nu)^{-1} \circ (\Lambda^\nu)^{-1}(z) < 0\}. \tag{3.8}$$

Note that $\mathcal{D}_\nu$ is convex and $(-i, 0') \in \mathcal{D}_\nu$ for all ν. Among other things, the following two claims were proved in [**11**] (cf. [**8**]): firstly, $\mathcal{D}_\nu$ converges to

$$\mathcal{G} = \{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1} : \Im(z_1) + P(z') < 0\}, \tag{3.9}$$

a convex domain of finite type at most $2m$. Moreover, $P(z')$ is a convex polynomial of degree at most $2m$. Secondly, for all large ν, $\mathcal{D}_\nu$ and hence $\mathcal{G}$ are all contained in the intersection $H_1 \cap H_2 \cap \cdots \cap H_n$ where each H_j is a half space of the form

$$\begin{aligned} H_1 &= \{z \in \mathbb{C}^n : \Im(z_1) < 0\}, \\ H_j &= \left\{ z \in \mathbb{C}^n : \Im\left(\alpha_j z_j + \sum_{k<j} \alpha_{j,k} z_k \right) < 0 \right\} \end{aligned} \tag{3.10}$$

for $j \geq 2$ with $\alpha_j \in \mathbb{R} \setminus \{0\}$ for all $j \geq 2$ and $\alpha_{j,k} \in \mathbb{R}$ for all j, k.

Proof of Theorem 1.2: (i) Suppose that the assertion is false. Then there exists $\eta \in (0,1)$, a sequence of correspondences $\hat{f}^\nu, q_\nu \in \hat{f}^\nu(0)$, and a sequence of radii $R_\nu \to \infty$ such that

$$\hat{f}^\nu_{0,q_\nu}(\eta \mathbb{B}^p) \not\subset B^k_{\mathcal{D}}(q_\nu, R_\nu) \tag{3.11}$$

for all ν. By Theorem 1.1 $\{q_\nu\}$ cannot cluster at any interior point of $\mathcal{D}$. After passing to a subsequence assume that $q_\nu \to q_\infty \in \partial\mathcal{D}$ and $q_\infty = 0$ after a translation of coordinates. Pick $\tau > 0$ so that $U := B(0,\tau)$ is such that $\mathcal{D} \cap U$ is a convex domain of finite type. Fix $\eta < \eta' < 1$. Proposition 5.1 in [**22**] shows the existence of $\delta > 0$ such that

$$\hat{f}^\nu_{0,q_\nu}(\eta' \mathbb{B}^p) \subset B(0,\tau) \tag{3.12}$$

provided $q_\nu \in B(0,\tau)$. This is true for all large ν as $q_\nu \to 0$. By (3.11)

$$\hat{f}^\nu_{0,q_\nu}(\eta \mathbb{B}^p) \not\subset B^k_{\mathcal{D}\cap U}(q_\nu, R_\nu) \tag{3.13}$$

for all ν. Note that $\Lambda^\nu \circ h^\nu(q_\nu) = (-i, 0') \in \mathcal{D}_\nu$ for all ν. Since biholomorphic mappings are isometries for the Kobayashi metric it follows that

$$(3.14) \qquad \big(\Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,(-i,0')}(\eta \mathbb{B}^p) \not\subset B^k_{\Lambda^\nu \circ h^\nu(\mathcal{D}\cap U)}\big((-i,0'), R_\nu\big)$$

for all large ν. Consider the family

$$(3.15) \quad \mathcal{F} = \big\{\big(\Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,(-i,0')}\big\} \subset \mathrm{Cor}\,\big(\eta'\mathbb{B}^p, \Lambda^\nu \circ h^\nu(\mathcal{D}\cap U), k\big)$$

of scaled correspondences. We will show that a subsequence of $\mathcal{F}$ converges to a limiting correspondence $\hat{f}^\infty \in \mathrm{Cor}\,\big(\eta'\mathbb{B}^p, \overline{\mathcal{G}}, k\big)$ where $\mathcal{G}$ is as in (3.9). The convergence of $\Lambda^\nu \circ h^\nu(\mathcal{D}\cap U)$ to $\mathcal{G}$ follows from [**11**] as described earlier. To show the convergence of correspondences fix an arbitrary sequence $0 < \eta_1 < \eta_2 < \cdots < \eta_n < \eta'$. We will first show that a subsequence of $\mathcal{F}$ converges on $\eta_1\mathbb{B}^p$. For this consider the projections $\pi_j(z) = z_j$ onto the coordinate axes. Then (3.10) shows that

$$(3.16) \qquad \pi_1 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu(\eta'\mathbb{B}^p) \subset H_1 = \{\Im(z_1) < 0\}$$

and moreover $\pi_1 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu(0) = -i$ for all ν. By Pinchuk's version of the Schwarz lemma (cf. [**18**]) it follows that the image

$$\big(\pi_1 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,-i}(\eta_n\mathbb{B}^p)$$

is uniformly bounded for all large ν. Now consider the correspondences $\big(\pi_2 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,0}\colon \eta'\mathbb{B}^p \to \mathbb{C}$. For $z \in \eta_n\mathbb{B}^p$ and $w \in \big(\pi_2 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,0}(z)$ we must have

$$(3.17) \qquad \Im(\alpha_2 w) < -\Im\Big(\alpha_{2,1}\big(\pi_1 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,-i}(z)\Big)$$

for all large ν by (3.10) again. The right side is uniformly bounded above if $z \in \eta_n\mathbb{B}^p$ and since $\alpha_2 \neq 0$, it follows from the Schwarz lemma in [**18**] that

$$\big(\pi_2 \circ \Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,0}(\eta_{n-1}\mathbb{B}^p)$$

is also uniformly bounded for all large ν. Proceeding inductively in the same way as above and using the form of the half spaces H_j we see that each component of $\big(\Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\big)_{0,(-i,0')}$ is uniformly bounded on $\eta_1\mathbb{B}^p$. Thus the family of symmetric functions of all the branches of each member of $\mathcal{F}$ is uniformly bounded on compact sets of $\eta'\mathbb{B}^p$. By Montel's theorem and by passing to the diagonal subsequence we get a well defined correspondence $\hat{f}^\infty$ defined on $\eta'\mathbb{B}^p$. Since the domains $\Lambda^\nu \circ h^\nu(\mathcal{D}\cap U)$ converge on compact sets to $\mathcal{G}$ it follows that $\hat{f}^\infty \in \mathrm{Cor}\,\big(\eta'\mathbb{B}^p, \overline{\mathcal{G}}, k\big)$.

If $z' \in \hat{f}^\infty(z) \cap \partial\mathcal{G}$ for some $z \in \mathbb{B}^p$, Theorem 1.1 can be used to peel off the 'constant' component of $\hat{f}^\infty$ that contains (z, z'). This peeling off process cannot exhaust all the branches of $\hat{f}^\infty$ as $(0, (-i, 0')) \in \hat{f}^\infty(0)$ is not in $\partial\mathcal{G}$. Moreover by Theorem 1.1 again, we must have

$$\hat{f}^\infty_{0,(-i,0')}(\eta\mathbb{B}^p) \subset U \tag{3.18}$$

where U is a relatively compact neighbourhood of $(-i, 0')$ in $\mathcal{G}$. Since the diagonal subsequence, which we will still denote by $\Lambda^\nu \circ h^\nu \circ \hat{f}^\nu$ converges uniformly to $\hat{f}^\infty$ on $\eta\mathbb{B}^p$, the continuity of roots of these correspondences implies that there exists a relatively compact neighbourhood of U in $\mathcal{G}$, say U' such that

$$\left(\Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\right)_{0,(-i,0')}(\eta\mathbb{B}^p) \subset U' \Subset \mathcal{G} \tag{3.19}$$

for all large ν. The convergence of the domains $\Lambda^\nu \circ h^\nu(\mathcal{D} \cap U)$ on compact sets implies that

$$\left(\Lambda^\nu \circ h^\nu \circ \hat{f}^\nu\right)_{0,(-i,0')}(\eta\mathbb{B}^p) \subset U' \Subset \Lambda^\nu \circ h^\nu(\mathcal{D} \cap U) \tag{3.20}$$

for all large ν. This shows that the distance between U' and the boundary of $\Lambda^\nu \circ h^\nu(\mathcal{D} \cap U)$ is uniformly bounded from below for all large ν. This contradicts the assumption made in (3.14) and thus the theorem follows.

(ii) For a given $R > 0$, let $U \subset B^k_{\mathcal{D}}(q, R)$ be a neighbourhood of q in the Euclidean metric. Then Theorem 1.1 (ii) shows the existence of $\eta \in (0, 1)$, independent of $\hat{f}$ such that

$$\hat{f}_{0,q}(\eta\mathbb{B}^p) \subset U \subset B^k_{\mathcal{D}}(q, R)$$

holds. □

Scaling near a point where the boundary is convex and of finite type also leads to the next theorem.

Proof of Theorem 1.3: Let $\mathcal{F} := \{\hat{f}^\nu\} \subset \mathrm{PropCor}(\mathcal{D}, \mathcal{D}, k, k')$ be such that the orbit of $z_0 \in \mathcal{D}$ with respect to $\mathcal{F}$ accumulates at $\zeta_0 \in \partial\mathcal{D}$. Fix an arbitrary $K \Subset \mathcal{D}$ containing z_0. Let $z_\nu \in \hat{f}^\nu(z_0)$ be such that $z_\nu \to \zeta_0$. Fix $\epsilon > 0$ so that $\partial\mathcal{D} \cap B(\zeta_0, \epsilon)$ is convex of finite type. Then by Proposition 5.1 in [**22**]

$$\hat{f}^\nu_{z_0,z_\nu}(K) \subset \mathcal{D} \cap B(\zeta_0, \epsilon) \tag{3.21}$$

for all large ν. From the previous theorem we can conclude two things: firstly, the domain $\mathcal{D} \cap B(\zeta_0, \epsilon)$ can be scaled to get the domain $\mathcal{G}$. Secondly, the family of correspondences given by

$$\hat{f}^{\nu}_{z_0, z_\nu} \colon K \to \mathcal{D} \cap B(\zeta_0, \epsilon) \tag{3.22}$$

converges to a correspondence $\hat{f}^{\infty}_K \in \mathrm{Cor}(K, \overline{\mathcal{G}}, k)$. By exhausting $\mathcal{D}$ with an increasing sequence of relatively compact sub-domains that contain z_0 and passing to the diagonal subsequence, we get a correspondence $\hat{f}^{\infty} \in \mathrm{Cor}(\mathcal{D}, \overline{\mathcal{G}}, k)$. Since $\mathcal{D}$ is bounded, the scaled family of the inverses of $\hat{f}^{\nu}$ also converges to a correspondence $\hat{g}^{\infty} \in \mathrm{Cor}(\mathcal{G}, \overline{\mathcal{D}}, k')$. Proposition 7 in [**14**] shows that $\hat{f}^{\infty} \in \mathrm{Cor}(\mathcal{D}, \mathcal{G}, k)$ and $\hat{g}^{\infty} \in \mathrm{Cor}(\mathcal{G}, \mathcal{D}, k')$ and that $\hat{f}^{\infty}$ and $\hat{g}^{\infty}$ are inverses of each other. The desired proper correspondence $\hat{F}$ is given by $\hat{f}^{\infty}$. Finally, let $\hat{f}^{\infty}_1, \dots, \hat{f}^{\infty}_k$ be the branches of $\hat{f}^{\infty}$ which are locally well defined away from an analytic set of codimension at least 1 in $\mathcal{D}$. If $\psi(z)$ is a plurisubharmonic exhaustion for $\mathcal{G}$, then $\max(\psi \circ \hat{f}^{\infty}_1, \dots, \psi \circ \hat{f}^{\infty}_k)$ is a plurisubharmonic exhaustion for $\mathcal{D}$ and hence $\mathcal{D}$ is pseudoconvex. $\square$

In case ζ_0 is a strongly pseudoconvex point, then $P(z')$ is a convex, positive definite Hermitian form of degree 2 (ζ_0 has finite type 2). After a change of coordinates, $P(z') = \sum_{j=1}^{n-1} |z_j|^2$. Thus $\mathcal{G}$ is equivalent to $\mathbb{B}^n$.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, let $s_i(z), 1 \le i \le n$ denote the elementary symmetric polynomial in z of degree i. That is, $s_i(z)$ is the sum of all possible products of $z_1, \dots, z_n$ taken i at a time. Define $\tau \colon \mathbb{C}^n \to \mathbb{C}^n$ in the following way:

$$\tau(z_1, \dots, z_n) = (s_1(z), \dots, s_n(z)). \tag{3.23}$$

It is known (for example [**15**, p. 86]) that τ is a finite-to-one, proper surjection.

Proof of Proposition 1.1: Let $\mathcal{F} = \{\hat{f}^{\nu}\} \in \mathrm{PropCor}(\mathcal{D}, \mathcal{D}, k, k')$ and suppose that $z_\nu \in \hat{f}^{\nu}(z)$ converges to ζ_0 for a given $z \in \mathcal{D}$. To obtain a contradiction, assume that $\zeta_0 \in \hat{\mathcal{D}}$. Consider the family of all symmetric functions of the various branches of $\left(\hat{f}^{\nu}\right)^{-1}$. The boundedness of $\mathcal{D}$ shows that this is a uniformly bounded family of holomorphic functions on $\mathcal{D}$. All these functions will extend to a small ball, say $B(\zeta_0, \epsilon)$ for some $\epsilon > 0$. Note that the extended functions which are now defined on $\mathcal{D} \cup B(\zeta_0, \epsilon)$ will still be uniformly bounded. Since τ is proper, it follows that

$$\left(\hat{f}^{\nu}\right)^{-1} \colon \mathcal{D} \cup B(\zeta_0, \epsilon) \to B(0, L) \tag{3.24}$$

for some large L. Moreover the number of branches is still at most k'. Consider now the correspondences

$$(\hat{f}^\nu)^{-1}_{z_\nu, z} \colon B(z_\nu, \epsilon) \to B(0, L). \tag{3.25}$$

Choose $r > 0$ small enough so that $B(z, r) \Subset \mathcal{D}$. Theorem 1.1 (ii) now yields $0 < \eta < \epsilon$, independent of ν, such that

$$(\hat{f}^\nu)^{-1}_{z_\nu, z}(B(z_\nu, \eta)) \subset B(z, r). \tag{3.26}$$

For large ν, the ball $B(z_\nu, \eta)$ intersects $\partial\mathcal{D}$ near ζ_0. By (3.26), a piece of $\partial\mathcal{D}$ is mapped inside $B(z, r)$. This contradicts the fact that $(\hat{f}^\nu)^{-1}$ is proper. $\square$

Note. Not all the references given below have been stated in the article. However, they all contain material that is relevant to the above presentation.

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